A STEP TOWARDS SYSTEMATIC STUDIES OF THE CUSP IN $\eta \to 3\pi^0$ DECAY*

Andrzej Kupść[a], Akaki Rusetsky[b], Carl-Oscar Gullström[a]

[a]Department of Physics and Astronomy, Uppsala University
75120 Uppsala, Sweden
[b]Helmholtz-Institut für Strahlen- und Kernphysik
and
Bethe Center for Theoretical Physics, University of Bonn
53115 Bonn, Germany

(Received February 4, 2009)

A realistic estimate of the cusp effect in the $\eta \to 3\pi^0$ decay is required for the forthcoming high precision experiments. An estimate for the size of this effect was given recently within the framework of non-relativistic effective field theory. Here we present the next step towards the systematic way of fixing the parameters of the effective non-relativistic Lagrangian and consistency tests of the existing data on $\eta \to \pi^0\pi^0\pi^0$ and $\eta \to \pi^+\pi^-\pi^0$ decays.

PACS numbers: 13.25.–k, 12.39.Fe, 13.75.Lb

1. Introduction

The invariant mass of two pions ($M_{\pi^0\pi^0}$) for $\eta \to 3\pi^0$ decay extends below the charged two-pion threshold. It means that a cusp structure should be visible in this distribution around $2M_{\pi^\pm}$, in analogy with the pronounced cusp in $K^+ \to \pi^+\pi^0\pi^0$ decay, observed recently by NA48/2 Collaboration [1]. In this paper, in particular, it has been shown that measuring charged kaon decays in the cusp region enables one to precisely determine S-wave $\pi\pi$ scattering lengths a_0 and a_2, provided an accurate theoretical parameterization of the invariant mass distribution in terms of these scattering lengths is known [2–5] (The strong impact of the unitarity cusp on $\pi^0\pi^0$ scattering was already mentioned in Ref. [6]). Moreover, the same logic applies to the neutral kaon decays into three pions, which have been studied in the recent experiment [7]. The theoretical framework for analysis of the neutral kaon decays is provided in Refs. [3, 4, 8] and the systematic inclusion of the electromagnetic effects both in charged and neutral kaon decays is considered in Ref. [9].

* Presented at the Symposium on Meson Physics, Kraków, Poland, October 1–4, 2008.

We further mention that the general structure of the amplitude in the neutral kaon decays is similar to the $\eta \to 3\pi^0$ decay amplitude. For this reason, *e.g.*, the two-loop representation of the amplitude in terms of the $\pi\pi$ effective-range expansion parameters, derived in Ref. [8], can be directly used to predict the cusp in the $\eta \to 3\pi^0$ decay, which is studied in KLOE, Crystal Ball and WASA Collaboration experiments [10–14]. Note that the cusp effect in the $\eta \to 3\pi^0$ decay has been addressed already in various settings, *e.g.* in Refs. [15–17].

It should be pointed out that the two-loop formula for the kaon and eta decay amplitudes have been obtained in Refs. [5, 8] within the non-relativistic effective field theory framework. This framework is ideally suited for parameterizing the final-state interactions in terms of the $\pi\pi$ *scattering lengths* (effective-range parameters, in general), whereas the expansion of the amplitudes in Chiral Perturbation Theory (ChPT) is performed in powers of the *quark masses* and is less convenient for expressing the amplitude in the cusp region in terms of the observable quantities. (Note that the non-relativistic approach has been successfully applied recently to study of the K_{e4} decays [18].)

The aim of the present work is to use the two-loop representation, derived in Ref. [8], to cross-check the size of the cusp in the invariant mass distribution for $\eta \to 3\pi^0$ decays, predicted in our previous paper [19]. In addition, we shall apply the same framework to study the consistency of the experimental data on the two $\eta \to 3\pi$ decay modes. At present, the theory and experiment have not yet converged to a common denominator on the Dalitz parameters for the decays. Specially, the quadratic slope for $\eta \to 3\pi^0$ decay, which was studied in detail by Crystal Ball [10, 13], WASA [14], KLOE [11] experiments, provides a window on the higher order ChPT effects. For example ChPT at one loop in the isospin symmetry limit [20] predicts a different sign for this parameter as compared to the experimentally measured one. At two loops, the sign of this quantity is no more fixed due to the large error bars coming from the unknown low-energy constants in ChPT [21] albeit the central value is still positive (The isospin-breaking corrections at one loop have been calculated in Refs. [17, 22, 23] and are found to be small.). However, the predicted sign in Ref. [24] where the calculations were done in the framework of unitarized ChPT, as well as the sign emerging in dispersive calculations [25, 26], agree with the existing experimental data. We believe that in the high-precision measurements of the slope parameter it will be very important to use as accurate a parameterization of the decay amplitude, as possible. The parameterization should be based on solid theoretical ground and, in particular, should take into account the cusp phenomenon which emerges at the physical values of the pion masses.

2. Theoretical framework

Below we display the main formulae from our previous work [19], which follows the notations of Ref. [8]. The tree-level amplitudes are expressed in terms of the quantities X_i

$$X_i = E_i - M_{\pi^0}\,, \tag{1}$$

where E_i denote the pion energies in the eta rest frame. Up to the quadratic terms in X_i,

$$\begin{aligned} \mathcal{M}_{000}^{\text{tree}} &= K_0 + K_1(X_1^2 + X_2^2 + X_3^2)\,, \\ \mathcal{M}_{+-0}^{\text{tree}} &= L_0 + L_1 X_3 + L_2 X_3^2 + L_3(X_1 - X_2)^2\,, \end{aligned} \tag{2}$$

where L_i, K_i are the effective couplings in the non-relativistic Lagrangian that describe $\eta \to 3\pi$ decays at tree level. Note that we use the same notation for these couplings as in Ref. [8], where they denote the couplings describing the 3-pion decays of the neutral kaons.

Assuming $\Delta I = 1$ rule in the $\eta \to 3\pi$ vertex, the isospin symmetry relates the amplitudes for $\eta \to 3\pi^0$ and $\eta \to \pi^+\pi^-\pi^0$ (we use Condon–Shortley phase convention)

$$\begin{aligned} \mathcal{M}_{000}(s_1, s_2, s_3) &= -\mathcal{M}_{+-0}(s_1, s_2, s_3) \\ &\quad - \mathcal{M}_{+-0}(s_2, s_3, s_1) - \mathcal{M}_{+-0}(s_3, s_1, s_2)\,. \end{aligned} \tag{3}$$

At tree level, this allows one to express the couplings K_i through L_i

$$\begin{aligned} K_0 &= -(3L_0 + L_1 Q - L_3 Q^2)\,, \\ K_1 &= -(L_2 + 3L_3)\,, \end{aligned} \tag{4}$$

where $Q = M_\eta - 3M_{\pi^0}$. In real world, the relations (4) hold up to isospin-breaking corrections.

In general, $\eta \to 3\pi$ decay amplitudes are given in a form of a sum of the tree, one-loop, two-loop, ... contributions $\mathcal{M}_{000} = \mathcal{M}_{000}^{\text{tree}} + \mathcal{M}_{000}^{1-\text{loop}} + \mathcal{M}_{000}^{2-\text{loops}} + \ldots$, and similarly for $\mathcal{M}_{+-0}$. The pertinent (rather lengthy) expressions are given in Ref. [8]. We do not display them here. It can be checked that these amplitudes in the isospin symmetry limit explicitly obey the constraints (3) at one- and two-loop level.

The representations given in Refs. [5, 8] should be understood as a parameterization to be fitted to the data. In other words, the constants $L_i, a_0, a_2, \ldots$ are considered as free parameters to be fixed from the fit (K_i are determined from Eq. (3)). We have, in addition, fixed a_0, a_2 to their theoretical values [27] and neglected isospin breaking in the derivative 4-pion couplings, as well as the shape parameter and the P-waves. Contrary to our previous paper [19], where the first rough estimate of the size of the cusp effect has been provided by carrying out the tree-level matching for L_i, here we make an attempt to consistently determine the parameters L_i by fitting the two-loop representation for the charged amplitude to the KLOE data on the $\eta \to \pi^+\pi^-\pi^0$ decay [28].

The KLOE results are given in terms of a polynomial parameterization using Dalitz plot variables x and y:

$$x = \frac{\sqrt{3}(E_1 - E_2)}{Q_+}, \qquad y = \frac{3(E_3 - M_{\pi 0})}{Q_+} - 1\,. \tag{5}$$

where $Q_+ = M_\eta - 2M_{\pi^\pm} - M_{\pi^0}$ is the excess energy for the $\eta \to \pi^+\pi^-\pi^0$ decay. The following expression for the Dalitz plot density was used:

$$|\mathcal{M}_{+-0}|^2 \;=\; N\left(1 + ay + by^2 + dx^2 + fy^3\right)\,, \tag{6}$$

where a, b, d, f are fit parameters:

$$\begin{aligned} a &= -1.090 \pm 0.005(\text{stat})^{+0.008}_{-0.019}(\text{syst})\,,\\ b &= 0.124 \pm 0.006(\text{stat}) \pm 0.010(\text{syst})\,,\\ d &= 0.057 \pm 0.006(\text{stat})^{+0.007}_{-0.016}(\text{syst})\,,\\ f &= 0.14 \pm 0.01(\text{stat}) \pm 0.02(\text{syst})\,. \end{aligned} \tag{7}$$

The overall normalization N is also fitted. In the present attempt we reconstruct the experimental histogram by creating Dalitz plot histogram with the same binning as KLOE ($\Delta x = \Delta y = 0.125$ bins) and filling it with 1.3×10^6 synthetic events generated with the density given by the central values of Eq. (7). We ignore for the moment the spread of the parameters as well as the correlation matrix. For fitting of the parameters L_i we apply the same technique, considering only the bins contained fully inside the Dalitz plot boundary (with our method of selecting the bins we arrive at 152 bins — two bins less than KLOE). The procedure was checked by fitting back the parameterization from Eq. (6) and obtaining consistent values with Eq. (7).

3. Results

We use two-loop expression for the $\mathcal{M}_{+-0}$ amplitude in order to fit the couplings L_i to the KLOE Dalitz plot. The obtained value for χ^2/ndf is 129/148. The couplings, determined in this way, are then used in the two-loop expressions for the amplitude $\mathcal{M}_{000}$ in order to predict the differential

$d\Gamma/dM_{\pi^0\pi^0}$ and $d\Gamma/dz$ distributions for the $\eta \to 3\pi^0$ decay. The variable z is given by:

$$z = x^2 + y^2 \,, \tag{8}$$

with x and y evaluated now using the excess energy Q appropriate for the $\eta \to \pi^0\pi^0\pi^0$ decay. The plotted $d\Gamma/dM_{\pi^0\pi^0}$ and $d\Gamma/dz$ distributions are divided by the phase space. The decay amplitude is normalized in the center of the Dalitz plot

$$|\mathcal{M}_{000}(s_0, s_0, s_0)|^2 = 1 \,, \qquad s_0 = \frac{M_\eta^2}{3} + M_{\pi^0}^2 \,. \tag{9}$$

The results are given by the dashed line in Fig. 1 and Fig. 2. We compare the predictions to the recent precise Crystal Ball data [13].

The resulting cusp in the $d\Gamma/dM_{\pi^0\pi^0}$ distribution amounts roughly up to a 2% effect. We would like to mention that the sign of the cusp effect is fixed by the isospin symmetry, see Eqs. (3) and (4) and is thus a robust theoretical prediction.

We have checked the convergence of the non-relativistic expansion by using only one-loop expressions for the L_i fit and for the predictions for the $\eta \to 3\pi^0$ decay. The results are practically identical to the two-loop case. The predictions give steeper z dependence than in the Crystal Ball data. As a further cross-check, we have fitted the couplings using only tree expression from Eq. (2). The results are given by the dotted lines in Fig. 1 and Fig. 2. The predicted distributions are significantly different from the one- and two-loop result.

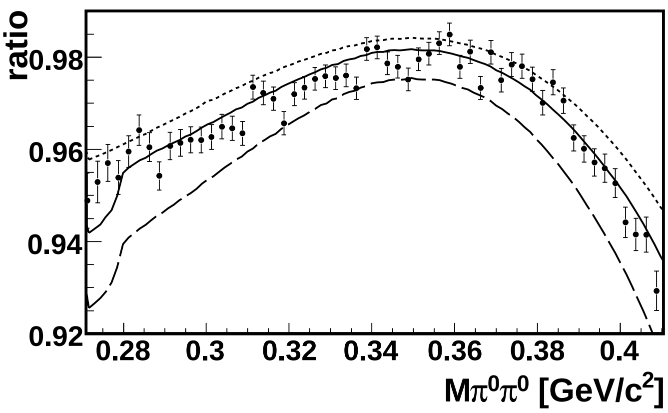

Fig. 1. Invariant mass distribution $d\Gamma/dM_{\pi^0\pi^0}$ divided by the phase space: (1) 2-loop result with fit to KLOE data (dashed line); (2) 2-loop result with combined fit to KLOE and Crystal Ball data (solid line) (3) tree result fitted to KLOE data (dotted line). The data points are from the Crystal Ball experiment [13].

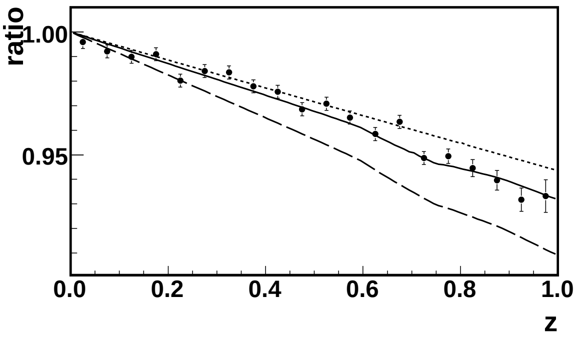

Fig. 2. $d\Gamma/dz$ distribution divided by the phase space: (1) 2-loop result with fit to KLOE data (dashed line); (2) 2-loop result with combined fit to KLOE and Crystal Ball data (solid line) (3) tree result fitted to KLOE data (dotted line). The data points are from the Crystal Ball experiment [13].

Finally, we have made an attempt to check, if a simultaneous description of the KLOE and Crystal Ball data is possible. For this purpose, we have included 10 Crystal Ball data points of the z distribution in the range $0 < z < 0.5$, (*i.e.*, outside the cusp region) into simultaneous fit using two-loop expressions for the amplitudes. The results are given in Fig. 1 and Fig. 2 by the solid curves. The obtained values for χ^2/ndf are 158/158 for all data points of the simultaneous fit and 141/148 for the KLOE data only. This preliminary result indicates that — assuming the isospin symmetry in the couplings — the two data sets are consistent within the experimental errors.

4. Conclusions

Performing a systematic fit of the two-loop amplitude in the non-relativistic EFT to the existing data for $\eta \to \pi^+\pi^-\pi^0$ decay enables one to determine $\eta \to 3\pi$ decay couplings in the non-relativistic Lagrangian. Using these values for the couplings, we have confirmed that the size of the cusp effect in the invariant mass distribution for the process $\eta \to 3\pi^0$ is consistent with our previous estimate [19].

We thank J. Bijnens, J. Gasser, B. Kubis, U.-G. Meissner, R. Nissler and S. Prakhov for useful discussions. We are grateful to Sergey Prakhov for providing the numerical values of the Crystal Ball experimental data points. We appreciate help of Biagio Di Micco in the interpretation of the KLOE results. Partial financial support under the EU Integrated Infrastructure Initiative Hadron Physics Project (contract number RII3–CT–2004–506078) and DFG (SFB/TR 16, "Subnuclear Structure of Matter") is gratefully acknowledged. This work was supported by EU MRTN–CT–2006–035482 (FLAVIAnet). One of us (A.K.) would like to thank Paweł Moskal for the invitation to the Symposium.

REFERENCES

[1] J.R. Batley *et al.* [NA48/2 Collaboration], *Phys. Lett.* **B633**, 173 (2006) [arXiv: hep-ex/0511056].

[2] N. Cabibbo, *Phys. Rev. Lett.* **93**, 121801 (2004) [arXiv: hep-ph/0405001].

[3] N. Cabibbo, G. Isidori, *J. High Energy Phys.* **0503**, 021 (2005) [arXiv: hep-ph/0502130].

[4] E. Gamiz, J. Prades, I. Scimemi, *Eur. Phys. J.* **C50**, 405 (2007) [arXiv: hep-ph/0602023].

[5] G. Colangelo, J. Gasser, B. Kubis, A. Rusetsky, *Phys. Lett.* **B638**, 187 (2006) [arXiv: hep-ph/0604084].

[6] U.-G. Meissner, G. Müller, S. Steininger, *Phys. Lett.* **B406**, 154 (1997) Erratum **B407**, 454 (1997), [arXiv: hep-ph/9704377].

[7] E. Abouzaid *et al.* [KTeV Collaboration], *Phys. Rev.* **D78**, 032009 (2008) [arXiv:0806.3535 [hep-ex]].

[8] M. Bissegger, A. Fuhrer, J. Gasser, B. Kubis, A. Rusetsky, *Phys. Lett.* **B659**, 576 (2008) [arXiv:0710.4456 [hep-ph]].

[9] M. Bissegger, A. Fuhrer, J. Gasser, B. Kubis, A. Rusetsky, *Nucl. Phys.* **B806**, 178 (2009) [arXiv:0807.0515 [hep-ph]].

[10] W.B. Tippens *et al.* [Crystal Ball Collaboration], *Phys. Rev. Lett.* **87**, 192001 (2001).

[11] F. Ambrosino *et al.* [KLOE Collaboration], arXiv:0707.4137 [hep-ex].

[12] M. Bashkanov *et al.*, *Phys. Rev.* **C76**, 048201 (2007) [arXiv:0708.2014 [nucl-ex]].

[13] S. Prakhov *et al.*, arXiv:0812.1999 [hep-ex].

[14] C. Adolph *et al.* [WASA-at-COSY Collaboration], arXiv:0811.2763 [nucl-ex].

[15] J. Belina, Diploma thesis, University of Bern, 2006.

[16] R. Nissler, PhD thesis, University of Bonn, 2007.

[17] C. Ditsche, B. Kubis, U.-G. Meissner, *Eur. Phys. J.* **C60**, 83 (2009) [arXiv:0812.0344 [hep-ph]].

[18] G. Colangelo, J. Gasser, A. Rusetsky, *Eur. J. Phys.* **C59**, 777 (2009) [arXiv:0811.0775 [hep-ph]].

[19] C.O. Gullstrom, A. Kupsc, A. Rusetsky, arXiv:0812.2371 [hep-ph].

[20] J. Gasser, H. Leutwyler, *Nucl. Phys.* **B250**, 539 (1985).

[21] J. Bijnens, K. Ghorbani, *J. High Energy Phys.* **0711**, 030 (2007) [arXiv:0709.0230 [hep-ph]].

[22] R. Baur, J. Kambor, D. Wyler, *Nucl. Phys.* **B460**, 127 (1996) [arXiv: hep-ph/9510396].

[23] A. Deandrea, A. Nehme, P. Talavera, *Phys. Rev.* **D78**, 034032 (2008) [arXiv:0803.2956 [hep-ph]].

[24] B. Borasoy, R. Nissler, *Eur. Phys. J.* **A26**, 383 (2005) [arXiv: hep-ph/0510384].

[25] J. Kambor, C. Wiesendanger, D. Wyler, *Nucl. Phys.* **B465**, 215 (1996) [arXiv:9509374 [hep-ph]].

[26] J. Bijnens, J. Gasser, *Phys. Scr.* **T99**, 34 (2002) [arXiv: hep-ph/0202242].

[27] G. Colangelo, J. Gasser, H. Leutwyler, *Nucl. Phys.* **B603**, 125 (2001) [arXiv: hep-ph/0103088].

[28] A. Antonelli *et al.* [KLOE Collaboration], *J. High Energy Phys.* **0805**, 006 (2008) [arXiv:0801.2642 [hep-ex]].