RESUMMATION IN FRACTIONAL APT: HOW MANY LOOPS DO WE NEED TO TAKE INTO ACCOUNT?*

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We give a short introduction to the Analytic Perturbation Theory (APT) [D.V. Shirkov, I.L. Solovtsov, *JINR Rapid Commun.* **2**, 5 (1996); *Phys. Rev. Lett.* **79**, 1209 (1997); *Theor. Math. Phys.* **150**, 132 (2007)] and its generalization to fractional powers — FAPT [A.P. Bakulev, S.V. Mikhailov, N.G. Stefanis, *Phys. Rev.* **D72**, 074014 (2005), 119908(E); **75**, 056005 (2007); **77**, 079901(E) (2008) and A.P. Bakulev, A.I. Karanikas, N.G. Stefanis, *Phys. Rev.* **D72**, 074015 (2005)]. We describe how to treat heavy-quark thresholds in FAPT and then show how to resume perturbative series in both the one-loop APT and FAPT. As an application we consider FAPT description of the Higgs boson decay $H^0 \to b\bar{b}$.

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1. APT and FAPT in QCD

In the standard QCD Perturbation Theory (PT) we know the Renormalization Group (RG) equation $da_{\rm s}[L]/dL = -a_{\rm s}^2 - \ldots$ for the effective coupling $\alpha_{\rm s}(Q^2) = a_{\rm s}[L]/\beta_f$ with $L = \ln(Q^2/\Lambda^2)$, $\beta_f = b_0(N_f)/(4\pi) = (11 - 2N_f/3)/(4\pi)$[1]. Then the one-loop solution generates Landau pole singularity, $a_{\rm s}[L] = 1/L$.

In the Analytic Perturbation Theory we have different effective couplings in Minkowskian (Radyushkin [4], and Krasnikov, Pivovarov [5]) and Euclidean (Shirkov, Solovtsov [1]) regions. In Euclidean domain, $-q^2 = Q^2$, $L = \ln Q^2/\Lambda^2$, APT generates the following set of images for the effective coupling and its n-th powers, $\{\mathcal{A}_n[L]\}_{n\in\mathbb{N}}$, whereas in Minkowskian domain, $q^2 = s$, $L_{\rm s} = \ln s/\Lambda^2$, it generates another set, $\{\mathfrak{A}_n[L_{\rm s}]\}_{n\in\mathbb{N}}$. APT is based

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[1] We use notations $f(Q^2)$ and $f[L]$ in order to specify the arguments we mean — squared momentum Q^2 or its logarithm $L = \ln(Q^2/\Lambda^2)$, that is $f[L] = f(\Lambda^2\, e^L)$ and Λ^2 is usually referred to $N_f = 3$ region.

on the RG and causality that guaranties standard perturbative UV asymptotics and spectral properties. Power series $\sum_m d_m a_s^m[L]$ transforms into non-power series $\sum_m d_m \mathcal{A}_m[L]$ in APT.

By the analytization in APT for an observable $f(Q^2)$ we mean the "Källen–Lehman" representation

$$\left[f(Q^2)\right]_{\text{an}} = \int\limits_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma\,, \qquad \text{with} \qquad \rho_f(\sigma) = \frac{1}{\pi}\,\text{Im}\,[f(-\sigma)]\,. \qquad (1)$$

Then in the one-loop approximation for the running coupling its spectral density is $\rho_1(\sigma) = 1/\sqrt{L_\sigma^2 + \pi^2}$ and

$$\mathcal{A}_1[L] \;=\; \int\limits_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2}\, d\sigma \;=\; \frac{1}{L} - \frac{1}{e^L - 1}\,, \qquad (2a)$$

$$\mathfrak{A}_1[L_s] \;=\; \int\limits_s^\infty \frac{\rho_1(\sigma)}{\sigma}\, d\sigma \;=\; \frac{1}{\pi}\arccos\frac{L_s}{\sqrt{\pi^2 + L_s^2}}\,, \qquad (2b)$$

whereas analytic images of the higher powers ($n \geq 2, n \in \mathbb{N}$) are:

$$\begin{pmatrix}\mathcal{A}_n[L]\\ \mathfrak{A}_n[L_s]\end{pmatrix} \;=\; \frac{1}{(n-1)!}\left(-\frac{d}{dL}\right)^{n-1}\begin{pmatrix}\mathcal{A}_1[L]\\ \mathfrak{A}_1[L_s]\end{pmatrix}. \qquad (3)$$

In the standard QCD PT we have also:

(i) the factorization procedure in QCD that gives rise to the appearance of logarithmic factors of the type: $a_s^\nu[L]\,L$ [2];

(ii) the RG evolution that generates evolution factors of the type: $B(Q^2) = \left[Z(Q^2)/Z(\mu^2)\right]\, B(\mu^2)$, which reduce in the one-loop approximation to $Z(Q^2) \sim a_s^\nu[L]$ with $\nu = \gamma_0/(2b_0)$ being a fractional number.

All this means we need to construct analytic images of new functions: $a_s^\nu,\ a_s^\nu\, L^m,\ \ldots$.

In the one-loop approximation using recursive relation (3) we can obtain explicit expressions for $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$:

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{F(e^{-L}, 1-\nu)}{\Gamma(\nu)}\,; \quad \mathfrak{A}_\nu[L] = \frac{\sin\left[(\nu-1)\arccos\left(\frac{L}{\sqrt{\pi^2+L^2}}\right)\right]}{\pi(\nu-1)\,(\pi^2+L^2)^{(\nu-1)/2}}\,. \quad (4)$$

[2] First indication that a special "analytization" procedure is needed to handle these logarithmic terms appeared in [6].

Here $F(z,\nu)$ is the reduced Lerch transcendental function, which is an analytic function in ν. They have very interesting properties, which we discussed extensively in our previous papers [2, 7].

Construction of FAPT with fixed number of quark flavors, N_f, is a two-step procedure: we start with the perturbative result $\left[a_\text{s}(Q^2)\right]^\nu$, generate the spectral density $\rho_\nu(\sigma)$ using Eq. (1), and then obtain analytic couplings $\mathcal{A}_\nu[L]$ and $\mathfrak{A}_\nu[L]$ via Eqs. (2). Here N_f is fixed and factorized out. We can proceed in the same manner for N_f-dependent quantities: $\left[\alpha_\text{s}(Q^2;N_f)\right]^\nu \Rightarrow \bar{\rho}_\nu(\sigma;N_f) = \bar{\rho}_\nu[L_\sigma;N_f] \equiv \rho_\nu(\sigma)/\beta_f^\nu \Rightarrow \bar{\mathcal{A}}_\nu[L;N_f]$ and $\bar{\mathfrak{A}}_\nu[L;N_f]$ — here N_f is fixed, but not factorized out[3].

Global version of FAPT, which takes into account heavy-quark thresholds, is constructed along the same lines but starting from global perturbative coupling $[\alpha_\text{s}^\text{glob}(Q^2)]^\nu$, being a continuous function of Q^2 due to choosing different values of QCD scales Λ_f, corresponding to different values of N_f. We illustrate here the case of only one heavy-quark threshold at $s = m_4^2$, corresponding to the transition $N_f = 3 \to N_f = 4$. Then we obtain the discontinuous spectral density

$$\rho_n^\text{glob}(\sigma) = \theta\left(L_\sigma < L_4\right)\,\bar{\rho}_n\left[L_\sigma;3\right] + \theta\left(L_4 \leq L_\sigma\right)\,\bar{\rho}_n\left[L_\sigma + \lambda_4;4\right]\,, \tag{5}$$

with $L_\sigma \equiv \ln\left(\sigma/\Lambda_3^2\right)$, $L_f \equiv \ln\left(m_f^2/\Lambda_3^2\right)$ and $\lambda_f \equiv \ln\left(\Lambda_3^2/\Lambda_f^2\right)$ for $f = 4$, which is expressed in terms of fixed-flavor spectral densities with 3 and 4 flavors, $\bar{\rho}_n[L;3]$ and $\bar{\rho}_n[L+\lambda_4;4]$. However, it generates the continuous Minkowskian coupling

$$\begin{aligned}\mathfrak{A}_\nu^\text{glob}[L_\text{s}] = {}&\theta\left(L_\text{s} < L_4\right)\left(\bar{\mathfrak{A}}_\nu[L_\text{s};3] - \bar{\mathfrak{A}}_\nu[L_4;3] + \bar{\mathfrak{A}}_\nu[L_4+\lambda_4;4]\right)\\ &+ \theta\left(L_4 \leq L_\text{s}\right)\bar{\mathfrak{A}}_\nu[L_\text{s}+\lambda_4;4]\,,\end{aligned} \tag{6}$$

and the analytic Euclidean coupling $\mathcal{A}_\nu^\text{glob}[L]$ (for more detail see in [7]).

2. Resummation in the one-loop APT and FAPT

We consider now the perturbative expansion of a typical physical quantity, like the Adler function and the ratio R, in the one-loop APT. Due to the limited space of our presentation we provide all formulas only for quantities in Minkowski region:

$$\mathcal{R}[L] = d_0 + \sum_{n=1}^{\infty} d_n\,\mathfrak{A}_n[L]\,. \tag{7}$$

[3] Remind here that $\beta_f = b_0(N_f)/(4\pi)$.

We suggest that there exists the generating function $P(t)$ for coefficients $\tilde{d}_n = d_n/d_1$:

$$\tilde{d}_n = \int_0^\infty P(t)\, t^{n-1} dt \quad \text{with} \quad \int_0^\infty P(t)\, dt = 1\,. \tag{8}$$

To shorten our formulae, we use for the integral $\int_0^\infty f(t)P(t)dt$ the following notation: $\langle\langle f(t) \rangle\rangle_{P(t)}$. Then coefficients $d_n = d_1 \langle\langle t^{n-1} \rangle\rangle_{P(t)}$ and as has been shown in [8] we have the exact result for the sum in (7)

$$\mathcal{R}[L] = d_0 + d_1 \langle\langle \mathfrak{A}_1[L-t] \rangle\rangle_{P(t)}\,. \tag{9}$$

The integral in variable t here has a rigorous meaning, ensured by the finiteness of the coupling $\mathfrak{A}_1[t] \leq 1$ and fast fall-off of the generating function $P(t)$.

In our previous publications [7, 9] we have constructed generalizations of these results, first, to the case of the global APT, when heavy-quark thresholds are taken into account. Then one starts with the series of the type (7), where $\mathfrak{A}_n[L]$ are substituted by their global analogs $\mathfrak{A}_n^{\text{glob}}[L]$ (note that due to different normalizations of global couplings, $\mathfrak{A}_n^{\text{glob}}[L] \simeq \mathfrak{A}_n[L]/\beta_f$, the coefficients d_n should be also changed). Then

$$\begin{aligned}\mathcal{R}^{\text{glob}}[L] \;=\; & d_0 + d_1\Big\langle\Big\langle \theta(L<L_4)\Big[\Delta_4\bar{\mathfrak{A}}_1[t] + \bar{\mathfrak{A}}_1\Big[L-\frac{t}{\beta_3};3\Big]\Big]\Big\rangle\Big\rangle_{P(t)} \\ & + d_1\Big\langle\Big\langle \theta(L\geq L_4)\bar{\mathfrak{A}}_1\Big[L+\lambda_4-\frac{t}{\beta_4};4\Big]\Big\rangle\Big\rangle_{P(t)}\,,\end{aligned} \tag{10}$$

where $\Delta_4\bar{\mathfrak{A}}_\nu[t] \equiv \bar{\mathfrak{A}}_\nu\Big[L_4+\lambda_4-t/\beta_4;4\Big] - \bar{\mathfrak{A}}_\nu\Big[L_3-t/\beta_3;3\Big]$.

The second generalization has been obtained for the case of the global FAPT. Then the starting point is the series of the type $\sum_{n=0}^\infty d_n\, \mathfrak{A}_{n+\nu}^{\text{glob}}[L]$ and the result of summation is a complete analog of Eq. (10) with substitutions

$$P(t) \Rightarrow P_\nu(t) = \int_0^1 P\left(\frac{t}{1-x}\right)\frac{\nu\, x^{\nu-1}dx}{1-x}\,, \tag{11}$$

$d_0 \Rightarrow d_0\,\bar{\mathfrak{A}}_\nu[L]$, $\bar{\mathfrak{A}}_1[L-t] \Rightarrow \bar{\mathfrak{A}}_{1+\nu}[L-t]$, and $\Delta_4\bar{\mathfrak{A}}_1[t] \Rightarrow \Delta_4\bar{\mathfrak{A}}_{1+\nu}[t]$. All needed formulas have been also obtained in parallel for the Euclidean case.

3. Applications to Higgs boson decay

Here we analyze the Higgs boson decay to a $\bar{b}b$ pair. For its width we have

$$\begin{aligned}\Gamma(H\to b\bar{b}) \;&=\; \frac{G_{\text{F}}}{4\sqrt{2}\pi}\, M_H\, \widetilde{R}_{\text{S}}(M_H^2) \\ \text{with}\quad \widetilde{R}_{\text{S}}(M_H^2) \;&\equiv\; m_b^2(M_H^2)\, R_{\text{S}}(M_H^2)\end{aligned} \tag{12}$$

and $R_{\rm S}(s)$ is the R-ratio for the scalar correlator, see for details in [2, 10]. In the one-loop FAPT this generates the following non-power expansion[4]:

$$\widetilde{\mathcal{R}}_{\rm S}[L] = 3\,\hat{m}^2_{(1)} \left\{ \mathfrak{A}^{\rm glob}_{\nu_0}[L] + d_1^{\rm S} \sum_{n\geq 1} \frac{\tilde{d}_n^{\rm S}}{\pi^n}\, \mathfrak{A}^{\rm glob}_{n+\nu_0}[L] \right\}, \tag{13}$$

where $\hat{m}^2_{(1)} = 8.45\ {\rm GeV}^2$ is the RG-invariant of the one-loop $m_b^2(\mu^2)$ evolution $m_b^2(Q^2) = \hat{m}^2_{(1)}\, \alpha_{\rm s}^{\nu_0}(Q^2)$ with $\nu_0 = 2\gamma_0/b_0(5) = 1.04$ and γ_0 is the quark-mass anomalous dimension (for a discussion see in [11]). We take for the generating function $P(t)$ the Lipatov-like model of [9] with $\{c = 2.4,\ \beta = -0.52\}$

$$\tilde{d}_n^{\rm S} = c^{n-1}\frac{\Gamma(n+1) + \beta\,\Gamma(n)}{1+\beta}\,, \qquad P_{\rm S}(t) = \frac{(t/c)+\beta}{c\,(1+\beta)}\, e^{-t/c}\,. \tag{14}$$

It gives a very good prediction for $\tilde{d}_n^{\rm S}$ with $n = 2, 3, 4$, calculated in the QCD PT [10]: 7.50, 61.1, and 625 in comparison with 7.42, 62.3, and 620. Then we apply FAPT resummation technique to estimate how good is FAPT in approximating the whole sum $\widetilde{\mathcal{R}}_{\rm S}[L]$ in the range $L \in [11.5, 13.7]$ which corresponds to the range $M_H \in [60, 180]\ {\rm GeV}^2$ with $\Lambda^{N_f=3}_{\rm QCD} = 189$ MeV and $\mathfrak{A}^{\rm glob}_1(m_Z^2) = 0.122$. In this range we have ($L_6 = \ln(m_t^2/\Lambda_3^2)$)

$$\frac{\widetilde{\mathcal{R}}_{\rm S}[L]}{3\,\hat{m}^2_{(1)}} = \mathfrak{A}^{\rm glob}_{\nu_0}[L] + \frac{d_1^{\rm S}}{\pi} \left\langle\!\!\left\langle \bar{\mathfrak{A}}_{1+\nu_0}\Big[L+\lambda_5 - \frac{t}{\pi\beta_5}; 5\Big] + \Delta_6 \bar{\mathfrak{A}}_{1+\nu_0}\left[\frac{t}{\pi}\right] \right\rangle\!\!\right\rangle_{P^{\rm S}_{\nu_0}} \tag{15}$$

with $P^{\rm S}_{\nu_0}(t)$ defined via Eqs. (14) and (11). Now we analyze the accuracy of the truncated FAPT expressions

$$\widetilde{\mathcal{R}}_{\rm S}[L; N] \;=\; 3\,\hat{m}^2_{(1)} \left[\mathfrak{A}^{\rm glob}_{\nu_0}[L] + d_1^{\rm S} \sum_{n=1}^{N} \frac{\tilde{d}_n^{\rm S}}{\pi^n}\, \mathfrak{A}^{\rm glob}_{n+\nu_0}[L] \right] \tag{16}$$

and compare them with the total sum $\widetilde{\mathcal{R}}_{\rm S}[L]$ in Eq. (15) using relative errors $\Delta^{\rm S}_N[L] = 1 - \widetilde{\mathcal{R}}_{\rm S}[L; N]/\widetilde{\mathcal{R}}_{\rm S}[L]$. In the left panel of Fig. 1 we show these errors for $N = 2$, $N = 3$, and $N = 4$ in the analyzed range of $L \in [11, 13.8]$. We see that already $\widetilde{\mathcal{R}}_{\rm S}[L; 2]$ gives accuracy of the order of 2.5%, whereas $\widetilde{\mathcal{R}}_{\rm S}[L; 3]$ of the order of 1%. That means that there is no need to calculate further corrections: at the level of accuracy of 1% it is quite enough to take into account only coefficients up to d_3. This conclusion is stable with respect to the variation of parameters of the model $P_{\rm S}(t)$.

[4] Appearance of denominators π^n in association with the coefficients $\tilde{d}_n$ is due to d_n normalization.

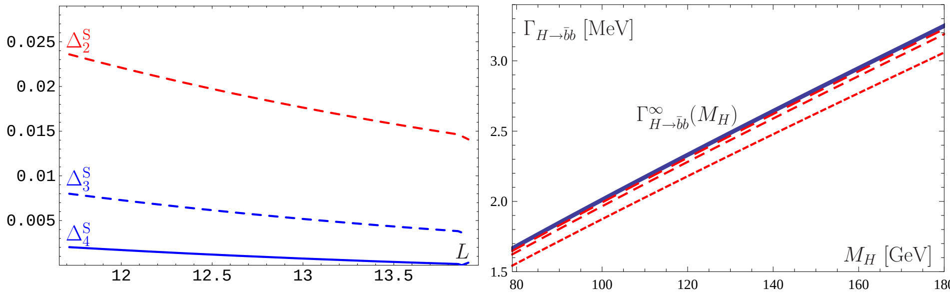

Fig. 1. Left: The relative errors $\Delta_N^{\rm S}[L]$, $N=2,3$ and 4, of the truncated FAPT in comparison with the exact summation result, Eq. (15). Right: The width $\Gamma^{\infty}_{H\to b\bar{b}}$ as a function of the Higgs boson mass M_H in the resummed FAPT (solid line).

4. Conclusions

In this report we described the resummation approach in the global versions of the one-loop APT and FAPT and argued that it produces finite answers, provided the generating function $P(t)$ of perturbative coefficients d_n is known. The main conclusion is: To achieve an accuracy of the order of 1% we do not need to calculate more than four loops and d_4 coefficients are needed only to estimate corresponding generating functions $P(t)$.

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