

ONCE AND TWICE SUBTRACTED DISPERSION
RELATIONS IN THE ANALYSIS OF $\pi\pi$ AMPLITUDES*

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Once and twice subtracted crossing symmetric dispersion relations applied to $\pi\pi \rightarrow \pi\pi$ scattering data are analyzed and compared. Both sets of dispersion relations can be used to test the $\pi\pi$ amplitudes in low partial waves up to about 1 GeV. We show how once subtracted dispersion relations can provide stronger constraints for $\pi\pi$ amplitudes than twice subtracted ones in the 400 to 1100 MeV range, given the same experimental input.

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1. Introduction

A set of dispersion relations for $\pi\pi \rightarrow \pi\pi$ scattering amplitudes incorporating crossing symmetry were presented by Roy in 1971 [1] (hereafter called the Roy equations). Two subtractions were used for faster convergence of the dispersive integrals. These equations are a very efficient tool in testing the $\pi\pi$ experimental data.

In the last few years, several analysis of $\pi\pi$ amplitudes using Roy equations have appeared with different aims. For instance, to predict the low energy S and P waves below 800 MeV with the aid of Chiral Perturbation Theory (ChPT), which is used to fix the threshold behavior of amplitudes [2, 3]. Or, for example, to eliminate the long standing “up-down”

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ambiguity in scalar–isoscalar $\pi\pi$ amplitudes below 1 GeV [4]. Later on, they were used by our group, together with forward dispersion relations (FDR) to describe data and also test predictions of ChPT [5, 6]. All these works provide very precise determinations of $\pi\pi$ amplitudes for the S and P waves below about 1 GeV and very precise predictions or determinations of scattering lengths and parameters of the σ meson [6–8].

More recently, together with our FDR and Roy equations analyses [6] we have developed a set of once subtracted dispersion relations that also incorporate $\pi\pi$ crossing symmetry (hereafter called GKPY). Due to the single subtraction, the higher $\pi\pi$ partial waves and the high energy region contributions are less suppressed in these equations than in Roy’s, although integrals in the GKPY equations still converge.

In this presentation we briefly analyze and compare the structure and cancellations that occur in Roy equations *versus* those in the GKPY equations.

2. Twice subtracted dispersion relations — Roy equations

Roy equations can be expressed as a sum

$$\text{Ref}_l^I(s) = \text{ST}(s) + \text{KT}(s) + \text{DT}(s), \quad (1)$$

where $\text{ST}(s)$, $\text{KT}(s)$ and $\text{DT}(s)$ are called subtraction, kernel and driving terms, respectively. The $\text{Ref}_l^I(s)$ on the left hand side is called “output” amplitude throughout this paper and can be calculated for the S and P waves up to about 1 GeV. The subtraction terms read

$$\begin{aligned} \text{ST}(s) = & a_0^0 \delta_{I0} \delta_{\ell 0} + a_0^2 \delta_{I2} \delta_{\ell 0} + \frac{s - 4m_\pi^2}{12} (2a_0^0 - 5a_0^2) \\ & \times \left(\delta_{I0} \delta_{\ell 0} + \frac{1}{6} \delta_{I1} \delta_{\ell 1} - \frac{1}{2} \delta_{I2} \delta_{\ell 0} \right), \end{aligned} \quad (2)$$

where a_0^0 and a_0^2 are the $S0$ and $S2$ scattering lengths. Let us remark that the $\text{ST}(s)$ contain a piece that grows with $s - 4m_\pi^2$. The $\text{KT}(s)$ are singular integrals of some known kernels $K_{\ell\ell'}^{II'}(s, s')$ multiplied by imaginary parts of the “input” amplitudes $\text{Im } f_{\ell'}^{I'}(s')$

$$\text{KT}(s) = \sum_{I'=0}^2 \sum_{\ell'=0}^1 \int_{4m_\pi^2}^{s_{\max}} ds' K_{\ell\ell'}^{II'}(s, s') \text{Im } f_{\ell'}^{I'}(s'). \quad (3)$$

For twice subtracted dispersion relations the kernels are proportional to $1/s'^2(s' - s)$. In our analysis (see for example [6]) $s_{\max}^{1/2} = 1.42$ GeV, which is the maximum energy for which we can parameterize experimental S and P

wave data in terms of phase shifts and inelasticities. The so called driving terms $DT(s)$ collect the $s^{1/2} > 1.42$ GeV contributions from all waves — parameterized in terms of Regge theory — together with the contributions from $\ell \geq 2$ partial waves below 1.42 GeV. Our fits of $\pi\pi$ amplitudes to data are then constrained to minimize, within uncertainties, the difference between “input” and “output” amplitudes for the S , and P waves at 25 equidistant energy values $s_i^{1/2}$ below 1 GeV [6].

A similar method using only forward dispersion relations was used in [9] to show that many of the presently available experimental data sets do not fulfill well enough the analyticity constraints.

3. Once subtracted dispersion relations — GKPY equations

The general structure of the GKPY equations is similar to that of the Roy equations in Eq. (1). However, the subtraction terms are now constant for each wave and are expressed as combinations of scattering lengths

$$ST(s) = \sum_{I'} C_{II'}^{\text{st}} a_0^{I'} \quad (4)$$

with $a_0 = (a_0^0, 0, a_0^2)$ and C^{st} the usual crossing matrix. Since there is just one subtraction, the integral kernels are now proportional to $1/s'(s' - s)$. Consequently, the high energy contributions to kernel and particularly to the driving terms are now less suppressed compared with the Roy equations case. Nevertheless, we will see in the next section that the effect of driving terms is still smaller than the kernel terms for the P , S_2 and S_0 waves in almost the whole energy region of interest, which allows us to obtain a reliable calculation.

4. Comparison of the Roy and GKPY dispersion relations

In Table I we compare the structure of the different terms of Roy and GKPY equations for threshold parameters (TP), defined as:

$$\text{Re } f_\ell^I(s \approx 4m_\pi^2) = (s - 4m_\pi^2)^\ell [a_\ell^I + b_\ell^I(s - 4m_\pi^2) + \dots] \quad (5)$$

As is seen from the Table I in the case of Roy equations, not only the linear terms in $ST(s)$, but also the whole kernel and driving terms vanish at $s = 4m_\pi^2$. Thus, the constant terms in $ST(s)$ are the ones that ensure the correct values of the threshold parameters in the “output” amplitudes. In contrast, for the GKPY equations the $KT(s)$ and $DT(s)$ do not vanish at threshold and it is a combination of their nonzero constant parts with the $ST(s)$ terms that yield the correct threshold value. Hence, GKPY uncertainties around threshold are expected to be larger than for Roy equations.

TABLE I

Comparison of the threshold expansion parameters (TP) of the “output” amplitudes for Roy and GKPY equations (m_π units). Greek letters stand for constants whose precise value is irrelevant for the discussion. Note that the GKPY value at threshold is obtained from a cancellation between ST, KT and DT terms.

wave	TP	Roy		GKPY	
		ST	KT & DT	ST	KT & DT
$S0$	a_0^0	$a_0^0 + \alpha(s - 4)$	$\beta(s - 4)$	$a_0^0 + 5a_0^2$	$\delta(s - 4) - 5a_0^2$
P	0	$\alpha'(s - 4)$	$\beta'(s - 4)$	$a_0^0 - \frac{5}{2}a_0^2$	$\delta'(s - 4) - a_0^0 + \frac{5}{2}a_0^2$
$S2$	a_0^2	$a_0^2 + \alpha''(s - 4)$	$\beta''(s - 4)$	$a_0^0 + \frac{1}{2}a_0^2$	$\delta''(s - 4) - a_0^0 + \frac{1}{2}a_0^2$

In Fig. 1 we show the decomposition of Roy and GKPY equations into subtraction, kernel and driving terms. Error bands are obtained from a Monte Carlo with 10^5 Gaussian samplings, within three standard deviations of their central values, of all parameters used to describe the different input (for details see [6]). Let us remark that, above $\sqrt{s} \simeq 450$ MeV ($s \simeq 10$ in m_π units), the ST(s) and KT(s) terms in Roy equations suffer a very strong cancellation to give the total output amplitudes that, as seen in Fig. 2, satisfy $|\text{Ref}_\ell^I| < 0.6$ (note that the vertical scale for the Roy equations figures is much bigger than 1). In contrast, for GKPY equations, the kernel terms are dominant and there is no such strong cancellation. It is also clear from Fig. 1 that, above $\sqrt{s} \simeq 400$ MeV, the errors in Roy equations are significantly bigger than those in the GKPY ones. The main source of Roy equations uncertainties is the experimental error on a_0^2 , that propagates through a term in ST(s) proportional to s . In contrast, the ST(s) and their errors in the GKPY equations are constant.

Fig. 2 compares the size of the uncertainties for a preliminary data fit constrained to satisfy forward dispersion relations, sum rules, Roy and GKPY equations (see [6]). We also give preliminary values of an average χ^2/N_P for each wave, where $N_P = 29$ is number of points where “input” and “output” amplitudes are compared.

In conclusion, for the same input, GKPY equations have much smaller errors than standard Roy equations above $\sqrt{s} \approx 400$ MeV and thus they become a very promising tool to obtain a precise data analysis of $\pi\pi$ amplitudes in the 400 to 1100 MeV region.

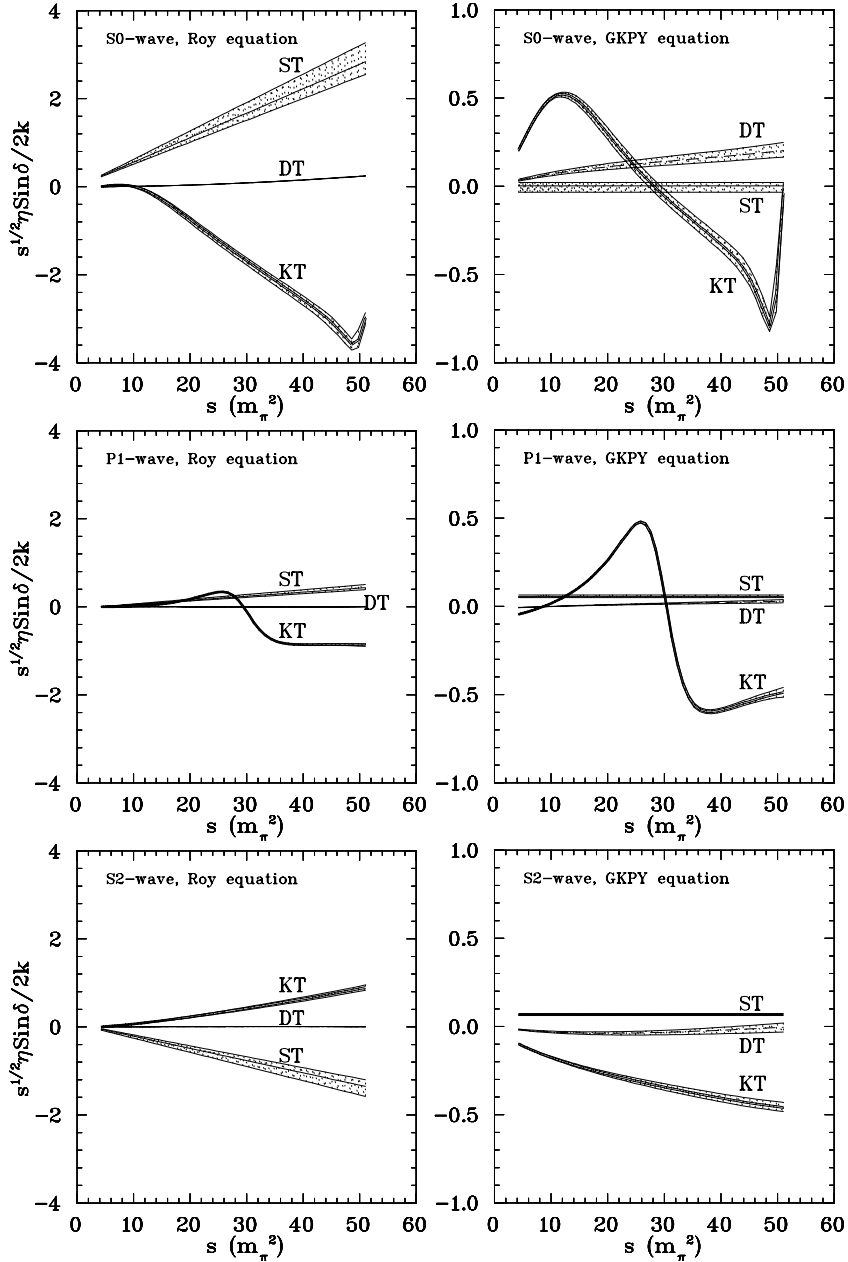


Fig. 1. Subtraction (ST), kernel (KT) and driving (DT) terms for the S_0 , P and S_2 waves from Roy equations (left panel) and from the GKP ones (right panel). Dashed bands denote the errors of these terms. Note that we use m_π units.

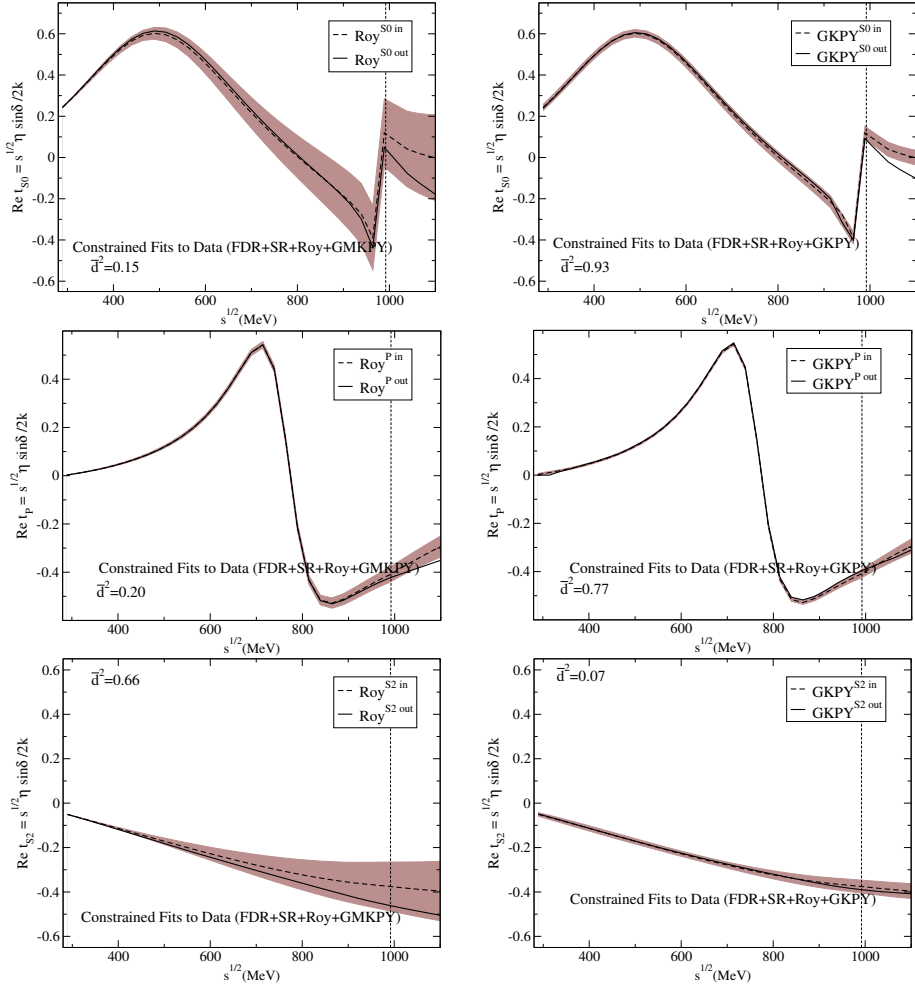


Fig. 2. Comparison of preliminary results from a data fit constrained to satisfy Forward dispersion relations, Roy and GKPY equations for the S_0 , P and S_2 waves. The dark bands show errors of these equations. Dashed and solid lines represent “input” and “output” amplitudes respectively. The \bar{d}^2 are averaged values of χ^2 for the Roy or GKPY equation corresponding to given wave.

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