# PHASE STRUCTURE OF LARGE $N$ LATTICE QCD ON AN $L^3$ TORUS*

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We consider large $N$ pure gauge lattice QCD on a cubical torus in $2+1$ dimensions. The theory has four continuum phases: $0c$, $1c$, $2c$ and $3c$, where the numeral denotes the number of lattice directions with broken center symmetry. We show through numerical calculation on lattices of size $2 \leq L \leq 8$ that $1c$–$2c$ and $2c$–$3c$ phase transitions are weak first order. Some remarks are made about the nature of the $0c$–$1c$ phase transition.

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## 1. Introduction

't Hooft showed in 1974 [1] that QCD-like field theory with gauge group SU($N$) is significantly simplified in the limit $N \to \infty$. The leading contribution to the perturbative expansion comes from pure gauge planar graphs, hence the other name of the theory — planar QCD. One can also naturally identify $1/N$ as an expansion parameter and hope to achieve "real" QCD as a well-behaved perturbation around the planar theory at $N = 3$.

Unfortunately, analytic solution of planar QCD has been unattainable so far, despite substantial progress made in its understanding (see *e.g.* [2, 3]). One can, however, use the lattice regularization to investigate many aspects of the theory. This approach leads to very interesting theoretical properties (like the Eguchi–Kawai reduction [4]) and allows numerical computations on finite lattices. Although large $N$ lattice calculations have been made for over 3 decades, rising computational power has given a new perspective to this field in the last few years.

In this paper we restrict ourselves to pure gauge theory — in large $N$ limit the fermions are quenched [1] and one can expect that dynamics of the theory will be determined by the bosonic sector. We also consider lattices

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in $2+1$ dimensions, mostly for numerical reasons. As we will see in the next section, in the examined case, the three dimensional model exhibits qualitatively similar behaviour as the four dimensional one.

## 2. Large $N$ phase diagram

### *2.1. Polyakov loops*

We consider $\mathrm{SU}(N)$ gauge theory on an isotropic $L^3$ periodic lattice with standard Wilson plaquette action. Note that a line of length $L$ in any given direction $\mu$ is a closed contour. We define Polyakov loop matrix $P_\mu$ as a product of link variables along such a contour. Its trace, called the Polyakov loop, is a gauge invariant quantity — a kind of uncontractible Wilson loop.

The theory exhibits additional global symmetry — select all the links $U_\mu(x)$ with set $x_\mu$ coordinate and modify them in a following manner:

$$U_\mu(x) \to C U_\mu(x)\,, \tag{1}$$

where $C$ is an element of the center of $\mathrm{SU}(N)$, namely $\mathbb{Z}_N$. All the contractible Wilson loops are invariant under this transformation but the Polyakov loops in the direction $\mu$ get multiplied by $C$.

Therefore, if the center symmetry is not spontaneously broken in any given direction $\mu$ then all the Polyakov loops $\mathrm{Tr}P_\mu$ must be zero (as well as any other uncontractible Wilson loops winding around that direction).

Also, as Narayanan and Neuberger have noticed [5], if this symmetry holds in the large $N$ limit (where $\mathbb{Z}_N \to \mathrm{U}(1)$) then no physical quantity can depend on the size of the lattice in that direction — this is a simple generalization of the Eguchi–Kawai reduction [4] (for more details on this concept, see [6]).

### *2.2. Center symmetry breaking phase transitions*

There is another interesting property related to the spontaneous breaking of the center symmetry. It was shown [7] that in finite temperature field theory one can treat the average Polakov loop in the temporal direction as the order parameter of quark confinement–deconfinement transition (see [8] for a more detailed discussion).

On an isotropic torus, which corresponds to a zero temperature euclidean field theory in the limit $L \to \infty$, the situation is more complex because none of the directions is *a priori* distinguished. Therefore, we may suspect that the center symmetry might be broken in every direction of the lattice.

Indeed, numerical analyses in both three [5,9] and four [10,11] dimensions show that there is a cascade of phase transitions which break the center symmetry in consecutive directions. On the other hand, there is no symmetry breaking in two dimensional case [12].

A schematic three dimensional phase diagram is pictured in Fig. 1. In four dimensions, the diagram is very similar [11], with phase transitions breaking the U(1) symmetry in all four directions.

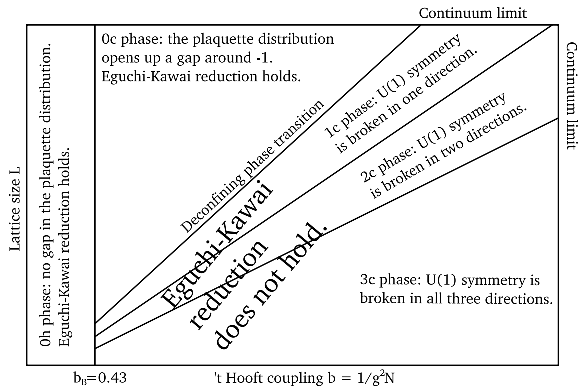


Fig. 1. Schematic phase diagram of planar QCD on $L^3$ lattice (after [13]).

The $0h-Xc$ is a transition in the plaquette distribution. It is an equivalent of two-dimensional Gross–Witten transition [12]. The lack of scaling of critical $b$ with $L$ indicates that this transition is a lattice artifact [9].

The $Xc-(X+1)c$ transitions break the center symmetry. The $0c-1c$ transition is the equivalent of the deconfining phase transition in finite temperature case. This transition is first order in large $N$ finite temperature model [14, 15]. We might expect similar behaviour in zero temperature.

The other two transitions ($1c$–$2c$ and $2c$–$3c$) are specific to zero temperature model. Arguments were given [9] that they are also most likely first order. We will check these hypotheses numerically in the next section.

## 3. Numerical results

### *3.1. Simulation method*

We are using standard Monte Carlo setup: an $L^3$ lattice with periodic boundary conditions and Wilson action for the gauge fields. A typical update consists of one lattice sweep using a heatbath algorithm followed by one to nine overrelaxation sweeps.

The heatbath (or rather pseudo-heatbath) is the Cabbibo–Marinari update [16] of all $\frac{N(N-1)}{2}$ SU(2) subgroups. For practically every calculation in this work, the Kennedy–Pendleton version of the SU(2) heatbath [17] was used. Overrelaxation sweeps are meant to achieve faster decorrelation of observables. We are using full $\mathrm{SU}(N)$ overrelaxation, based on [10] (see also [18]).

We are also using APE smearing [19] to diminish ultraviolet renormalization of the observables. Although this operation is not crucial in $2+1$ dimensions, it makes the analysis in the vicinity of phase transitions easier.

The observables we use to identify the phases are: plaquette traces, Polyakov loops and Polyakov loop matrix eigenvalues. Note that the set of eigenvalues of a loop operator is gauge invariant since the gauge transformation

$$\mathrm{U}_{\Gamma} \rightarrow \omega \mathrm{U}_{\Gamma} \omega^{-1}\,, \qquad \omega \in \mathrm{SU}(N) \tag{2}$$

does not change the eigenvalues.

### *3.2. Phase diagram*

We find the phase diagram in the following manner: for every $L$ we create a "hot" random lattice corresponding to $b=0$ and cool it gradually increasing $b$ until we get a stable $3c$ phase. Then we heat it up to $0h$ again.

Next, for a more precise analysis of a phase transition we make two thermalised configurations on both sides of the transition region — let us call them "hot" and "cold". Then, we pick $b \in (b_{\mathrm{hot}}, b_{\mathrm{cold}})$ and make two runs at this value of $b$ — one starting from the "hot" configuration and another from the "cold" one. If both configurations clearly converge to the same phase we identify it as a stable phase. The calculated phase diagram is presented in Fig. 2.

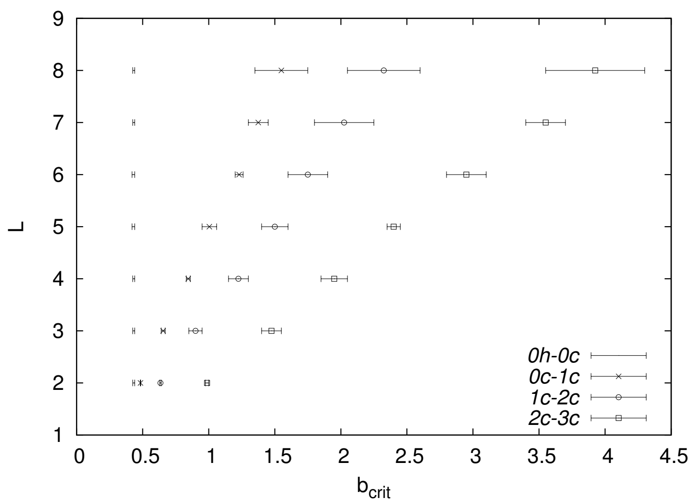


Fig. 2. Phase transitions on an $L^3$ lattice.

All the points on the phase diagram were calculated using $N = 55$. The exception is $L = 2$ where the calculations were done at $N = 70$. Using large values of $N$ is necessary especially in the vicinity of the phase transitions to diminish finite size effects on such small lattices. All the directions are *a priori* equal and the competition between them (observed as swapping of the broken directions with the unbroken ones) makes it hard to determine the transition point precisely.

Calculated results for $3 \le L \le 6$ can be compared to [9]. The results are consistent with each other but uncertainties in both cases are quite large.

### *3.3. Analysis of the 0c–1c phase transition*

The transition can be seen best in the Polyakov loops. For each configuration we calculate the moduli of the Polyakov loops. Then we take the average over the lattice sites for all three directions separately (denoted by $\overline{|\mathrm{Tr}P_\mu|}$). Next, we sort the values to eliminate the effect of exchange of directions. The order parameter for the transition is $\overline{|\mathrm{Tr}P_{\mathrm{max}}|}$ where "max" denotes the direction in which the modulus of the trace is the largest.

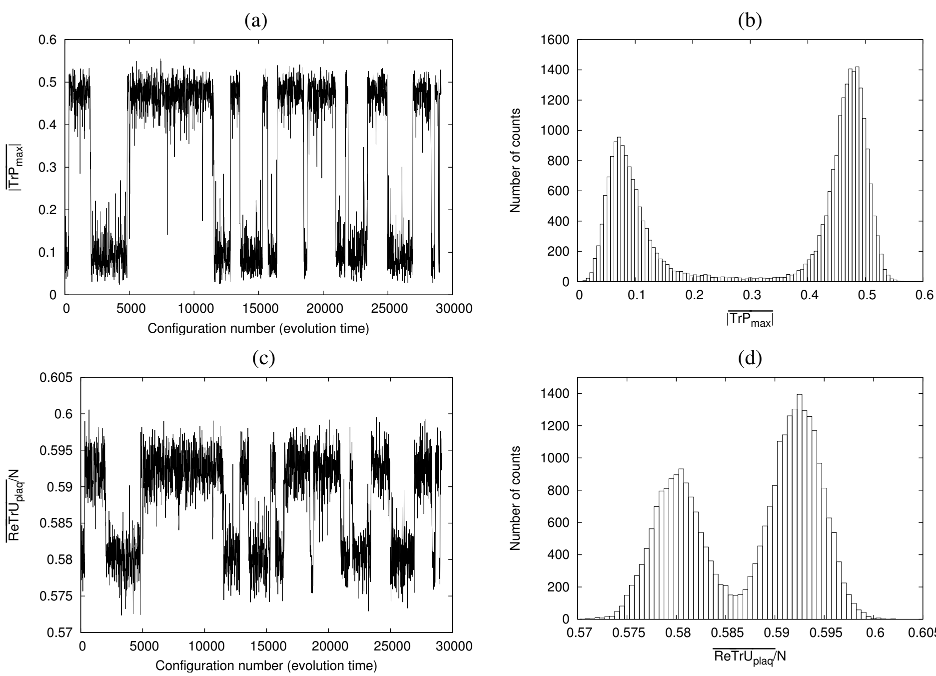


Fig. 3. 0$c$–1$c$ phase transition. Results of run $V = 2^3$, $N = 70$, $b = 0.4836$: evolution and probability distribution of maximal average Polyakov loop ((a) and (b), respectively) and average plaquette ((c) and (d), respectively).

For $L = 2$ we get a clear evidence of coexistence of phases in the transition region — a sample run is pictured in Fig. 3. That is a clear sign that the transition is likely to be first order.

In Fig. 3 we can also see the values of average plaquette. The peaks of the average plaquette are clearly separated. Yet, we are talking of an extremely small lattice so the measured latent heat per unit volume is very small and the transition is likely to be very weak. That is in agreement with finite temperature measurements [14].

Small latent heat of the transition indicates that there may be problems with calculations at larger $L$. That is indeed the case. As we get closer to the phase transition at $L > 2$ we can see very rapid swapping of the directions. Most configurations are calculated during a swap so, obviously, we get single-peaked distributions both in Polyakov loops and the plaquette. This behaviour makes it impossible to establish the order of the deconfining transition at considered values of $N$ with the methods we use.

### *3.4. Analysis of 1c–2c and 2c–3c phase transitions*

We use the same methods to analyse $1c$–$2c$ and $2c$–$3c$ phase transitions. The order parameters we use are $\overline{|\mathrm{Tr}P_{\mathrm{mid}}|}$ and $\overline{|\mathrm{Tr}P_{\mathrm{min}}|}$, respectively, (where "min" is the direction in which the modulus of the trace is the smallest and "mid" is the "middle" direction).

For both transitions we see the coexistence of phases for $2 \leq L \leq 8$. Example runs for $1c$–$2c$ and $2c$–$3c$ transitions are pictured in Fig. 4 and 5, respectively. However, the plaquette distributions are always single-peaked and already at $L = 4$ the latent heat is inextractible from the simulation noise. We conclude that both transitions are very weak first order.

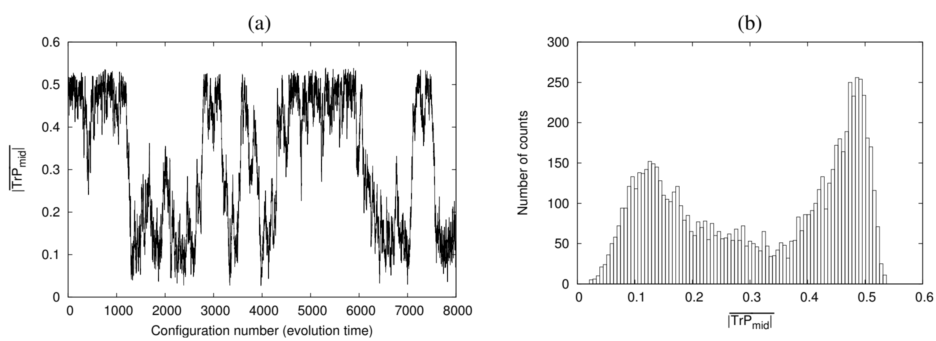


Fig. 4. $1c$–$2c$ phase transition. Results of run $V = 5^3$, $N = 55$, $b = 1.5$: evolution and probability distribution of "middle" average Polyakov loop ((a) and (b), respectively).

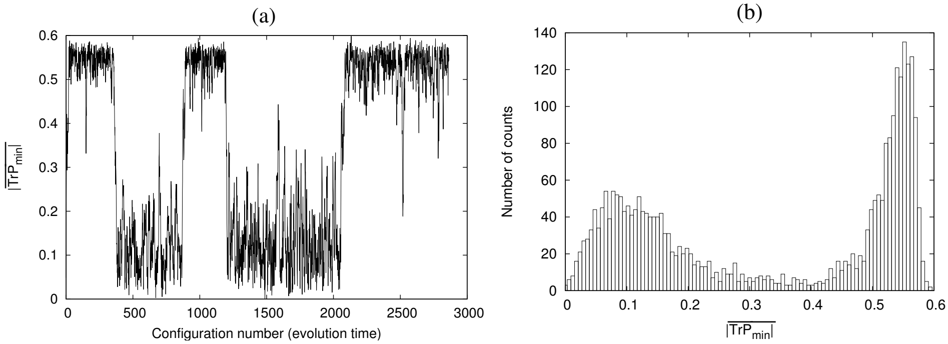


Fig. 5. $2c$–$3c$ phase transition. Results of run $V = 4^3$, $N = 55$, $b = 1.98$: evolution and probability distribution of minimal average Polyakov loop ((a) and (b), respectively).

## 4. Summary

We analyse the phase diagram of large $N$ lattice pure gauge QCD on an isotropic torus in 2+1 dimensions for lattice size $2 \leq L \leq 8$. We observe four continuum phases: $0c$, $1c$, $2c$, $3c$ corresponding to broken U(1) symmetry in respectively zero, one, two and three directions.

The $0c$–$1c$ phase transition is the hardest to analyse — for $L = 2$ we get double-peaked distributions for both the order parameter (Polyakov loops) and the plaquettes but for larger $L$ rapid changing of the broken direction makes it impossible to say whether we are dealing with a weak first order or continuous transition.

For $1c$–$2c$ and $2c$–$3c$ phase transitions we get double-peaked distributions of the Polyakov loops. However, in both cases the plaquette distribution is single-peaked — this signals that both transitions are weak first order.

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