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# LATE-TIME TAILS OF SELF-GRAVITATING MASSLESS FIELDS*

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In this talk I summarize briefly recent results of joint work with P. Bizoń, T. Chmaj and S. Zając, on the nonlinear origin of the power-law tails in the long-time evolution of self-gravitating massless fields. We focus on a spherically symmetric massless scalar field and wave map matter coupled to gravity. Using third-order perturbation method we derive explicit expressions for the tail (the decay rate and the amplitude) for solutions starting from small initial data and we verify this prediction via numerical integration of the full system of Einstein field equations. Our results show that the nonlinear effects can dominate the late time asymptotics.

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## 1. Introduction

Self-gravitating massless fields have been serving as a toy-model of gravitational collapse for a few decades, leading to valuable insights about the validity of the weak cosmic censorship and no-hair conjectures. In particular, for the case of spherically symmetric self-gravitating massless scalar field, Christodoulou proved that there are two generic endstates of evolution: Minkowski spacetime for small initial data [1] and Schwarzschild black hole for large initial data [2]. In both cases the upper bound for the rate of relaxation to the endstate inside the light cone is $t^{-3}$ (this was proved in [1] for the dispersive solutions and by Dafermos and Rodnianski [3] for the collapsing solutions). In view of these rigorous results one might wonder what is the point of studying this problem again.

In general our motivation is to understand the mechanism of relaxation to stable static solutions of nonlinear wave equations (for example Minkowski or Schwarzschild spacetime in the case of Einstein's equations). To illustrate

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the problem let us consider a spherically symmetric wave equation with a potential and the cubic power nonlinearity in 3 spatial dimensions:

$$\partial_t^2 F(t,r) - \Delta F(t,r) + V(r)F(t,r) + F^3(t,r) = 0\,. \tag{1}$$

Assume the potential to be spherically symmetric, nonnegative and decaying as $1/r^3$ for $r$ going to infinity. We prepare smooth, compactly supported and spherically symmetric Cauchy data and watch how the solution behaves at some fixed distance from the center at large time $t$. The potential is repulsive and the nonlinearity is defocussing as well, so at any fixed point $r$ the solution $F(t,r)$ tends to zero as times goes to infinity. In fact, at intermediate times the solution decays exponentially (this stage is usually called quasinormal ringing) but at late times polynomial decay is revealed (this stage is usually called a tail). Quasinormal ringing is a well understood linear phenomenon (related to resonances of the potential $V$) and since at later times the solution gets even smaller, it is tempting to describe the tail by the linearized theory as well. It turns out that this is wrong and the cubic term cannot be neglected. The heuristic picture which explains the tail phenomena in the linearized approximation is the following. In the first approximation the signal which arrives at late time $t$ at some small distance $r_{\mathrm{obs}}$, comes from a single scattering off the potential. The amplitude of the signal is proportional to the amplitude of the potential at the scattering point. The signal spends roughly one half of its time traveling forth and second half traveling back thus it gets scattered at the distance $r_{\mathrm{scatt}} \approx t/2$. As the potential falls off as $1/r^3$ the amplitude of the potential at the scattering point is proportional to $1/t^3$. Now we should compare two terms: a linear term $V\,F$ and a cubic term $F^3 = F^2\,F$, hence $V$ *versus* $F^2$. The general solution of a free wave equation decays as $1/r$ along the light-cone. Therefore, on the light-cone we have $V \sim 1/r^3$ *versus* $F^2 \sim 1/r^2$ and it is $F^2$ which decays slower and sets the decay rate of the tail to be $1/t^2$, while a linear tail decays as $1/t^3$!

Despite the above simple heuristics, the nonlinear effects in the late-time asymptotics have been frequently overlooked in the physical literature. Let us take a self-gravitating massless scalar field as an example. In 1972 Price [4] argued that the $\ell$ multipole of the massless scalar field evolving on a fixed Schwarzschild background in $3+1$ spacetime dimensions decays as $1/t^{3+2\ell}$. Then in 1994 Gundlach, Price and Pullin published two influential papers [5, 6]. In [5] they checked Price's prediction numerically and indeed obtained $1/t^{3+2\ell}$ decay for an $\ell$-multipole of the massless scalar field evolving on a fixed Schwarzschild background. In [6] they evolved the full system of Einstein equations for spherically symmetric self-gravitating massless scalar field and again found $1/t^3$ decay regardless of whether the scalar field collapses and forms a black hole or disperses to infinity. As this decay rate coincides with a decay rate of the tail on the fixed Schwarzschild background (linear approximation) they suggested that the linearized theory might apply also in the nonlinear regime: *We found that the predictions for power-law tails of perturbations of Schwarzschild spacetime hold to reasonable approximation, even quantitatively, in a variety of situations to which the predictions might seem initially not to apply.* As the time passed this rather cautious statement was repeated with increasing sureness in numerous citations of [6]. In fact, there are hints already in [6] that the decay rates of linear tails on static background are not generic and the true decay rates may be slower. In Sec. V of [6] two scalar fields are considered: a self-gravitating field and a test field evolving on a background provided by the self-gravitating field. For the test field the problem is linear (and it does make sense to expand the test field into $\ell$ multipoles), however, the background provided by self-gravitating field is not static and $1/t^{2\ell+3}$ prediction does not apply (see Fig. 12 in [6]). Another hint is that the tails exist for arbitrary small initial data and for such data one should not expect a tail in linearized approximation at all.

## 2. Self-gravitating massless scalar field

As the tails exist for arbitrary small initial data we approach the problem from Minkowski side. We generalise the problem slightly and work in an odd number of spatial dimensions $d = 3 + 2\ell$, parametrised with a nonnegative integer $\ell$. We use the letter $\ell$ deliberately to stress the link between the solutions of the spherically symmetric wave equation in $d = 3 + 2\ell$ spatial dimensions and the solutions of the wave equation in 3 spatial dimensions for an $\ell$ multipole:

$$\Box_{(d)} \frac{F(t,r)}{r^\ell} = \frac{1}{r^\ell} \Box_{(\ell)}\, F(t,r)\,, \tag{2}$$

where

$$\Box_{(d)} = \left(\partial_t^2 - \partial_r^2 - \frac{d-1}{r}\partial_r\right) \text{ and } \Box_{(\ell)} = \left(\partial_t^2 - \partial_r^2 - \frac{2}{r}\partial_r + \frac{\ell(\ell+1)}{r^2}\right)\,. \tag{3}$$

We assume spherical symmetry and parametrize the metric using two metric functions $m(t,r)$ and $\beta(t,r)$:

$$ds^2 = \left(1 - \frac{m}{r^{d-2}}\right)^{-1} \left(-e^{2\beta} dt^2 + dr^2\right) + r^2 d\Omega^2_{d-1}\,. \tag{4}$$

Einstein equations set a link between metric functions $m$ and $\beta$ and massless scalar field $\phi$

$$G_{\alpha\beta} = 8\pi T_{\alpha\beta}\,, \qquad T_{\alpha\beta} = \nabla_\alpha\phi\nabla_\beta\phi - \tfrac{1}{2}\, g_{\alpha\beta}\left(\nabla_\mu\phi\nabla^\mu\phi\right)\,, \tag{5}$$

which together with equation of motion for the massless field $\nabla_\mu \nabla^\mu \phi = 0$ constitutes a closed system of equations:

$$
\begin{aligned}
m' &= \kappa\, r^{d-1}\left(1-\frac{m}{r^{d-2}}\right)\left(\phi'^2+e^{-2\beta}\dot{\phi}^2\right), && \text{(Hamiltonian constraint)} \quad (6)\\
\dot{m} &= 2\kappa\, r^{d-1}\left(1-\frac{m}{r^{d-2}}\right)\dot{\phi}\,\phi', && \text{(momentum constraint)} \quad (7)\\
\beta' &= (d-2)\frac{m}{r^{d-1}}\left(1-\frac{m}{r^{d-2}}\right)^{-1}, && \text{(slicing condition)} \quad (8)\\
\left(e^{-\beta}\dot{\phi}\right)^{\cdot} &- \frac{1}{r^{d-1}}\left(r^{d-1}e^{\beta}\phi'\right)' = 0\,, && \text{(wave equation)} \quad (9)
\end{aligned}
$$

where $\kappa = 8\pi/(d-1)$, and primes and dots denote partial derivatives with respect to $r$ and $t$, respectively. We treat this system perturbatively, taking as perturbative parameter the amplitude of Cauchy data. Consider small and compactly supported initial data $(\phi,\, \dot{\phi})_{t=0} = (\varepsilon f(r),\, \varepsilon g(r))$. Then, up to the order $\mathcal{O}(\varepsilon^3)$, we have

$$
\phi = \varepsilon\phi_1 + \varepsilon^3\phi_3\,, \qquad m = \varepsilon^2 m_2\,, \qquad \beta = \varepsilon^2\beta_2\,, \tag{10}
$$

where $\phi_1$ satisfies the flat-space radial wave equation

$$
\Box_{(d)}\phi_1 = 0\,, \qquad (\phi_1, \dot{\phi}_1)_{t=0} = (f(r)\,,\, g(r))\,, \tag{11}
$$

the second-order perturbations of the metric functions satisfy

$$
m_2' = \kappa\, r^{d-1}\left(\dot{\phi}_1^2 + {\phi_1'}^2\right), \qquad \dot{m}_2 = 2\kappa\, r^{d-1}\dot{\phi}_1\phi_1'\,, \qquad \beta_2' = \frac{(d-2)m_2}{r^{d-1}} \tag{12}
$$

(these equations can be easily integrated once $\phi_1$ is known) and finally $\phi_3$ satisfies the inhomogeneous wave equation (with zero initial data)

$$
\Box_{(d)}\phi_3 = 2\beta_2\ddot{\phi}_1 + \dot{\beta}_2\dot{\phi}_1 + \beta_2'\phi_1' =: S(t,r)\,. \tag{13}
$$

Thus the tools to deal with the system (11)–(13) are elementary. They consist of the formula for the general solution of the equation (11)

$$
\phi_1(t,r) = \frac{1}{r^{\ell+1}}\sum_{k=0}^{\ell}\frac{(2\ell-k)!}{k!(\ell-k)!}\,\frac{\left(a^{(k)}(t-r)-(-1)^k a^{(k)}(t+r)\right)}{(2r)^{\ell-k}}\,, \tag{14}
$$

where a compactly supported function $a(x)$ is set by initial data in (11), and the Duhamel formula for the equation (13)

$$
\phi_3(t,r) = \frac{1}{2r^{\ell+1}}\int_0^t d\tau \int_{|t-r-\tau|}^{t+r-\tau}\rho^{\ell+1}P_\ell\left(\frac{r^2+\rho^2-(t-\tau)^2}{2r\rho}\right)S(\tau,\rho)\,d\rho \tag{15}
$$

(here $P_\ell(x)$ is a Legendre polynomial of degree $\ell$). In this way we can get not only the decay rate but also the amplitude of the tail both at timelike and null infinity. The asymptotic behavior at timlike infinity reads

$$\phi_3(t,r) = \frac{1}{t^{3\ell+3}} \left[ A_\ell + \mathcal{O}\left(\frac{1}{t}\right)\right] , \tag{16}$$

where

$$A_0 = -2^5\pi \int_{-\infty}^{+\infty} a(u) \int_{u}^{\infty} \left(a'(s)\right)^2 \, ds \, du \,, \tag{17}$$

$$A_\ell = (-1)^{\ell+1} 2^{3\ell+5}\pi \int_{-\infty}^{+\infty} a^{(\ell-1)}(u) \left(a^{(\ell+1)}(u)\right)^2 \, du \,, \qquad \text{for } \ell > 0 \tag{18}$$

are the only trace of initial data. We refer the reader to [7] for more details about this calculation and numerical evidence.

Now, it is interesting to compare our predictions with the predictions for the tails on a fixed Schwarzschild–Tangherlini black hole in $2\ell+3$ spatial dimensions. For dispersing solutions the tail has genuinely nonlinear origin, while for collapsing solutions it has two contributions: a nonlinear one and a linear one coming from the scattering on the fixed background curvature. In 3 spatial dimensions both tails decay as $1/t^3$, but for $\ell > 0$ the nonlinear tail decays as $1/t^{3\ell+3}$, while the linear tail decays as $1/t^{6\ell+4}$ [8]. Thus in higher dimensions it is the nonlinear tail which dominates.

## 3. Wave map matter

As a second toy-model to study late time asymptotics in gravitating system we considered wave map matter coupled to gravity [9]. Let $U : \mathcal{M} \to \mathcal{N}$ be a map from a spacetime $(\mathcal{M}, g_{ab})$ into a Riemannian manifold $(\mathcal{N}, G_{AB})$. A pair $(U, g_{ab})$ is said to be a wave map coupled to gravity if it is a critical point of the action functional

$$S = \int_{\mathcal{M}} \left( \frac{R}{16\pi G} - \frac{\lambda}{2} g^{ab} \partial_a U^A \partial_b U^B G_{AB} \right) dv \,, \tag{19}$$

where $R$ is the scalar curvature of the metric $g_{ab}$, $G$ is Newton's constant, $\lambda$ is the wave map coupling constant, and $dv$ is the volume element on $(\mathcal{M}, g_{ab})$. The field equations derived from (19) are the Einstein equations $R_{ab} - \frac{1}{2}\, g_{ab} R = 8\pi G T_{ab}$ with the stress-energy tensor

$$T_{ab} = \lambda \left( \partial_a U^A \partial_b U^B - \tfrac{1}{2}\, g_{ab} \left( g^{cd} \partial_c U^A \partial_d U^B \right)\right) G_{AB} \,, \tag{20}$$

and the wave map equation

$$\Box_g U^A + \Gamma^A_{BC}(U)\partial_a U^B \partial_b U^C g^{ab} = 0\,, \tag{21}$$

where $\Gamma^A_{BC}$ are the Christoffel symbols of the target metric $G_{AB}$ and $\Box_g$ is the wave operator associated with the metric $g_{ab}$. As a target manifold we take the three-sphere with the round metric in polar coordinates $U^A = (F, \Theta, \Phi)$

$$G_{AB} dU^A dU^B = dF^2 + \sin^2 F \left(d\Theta^2 + \sin^2\Theta\, d\Phi^2\right)\,. \tag{22}$$

For the four dimensional spacetime $\mathcal{M}$ we assume spherical symmetry and use the following ansatz for the metric

$$g_{ab} dx^a dx^b = \left(1 - \frac{2m}{r}\right)^{-1}\left(-e^{2\beta(t,r)} dt^2 + dr^2\right) + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\,. \tag{23}$$

In addition, we assume that the map $U$ is spherically $\ell$-equivariant, that is

$$F = F(t,r)\,, \qquad (\Theta, \Phi) = \Omega_\ell(\theta, \phi)\,, \tag{24}$$

where $\Omega_\ell : S^2 \to S^2$ is a harmonic eigenmap with eigenvalue $\ell(\ell+1)$ (the components of $\Omega_\ell$ are homogeneous harmonic polynomials of degree $\ell$). The map index $\ell$ plays a role similar to angular momentum, but can be built consistently into fully nonlinear Einstein equations (as the energy-momentum tensor (20) for the ansatz (24) does not depend on angles). Thus it may serve as a toy-model to study non spherically symmetric gravitational collapse.

The coupled Einstein-wave map system (here $\kappa = 4\pi G\lambda$ is a dimensionless parameter):

$$m' = \frac{\kappa}{2} r^2 \left(1 - \frac{2m}{r}\right)\left(F'^2 + e^{-2\beta}\dot{F}^2\right) + \kappa \frac{\ell(\ell+1)}{2} \sin^2 F\,, \tag{25}$$

$$\dot{m} = \kappa\, r^2 \left(1 - \frac{2m}{r}\right) \dot{F}\, F'\,, \tag{26}$$

$$\beta' = \frac{2m}{r^2}\left(1 - \frac{2m}{r}\right)^{-1} - \kappa\ell(\ell+1)\left(1 - \frac{2m}{r}\right)^{-1} \frac{\sin^2 F}{r}\,, \tag{27}$$

$$\left(e^{-\beta}\dot{F}\right)^{\cdot} - \frac{1}{r^2}\left(r^2 e^{\beta} F'\right)' + \left(1 - \frac{2m}{r}\right) e^{\beta}\ell(\ell+1)\frac{\sin 2F}{2r^2} = 0\,, \tag{28}$$

with initial condition $(F, \dot{F})_{t=0} = (\varepsilon f(r), \varepsilon g(r))$ can be treated similarly to the massless scalar field case (Sec. 2). Up to the order $\mathcal{O}(\varepsilon^3)$ we have $F(t,r) = \varepsilon F_1(t,r) + \varepsilon^3 F_3(t,r)$ where

$$F_1(t,r) = \frac{1}{r}\sum_{k=0}^{\ell} \frac{(2\ell-k)!}{k!(\ell-k)!} \frac{\left(a^{(k)}(t-r) - (-1)^k a^{(k)}(t+r)\right)}{(2r)^{\ell-k}} \tag{29}$$

(with a compactly supported function $a(x)$ set by the initial data in (28)), and for $\ell > 0$

$$F_3(t,r) = \frac{r^\ell}{(t^2-r^2)^{\ell+1}} \left[A_\ell \kappa + B_\ell + \mathcal{O}\left(\frac{t}{t^2-r^2}\right)\right] \tag{30}$$

and coefficients $A_\ell$, $B_\ell$ are the only trace of initial data (for $\ell = 0$ the equations (25)–(28) reduce to the equations (6)–(9) for $d = 3$ spatial dimensions). Thus for $\ell > 0$ the map decays as $1/t^{2\ell+2}$ at time-like infinity. We refer the reader to [9] for more details about this calculation and numerical evidence.

This result suggest that in $3+1$ spacetime dimensions the linear tail of an $\ell > 0$ multipole of a massless scalar field evolving on a fixed but time dependent background decays generically as $1/t^{2\ell+2}$ (as numerically reported in [6] Sec. V), that is by one power slower then Price's law prediction $1/t^{2\ell+3}$ for a fixed static background [4,5]. This issue will be discussed elsewhere [10].

## 4. Summary

We discussed the non-linear origin of the power-law tails in the long-time evolution of a spherically symmetric self-gravitating massless fields in even-dimensional spacetime. Using third-order perturbation method, we derived explicit expressions for the tail (the decay rate and the amplitude) for solutions starting from small initial data and then we confirmed this prediction via numerical integration of the full set of Einstein equations. Our results show that the agreement between decay rates of linear (on static background) and nonlinear tails of self-gravitating massless scalar field observed in $3+1$ spacetime dimensions [6] is accidental and holds neither in higher dimensions nor in other models (*e.g.* wave maps).

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