α_{s} DETERMINATION FROM τ DECAYS: THEORETICAL STATUS*

Antonio Pich

Departament de Física Teòrica, IFIC, Universitat de València — CSIC
Apt. Correus 22085, 46071 València, Spain

The total τ hadronic width can be accurately calculated using analyticity and the operator product expansion. The result turns out to be very sensitive to the value of $\alpha_{\mathrm{s}}(m_\tau^2)$, providing a precise determination of the strong coupling constant. The theoretical description of this observable is updated, including the recently computed $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ contributions. The experimental determination of $\alpha_{\mathrm{s}}(m_\tau^2)$ and its actual uncertainties are discussed.

PACS numbers: 12.38.–t, 12.38.Bx, 12.38.Qk

1. Introduction

The inclusive hadronic decay width of the τ lepton provides one of the most precise measurements of the strong coupling [1–7]. Moreover, the comparison of $\alpha_{\mathrm{s}}(m_\tau^2)$ with α_{s} determinations at higher energies constitutes the most accurate test of asymptotic freedom, successfully confirming the predicted running of the QCD coupling at the four-loop level.

The calculation of the $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ contribution [8] has triggered a renewed theoretical interest on the $\alpha_{\mathrm{s}}(m_\tau^2)$ determination, since it allows to push the accuracy to the four-loop level [8–13]. However, the recent theoretical analyses slightly disagree on the final result, giving rise to a range of different values for $\alpha_{\mathrm{s}}(m_\tau^2)$. The differences among these results, shown in Table I, are too large compared with the claimed $\mathcal{O}(\alpha_{\mathrm{s}}^4)$ accuracy and originate in the different inputs or theoretical procedures which have been adopted.

In the following I try to clarify the reasons behind these numerical discrepancies and reassess the actual uncertainties of the τ decay determination of α_{s}. Using all present experimental and theoretical knowledge, I derive the value

$$\alpha_{\mathrm{s}}\left(m_\tau^2\right) = 0.342 \pm 0.012\,. \tag{1}$$

TABLE I

$\mathcal{O}(\alpha_\mathrm{s}^4)$ determinations of $\alpha_\mathrm{s}(m_\tau^2)$ and associated pQCD contribution δ_P.

Reference	Method	δ_P	$\alpha_\mathrm{s}(m_\tau^2)$
Baikov *et al.* [8]	CIPT, FOPT	0.1998 ± 0.0043	0.332 ± 0.016
Davier *et al.* [9]	CIPT	0.2066 ± 0.0070	0.344 ± 0.009
Beneke–Jamin [10]	BSR + FOPT	0.2042 ± 0.0050	0.316 ± 0.006
Maltman–Yavin [11]	PWM + CIPT	—	0.321 ± 0.013
Menke [12]	CIPT, FOPT	0.2042 ± 0.0050	$0.342^{+0.011}_{-0.010}$
Caprini–Fischer [13]	BSR + CIPT	0.2042 ± 0.0050	$0.320^{+0.011}_{-0.009}$

2. Theoretical framework

The inclusive character of the total τ hadronic width renders possible an accurate calculation of the ratio [1–5]

$$R_\tau \equiv \frac{\Gamma[\tau^- \to \nu_\tau \,\text{hadrons}\,(\gamma)]}{\Gamma[\tau^- \to \nu_\tau e^- \bar{\nu}_e(\gamma)]} = R_{\tau,\mathrm{V}} + R_{\tau,\mathrm{A}} + R_{\tau,\mathrm{S}} . \tag{2}$$

The theoretical analysis involves the two-point correlation functions for the left-handed quark currents $L_{ij}^\mu = \bar{\psi}_j \gamma^\mu (1 - \gamma_5)\psi_i$ $(i,j = u,d,s)$:

$$\Pi_{ij}^{\mu\nu}(q) \equiv i \int d^4 x\, e^{iqx}\, \langle 0| T\left(L_{ij}^\mu(x) L_{ij}^\nu(0)^\dagger \right) |0\rangle$$

$$= \left(-g^{\mu\nu} q^2 + q^\mu q^\nu \right) \Pi_{ij}^{(1)}\left(q^2\right) + q^\mu q^\nu\, \Pi_{ij}^{(0)}\left(q^2\right) . \tag{3}$$

Using analyticity, R_τ can be written as a contour integral in the complex s-plane running counter-clockwise around the circle $|s| = m_\tau^2$:

$$R_\tau = 6\pi i \oint_{|s|=m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) \Pi^{(0+1)}(s) - 2\frac{s}{m_\tau^2} \Pi^{(0)}(s) \right] , \tag{4}$$

where $\Pi^{(J)}(s) \equiv |V_{ud}|^2\, \Pi_{ud}^{(J)}(s) + |V_{us}|^2\, \Pi_{us}^{(J)}(s)$.

This expression requires the correlators only for complex s of the order of m_τ^2, which is significantly larger than the scale associated with non-perturbative effects. Using the Operator Product Expansion (OPE) to evaluate the contour integral, R_τ can be expressed as an expansion in powers of $1/m_\tau^2$. The uncertainties associated with the use of the OPE near the time-like axis are heavily suppressed by the presence in (4) of a double zero at $s = m_\tau^2$ [1, 9, 14].

The theoretical prediction for the Cabibbo-allowed decay width can be written as [1]

$$R_{\tau,\text{V+A}} = N_C \left|V_{ud}\right|^2 S_{\text{EW}} \left\{1 + \delta_\text{P} + \delta_\text{NP}\right\}, \tag{5}$$

where $N_C = 3$ is the number of quark colours and $S_{\text{EW}} = 1.0201 \pm 0.0003$ contains the electroweak radiative corrections [15–17]. The dominant correction ($\sim 20\%$) is the perturbative QCD contribution in the massless-quark limit δ_P, which is already known to $O(\alpha_\text{s}^4)$ [1,8]. Quark mass effects [1,18,19] are tiny for the Cabibbo-allowed current and amount to a negligible correction smaller than 10^{-4} [1,10].

Non-perturbative contributions are suppressed by six powers of the τ mass [1] and, therefore, are very small. Their numerical size has been determined from the invariant-mass distribution of the final hadrons in τ decay, through the study of weighted integrals [20] which can be calculated theoretically in the same way as R_τ. The predicted suppression [1] of the non-perturbative corrections has been confirmed by ALEPH [21], CLEO [22] and OPAL [23]. The most recent analysis [6] gives

$$\delta_\text{NP} = -0.0059 \pm 0.0014. \tag{6}$$

The measured values of the τ lifetime and leptonic branching ratios imply $R_\tau = 3.640 \pm 0.010$ [9]. Subtracting the Cabibbo-suppressed contribution $R_{\tau,\text{S}} = 0.1615 \pm 0.0040$ [9], one obtains $R_{\tau,\text{V+A}} = 3.479 \pm 0.011$. Using $\left|V_{ud}\right| = 0.97418 \pm 0.00027$ [24] and (6), the pure perturbative contribution to R_τ is determined to be:

$$\delta_\text{P} = 0.2038 \pm 0.0040. \tag{7}$$

3. Perturbative contribution to R_τ

In the chiral limit ($m_u = m_d = m_s = 0$), the vector and axial-vector currents are conserved. This implies $s\, \Pi^{(0)}(s) = 0$; therefore, only the correlator $\Pi^{(0+1)}(s)$ contributes to Eq. (4). The result is more conveniently expressed in terms of the logarithmic derivative of the two-point correlation function of the vector (axial) current, $\Pi(s) = \frac{1}{2} \Pi^{(0+1)}(s)$, which satisfies an homogeneous renormalization–group equation:

$$D\left(Q^2\right) \equiv -Q^2 \frac{d}{dQ^2}\Pi\left(Q^2\right) = \frac{N_C}{12\pi^2} \sum_{n=0} K_n \left(\frac{\alpha_\text{s}\left(Q^2\right)}{\pi}\right)^n. \tag{8}$$

With the choice of renormalization scale $\mu^2 = Q^2 \equiv -s$ all logarithmic corrections, proportional to powers of $\log\left(-s/\mu^2\right)$, have been summed into the running coupling. The K_n coefficients are known to order α_s^4. For $n_f = 3$ quark flavours, one has [8,25–29]: $K_0 = K_1 = 1$, $K_2 = 1.63982$, $K_3^{\overline{\text{MS}}} = 6.37101$ and $K_4^{\overline{\text{MS}}} = 49.07570$.

The perturbative component of R_τ is given by

$$\delta_\mathrm{P} = \sum_{n=1} K_n A^{(n)}(\alpha_\mathrm{s})\,, \qquad (9)$$

where the functions [2]

$$A^{(n)}(\alpha_\mathrm{s}) \equiv \frac{1}{2\pi i} \oint_{|s|=m_\tau^2} \frac{ds}{s} \left(\frac{\alpha_\mathrm{s}(-s)}{\pi}\right)^n \left(1 - 2\frac{s}{m_\tau^2} + 2\frac{s^3}{m_\tau^6} - \frac{s^4}{m_\tau^8}\right) \quad (10)$$

are contour integrals in the complex plane which only depend on the strong coupling. Using the exact solution (up to unknown $\beta_{n>4}$ contributions) for $\alpha_\mathrm{s}(s)$ given by the renormalization-group β-function equation, they can be numerically computed with a very high accuracy [2].

Table II gives the numerical values for $A^{(n)}(\alpha_\mathrm{s})$ ($n \leq 5$) obtained at the one-, two-, three- and four-loop approximations (i.e. $\beta_{n>1} = 0$, $\beta_{n>2} = 0$, $\beta_{n>3} = 0$ and $\beta_{n>4} = 0$, respectively), together with the corresponding results for $\delta_\mathrm{P} = \sum_{n=1}^4 K_n A^{(n)}(\alpha_\mathrm{s})$, taking $\alpha_\mathrm{s}(m_\tau^2)/\pi = 0.1$. The perturbative convergence is very good. The error induced by the truncation of the β function at fourth order can be conservatively estimated through the variation of the results at five loops, assuming $\beta_5 = \pm\beta_4^2/\beta_3 = \mp 443$, i.e. a geometric growth of the β function. Higher-order contributions to the Adler function $D(Q^2)$ will be taken into account adding the fifth-order term $K_5 A^{(5)}(\alpha_\mathrm{s})$ with $K_5 = 275\pm400$. Including, moreover, the 5-loop variation with changes of the renormalization scale in the range $\mu^2 \in [0.5, 1.5]$, one gets the final result $\delta_\mathrm{P} = 0.1810 \pm 0.0045_{K_5} \pm 0.0013_{\beta_5} \pm 0.0013_\mu = 0.1810 \pm 0.0049$ for $\alpha_\mathrm{s}(m_\tau^2)/\pi = 0.1$.

Adopting this very conservative procedure, the experimental value of δ_P given in Eq. (7) implies the strong coupling determination quoted in Eq. (1).

TABLE II

Exact results for $A^{(n)}(\alpha_\mathrm{s})$ ($n \leq 5$) at different β-function approximations, and corresponding values of $\delta_\mathrm{P} = \sum_{n=1}^4 K_n A^{(n)}(\alpha_\mathrm{s})$, for $\alpha_\mathrm{s}(m_\tau^2)/\pi = 0.1$.

Loops	$A^{(1)}(\alpha_\mathrm{s})$	$A^{(2)}(\alpha_\mathrm{s})$	$A^{(3)}(\alpha_\mathrm{s})$	$A^{(4)}(\alpha_\mathrm{s})$	$A^{(5)}(\alpha_\mathrm{s})$	δ_P
1	0.13247	0.01570	0.001698	0.000169	0.0000154	0.1773
2	0.13523	0.01575	0.001629	0.000151	0.0000124	0.1788
3	0.13540	0.01565	0.001597	0.000145	0.0000115	0.1784
4	0.13529	0.01557	0.001579	0.000142	0.0000112	0.1779

4. Discussion on previous $\mathcal{O}(\alpha_{\rm s}^4)$ determinations of $\alpha_{\rm s}(m_\tau^2)$

The integrals $A^{(n)}(\alpha_{\rm s})$ can be expanded in powers of $a_\tau \equiv \alpha_{\rm s}(m_\tau^2)/\pi$, $A^{(n)}(\alpha_{\rm s}) = a_\tau^n + \mathcal{O}(a_\tau^{n+1})$. One recovers in this way the naive perturbative expansion $\delta_{\rm P} = \sum_{n=1} (K_n + g_n)\, a_\tau^n \equiv \sum_{n=1} r_n\, a_\tau^n$ [2]. This approximation is known as *fixed-order perturbation theory* (FOPT), while the improved expression (9), keeping the non-expanded values for the integrals $A^{(n)}(\alpha_{\rm s})$, is usually called *contour-improved perturbation theory* (CIPT) [2,30]. FOPT gives rise to a pathological non-convergent series (its radius of convergence is slightly smaller than the physical value of a_τ), because the long running of $\alpha_{\rm s}(s)$ along the circle $|s| = m_\tau^2$ generates very large g_n coefficients, which depend on $K_{m<n}$ and $\beta_{m<n}$ [2]: $g_1 = 0$, $g_2 = 3.56$, $g_3 = 19.99$, $g_4 = 78.00$, $g_5 = 307.78$. These corrections are much larger than the original K_n contributions and lead to values of $\alpha_{\rm s}(m_\tau^2)$ smaller than (1). FOPT suffers from a huge renormalization-scale dependence [2]. As shown in a recent detailed analysis [12], the actual FOPT uncertainties are much larger than usually estimated. Once this is taken properly into account, the FOPT results are consistent with CIPT, but their huge uncertainties make them irrelevant.

An *ad hoc* model of higher-order coefficients for the Adler function has been recently advocated [10]. The model mixes three different types of renormalons ($n = -1$, 2 and 3) plus a linear polynomial, trying to enforce a cancellation of the K_n and g_n coefficients in order to get a better behaviour of the FOPT series. It contains 5 free parameters which are determined by the known coefficients of the Adler series and the chosen value of K_5. Making a Borel summation of the full renormalon series (BSR), one gets a sizeable positive contribution to $\delta_{\rm P}$ from higher orders, implying a smaller value for $\alpha_{\rm s}(m_\tau^2)$ independently of the adopted FOPT [10] or CIPT [13] procedure. This model constitutes an interesting example of possible higher-order corrections, making apparent that the associated uncertainties have to be carefully estimated. However, it cannot be used to determine the physical value of $\alpha_{\rm s}$, because the result is model dependent. Different assumptions about the unknown K_n coefficients would lead to different central values for $\alpha_{\rm s}$.

The determination of $\alpha_{\rm s}(m_\tau^2)$ in Eq. (1) agrees with Refs. [6,12] and with the CIPT value in Ref. [8]. It is somewhat larger than the (CIPT) result extracted in Ref. [11] from pinched-weight moments (PWM) of the hadronic distribution, which seems to correspond to a very different value of $\delta_{\rm NP}$. After evolution up to the scale M_Z [31], the strong coupling in Eq. (1) decreases to

$$\alpha_{\rm s}(M_Z^2) = 0.1213 \pm 0.0014\,, \tag{11}$$

in excellent agreement with the direct measurements at the Z peak.

REFERENCES

[1] E. Braaten, S. Narison, A. Pich, *Nucl. Phys.* **B373**, 581 (1992).

[2] F. Le Diberder, A. Pich, *Phys. Lett.* **B286**, 147 (1992).

[3] S. Narison, A. Pich, *Phys. Lett.* **B211**, 183 (1988).

[4] E. Braaten, *Phys. Rev. Lett.* **60**, 1606 (1988); *Phys. Rev.* **D39**, 1458 (1989).

[5] A. Pich, *Nucl. Phys. Proc. Suppl.* **B,C39**, 326 (1995); arXiv:hep-ph/9701305.

[6] M. Davier, A. Höcker, Z. Zhang, *Rev. Mod. Phys.* **78**, 1043 (2006).

[7] A. Pich, *Nucl. Phys. Proc. Suppl.* **B186**, 187 (2009).

[8] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, *Phys. Rev. Lett.* **101**, 012002 (2008).

[9] M. Davier *et al.*, *Eur. Phys. J.* **C56**, 305 (2008).

[10] M. Beneke, M. Jamin, *J. High Energy Phys.* **0809**, 044 (2008).

[11] K. Maltman, T. Yavin, *Phys. Rev.* **D78**, 094020 (2008).

[12] S. Menke, arXiv:0904.1796 [hep-ph].

[13] I. Caprini, J. Fischer, *Eur. Phys. J.* **C64**, 35 (2009).

[14] O. Cata, M. Golterman, S. Peris, *Phys. Rev.* **D77**, 093006 (2008).

[15] W.J. Marciano, A. Sirlin, *Phys. Rev. Lett.* **61**, 1815 (1988).

[16] E. Braaten, C.S. Li, *Phys. Rev.* **D42**, 3888 (1990).

[17] J. Erler, *Rev. Mex. Phys.* **50**, 200 (2004).

[18] A. Pich, J. Prades, *J. High Energy. Phys.* **9910**, 004 (1999); **9806**, 013 (1998).

[19] P.A. Baikov, K.G. Chetyrkin, J.H. Kühn, *Phys. Rev. Lett.* **95**, 012003 (2005).

[20] F. Le Diberder, A. Pich, *Phys. Lett.* **B289**, 165 (1992).

[21] [ALEPH Collaboration], *Phys. Rep.* **421**, 191 (2005); *Eur. Phys. J.* **C4**, 409 (1998); *Phys. Lett.* **B307**, 209 (1993).

[22] [CLEO Collaboration], *Phys. Lett.* **B356**, 580 (1995).

[23] [OPAL Collaboration], *Eur. Phys. J.* **C7**, 571 (1999).

[24] C. Amsler *et al.*, *Phys. Lett.* **B667**, 1 (2008).

[25] K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, *Phys. Lett.* **B85**, 277 (1979).

[26] M. Dine, J. Sapirstein, *Phys. Rev. Lett.* **43**, 668 (1979).

[27] W. Celmaster, R. Gonsalves, *Phys. Rev. Lett.* **44**, 560 (1980).

[28] S.G. Gorishny, A.L. Kataev, S.A. Larin, *Phys. Lett.* **B259**, 144 1991.

[29] L.R. Surguladze, M.A. Samuel, *Phys. Rev. Lett.* **66**, 560 1991.

[30] A.A. Pivovarov, *Z. Phys.* **C53**, 461 (1992).

[31] G. Rodrigo, A. Pich, A. Santamaria, *Phys. Lett.* **B424**, 367 (1998).