PRECISION CALCULATIONS IN BR($\bar{B} \to X_s\gamma$)*

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We briefly summarize the current status of perturbative calculations at next-to-next-to-leading-order (NNLO) accuracy in the $\bar{B} \to X_s\gamma$ decay rate as well as that of non-perturbative power-corrections.

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1. Introduction

Corrections to the $\bar{B} \to X_s\gamma$ decay are usually described in the framework of an effective theory[1],

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm QCD\times QED}(u,d,s,c,b) + \frac{4G_{\rm F}}{\sqrt{2}} V_{tb}V_{ts}^* \sum_{i=1}^{8} C_i(\mu) O_i \,. \tag{1}$$

Here, C_i are renormalization scale dependent effective couplings, the so-called Wilson coefficients, which encode the heavy gauge boson and the heavy top quark effects. The b-quark scale contributions, on the other hand, are seen as matrix elements of flavor changing operators,

$$\begin{aligned}
O_1 &= (\bar{s}_L\gamma_\mu T^a c_L)\,(\bar{c}_L\gamma^\mu T^a b_L)\;, \qquad O_2 = (\bar{s}_L\gamma_\mu c_L)\,(\bar{c}_L\gamma^\mu b_L)\;,\\
O_{3,5} &= (\bar{s}_L\Gamma_{3,5} b_L)\sum\nolimits_q \left(\bar{q}\Gamma'_{3,5} q\right)\;, \quad O_{4,6} = (\bar{s}_L\Gamma_{3,5} T^a b_L)\sum\nolimits_q \left(\bar{q}\Gamma'_{3,5} T^a q\right)\;,\\
O_7 &= \frac{\alpha_{\rm em}}{4\pi}\, m_b\,(\bar{s}_L\sigma^{\mu\nu} b_R)\, F_{\mu\nu}\,, \qquad O_8 = \frac{\alpha_s}{4\pi}\, m_b\,(\bar{s}_L\sigma^{\mu\nu} T^a b_R)\, G^a_{\mu\nu}\,,
\end{aligned} \tag{2}$$

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[1] In writing (1) we discarded terms proportional to $V_{ub}V_{us}^*$ since they give only small contributions to the branching ratio that start at next-to-leading-order (NLO). Similar NNLO corrections can therefore be safely neglected.

where $\Gamma_3 = \gamma_\mu$, $\Gamma_3' = \gamma^\mu$, $\Gamma_5 = \gamma_\mu\gamma_\nu\gamma_\lambda$ and $\Gamma_5' = \gamma^\mu\gamma^\nu\gamma^\lambda$. Using (1), the differential decay rate for $\bar{B} \to X_s\gamma$ can be written as follows,

$$d\Gamma = \frac{G_{\rm F}^2 \alpha_{\rm em} m_b^2}{256\pi^6 m_B} |V_{tb}V_{ts}^*|^2 \, \frac{d^3q}{E_\gamma} \sum_{i,j} C_i^{\rm eff}(\mu) C_j^{\rm eff}(\mu) W_{ij}(\mu) \, . \qquad (3)$$

In the equation displayed above, q denotes the momentum of the photon, $C_i^{\rm eff}$ are certain linear combinations of C_i (see *e.g.* [1]), and the W_{ij} describe the hadronic dynamics. For $(ij) = (77)$ the latter can be written as imaginary part of a forward scattering amplitude,

$$W_{77}(\mu) = 2\,{\rm Im}\left(i \int d^4x \, e^{-iq\cdot x} \langle \bar{B} | \, T\Big\{ O_7^\dagger(x) O_7(0) \Big\} \, | \bar{B} \rangle \right) . \qquad (4)$$

Since the mass of the b-quark is much larger than the binding energy of the B-meson, which is of the order of $\Lambda \equiv \Lambda_{\rm QCD}$, we can perform an operator product expansion (OPE) of this time ordered product. Doing so, one finds that the leading term is the partonic decay rate which gives the dominant contribution, while the non-leading terms, the so-called power-corrections, are suppressed by powers of Λ/m_b and give non-vanishing contributions starting from $O(\Lambda^2/m_b^2)$ [2]. In what follows we describe the state-of-the-art of perturbative and non-perturbative corrections in the $\bar{B} \to X_s\gamma$ decay.

2. Perturbative corrections

The calculation of the perturbative corrections can be divided into three steps. In the first step one has to evaluate the effective couplings $C_i^{\rm eff}$ at the high-energy scale $\mu \sim M_{\rm W}$ by requiring equality of the Standard Model and the effective theory Green functions. Defining $\tilde{\alpha}_{\rm s}(\mu) = \alpha_{\rm s}(\mu)/(4\pi)$, the effective couplings can be expanded as follows,

$$C_i^{\rm eff}(\mu) = C_i^{(0)\rm eff}(\mu) + \tilde{\alpha}_{\rm s}(\mu) C_i^{(1)\rm eff}(\mu) + \tilde{\alpha}_{\rm s}^2(\mu) C_i^{(2)\rm eff}(\mu) + \ldots \, . \qquad (5)$$

At NNLO accuracy one has to determine the coefficients $C_i^{(2)\rm eff}(\mu)$. For $i = 7, 8$ it required performing a three-loop calculation [2] whereas for the remaining cases $i = 1, \ldots, 6$ a two-loop calculation was sufficient [3].

The second step involves the calculation of the anomalous dimension matrix $\gamma^{\rm eff}$ which describes the mixing of the operators under renormalization. Its knowledge is necessary to solve the effective theory renormalization group equations for the effective couplings,

[2] We stress that equation (4) and its OPE hold only for W_{77}. In all other cases the W_{ij} defined in (3) contain contributions in which the photon couples to light quarks (u, d, s, c), and this leads to non-perturbative effects different from that mentioned above (see Section 3).

$$\mu \frac{d}{d\mu} C_i^{\rm eff}(\mu) = \sum_j \gamma_{ji}^{\rm eff} C_j^{\rm eff}(\mu)\,, \tag{6}$$

and to evolve the latter down to the low-energy scale $\mu \sim m_b$. Performing a perturbative expansion in the strong coupling constant, the anomalous dimension matrix takes the following form,

$$\gamma^{\rm eff} = \tilde{\alpha}_{\rm s}(\mu)\gamma^{(0){\rm eff}} + \tilde{\alpha}_{\rm s}^2(\mu)\gamma^{(1){\rm eff}} + \tilde{\alpha}_{\rm s}^3(\mu)\gamma^{(2){\rm eff}} + \dots\,. \tag{7}$$

At NNLO one has to determine $\gamma^{(2){\rm eff}}$ which is actually a 8×8 matrix,

$$\gamma^{(2){\rm eff}} = \begin{pmatrix} A^{(2)}_{6\times 6} & B^{(2)}_{6\times 2} \\ 0_{2\times 6} & C^{(2)}_{2\times 2} \end{pmatrix}\,. \tag{8}$$

The block matrices A and C describing the self-mixing of the four-quark operators and the self-mixing of the dipole operators at three loops, respectively, have been calculated in [4]. The block matrix B describing the mixing of the four-quark operators into the dipole operators at four loops has been determined in [1]. After this calculation the first two steps of the perturbative calculation were completed, that is the effective couplings at the low-energy scale $\mu \sim m_b$ with resummed logarithms are now known at NNLO accuracy[3].

In the last step one has to calculate on-shell amplitudes of the operators at the low-energy scale. This is the most difficult part of the NNLO enterprise and it is still under investigation. In order to see what has been done so far, and what still has to be done, we write the decay rate for the partonic decay $b \to X_s^{\rm partonic}\gamma$ as follows,

$$\Gamma^{\rm partonic}\big|_{E_\gamma > E_0} = \frac{G_{\rm F}^2 \alpha_{\rm em} m_b^5}{32\pi^4}\,|V_{tb}V_{ts}^*|^2 \sum_{i,j} C_i^{\rm eff}(\mu) C_j^{\rm eff}(\mu) G_{ij}(E_0,\mu)\,, \tag{9}$$

where $G_{ij}(E_0,\mu)$ can again be expanded in terms of $\tilde{\alpha}_{\rm s}$,

$$G_{ij}(E_0,\mu) = \delta_{i7}\delta_{j7} + \tilde{\alpha}_{\rm s}(\mu) Y_{ij}^{(1)}(E_0,\mu) + \tilde{\alpha}_{\rm s}^2(\mu) Y_{ij}^{(2)}(E_0,\mu) + \dots\,. \tag{10}$$

At NNLO one has to determine the coefficients of $\tilde{\alpha}_{\rm s}^2(\mu)$ which, however, has only been done in a complete manner for $i = j = 7$ [5,6]. Once we neglect on-shell amplitudes that are proportional to the small Wilson coefficients of the four-quark penguin operators O_3–O_6, the remaining cases to be considered are $(ij) = (11), (12), (22), (17), (18), (27), (28), (78)$, and (88). The large-β_0

[3] This means large logarithms have been resummed up to $O(\alpha_{\rm s}^{n+2}\ln^n(m_b/M_{\rm W}))$.

corrections are known in all these cases except for (18) and (28) [7–9]. In addition, effects of the charm and bottom quark masses on the gluon lines are known in all the cases [10, 11]. The other beyond-large-β_0 corrections have been found only in the limit $m_c \gg m_b/2$, except for the (78) and (88) cases [12]. This limit has been used to interpolate the unknown beyond-large-β_0 corrections at $O(\alpha_s^2)$ to the measured value of $m_c \approx m_b/4$ [12]. The result for the branching ratio, for $E_0 = 1.6$ GeV, is given by [13][4]

$$\mathrm{BR}\left(\bar{B} \to X_s\gamma\right)_{\mathrm{SM}} = (3.15 \pm 0.23) \times 10^{-4} \qquad (\text{1S scheme})\,. \tag{11}$$

The theoretical uncertainty of this NNLO estimate is at the same level as the uncertainty of the current world average reported by HFAG [14][5],

$$\mathrm{BR}\left(\bar{B} \to X_s\gamma\right)_{\mathrm{exp}} = (3.52 \pm 0.23 \pm 0.09) \times 10^{-4}\,, \tag{12}$$

which is furthermore expected to come down to the 5% level at the end of the B-factory era.

Here a remark concerning the overall normalization of the theoretical prediction is in order. To reduce parametric uncertainties stemming from the CKM angles as well as from the c- and b-quark masses, the partonic decay rate given in (9) is usually normalized using a combination of the $\bar{B} \to X_c l\bar{\nu}$ and $\bar{B} \to X_u l\bar{\nu}$ decay rates which is reflected by the appearance of the semileptonic phase-space factor

$$C = \left|\frac{V_{ub}}{V_{cb}}\right|^2 \frac{\Gamma\left(\bar{B} \to X_c l\bar{\nu}\right)}{\Gamma\left(\bar{B} \to X_u l\bar{\nu}\right)} \tag{13}$$

in the analytical expressions [17][6]. Unfortunately, the determination of m_c and C from a fit to the measured spectrum of the $\bar{B} \to X_c l\bar{\nu}$ decay in the 1S scheme [19] differs from that in the kinetic scheme [20][7]. Using the values for m_c and C of the latter determination results in a higher central value for the $\bar{B} \to X_s\gamma$ decay rate [21],

$$\mathrm{BR}\left(\bar{B} \to X_s\gamma\right)_{\mathrm{SM}} = (3.25 \pm 0.24) \times 10^{-4} \qquad (\text{kinetic scheme})\,. \tag{14}$$

The difference of m_c and C in the 1S and kinetic scheme is likely to be due to different input data, differences in the fit method, and treatment

[4] For a discussion of the residual renormalization scale dependence of the branching ratio at NNLO we refer the reader to [13].

[5] This average includes the measurements from CLEO and BaBar and Belle [15]. The recently published update by Belle [16] has not been taken into account.

[6] The denominator $\Gamma(b \to u l\bar{\nu})$ is already known at NNLO accuracy [18].

[7] In [13] the values for m_c and C from [19] were adopted.

of theory errors. In this respect, supplementing the fit, for example, by the determination of the c- and b-quark masses from sumrules [22] could possibly be helpful to reduce the discrepancy of C in both schemes.

We should also remark that not all of the aforementioned contributions to the function G_{ij} entered the analysis of [13]. These are the massive fermionic corrections presented in [6, 10, 11] and the large-β_0 contributions for $(ij) \neq (77)$ from [8, 9][8]. These contributions will be included in a future update together with so far unknown contributions, which is, for example, the complete knowledge of G_{78} [23]. Also the complete calculation of G_{27} for $m_c = 0$ is underway [24]. Especially the latter will prove useful in the reduction of the uncertainty stemming from the interpolation in m_c. Apart from the NNLO corrections also tree-level diagrams with the u-quark analogues of $O_{1,2}$ and the four-quark operators O_{3-6} have been neglected so far [25]. The numerical effect of all of these contributions on the branching ratio is or is expected to remain within the uncertainty of the NNLO estimate given in (11).

Finally, we should mention that there are also cutoff-enhanced corrections which matter close to the endpoint [26]. However, as demonstrated in [21], the resummation of the cutoff-enhanced logarithms overestimates the effect of the $O(\alpha_{\rm s}^3)$-terms for $E_0 \lesssim 1.6$ GeV. Therefore, the prediction for the branching ratio given in (11) should, at present, be considered as more reliable.

3. Non-perturbative corrections

Since the perturbative calculations in $\bar{B} \to X_s\gamma$ are now performed at NNLO, the non-perturbative corrections become more important. In general well under control are the power-corrections stemming from the OPE of the time ordered product contained in (4); they are known at $O(\Lambda^2/m_b^2)$ [27] and $O(\Lambda^3/m_b^3)$ [28]. All the other W_{ij} with $(ij) \neq (77)$ contain contributions in which the photon couples to light quarks, and that causes the breakdown of an analogous OPE. In this case non-perturbative collinear effects [29] as well as power-corrections at $O(\Lambda^2/m_c^2)$ show up [30]. The combined effect of all of the aforementioned non-perturbative corrections is of around 3% in the branching ratio. Besides, non-perturbative effects appearing at $O(\alpha_{\rm s}\Lambda/m_b)$ show up when the photon couples to light quarks. Their size is not known at present, and hence a 5% uncertainty related to all the unknown non-perturbative effects has been included in (11). The size of this uncertainty is supported by the estimate of the $O(\alpha_{\rm s}\Lambda/m_b)$-corrections in the interference of the electro- and chromomagnetic dipole operators performed in [31]. As

[8] Also the mixing of the four-quark operators O_{1-6} into the chromomagnetic dipole operator O_8 [1] was not included in [13].

pointed out in references [31, 32], the magnitude of the effect considered in [31] could be probed by an improved measurement [33] of the isospin asymmetry

$$\Delta_{0-} = \frac{\Gamma\left(\bar{B}^0 \to X_s\gamma\right) - \Gamma\left(\bar{B}^- \to X_s\gamma\right)}{\Gamma\left(\bar{B}^0 \to X_s\gamma\right) + \Gamma\left(\bar{B}^- \to X_s\gamma\right)}\,. \tag{15}$$

Finally, we note that a non-perturbative uncertainty appears also when extrapolating the three different measurements performed at CLEO, BaBar and Belle down to the common lower cut $E_0 = 1.6$ GeV in the photon energy. It is accounted for in the error of the world average given in (12).

4. Conclusions

At present the uncertainties in the branching ratio of $\bar{B} \to X_s\gamma$ are on the same level on both the theoretical and experimental side. Thanks to the ongoing calculations of the perturbative corrections, the uncertainty stemming from this part will further reduce. However, to reach the 5% level or even less on the theoretical side, a better understanding of the non-perturbative power-corrections at $O(\alpha_{\rm s}\Lambda/m_b)$ is required.

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