ASSORTATIVITY IN RANDOM LINE GRAPHS*

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We investigate the degree–degree correlations in the Erdös–Rényi networks, the growing exponential networks and the scale-free networks. We demonstrate that these correlations are the largest for the exponential networks. We calculate also these correlations in the line graphs, formed from the considered networks. Theoretical and numerical results indicate that all the line graphs are assortative, *i.e.* the degree–degree correlation is positive.

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1. Introduction

A network of tennis players is formed when we link two players who met in the same game. Alternatively we can form a network of tennis games; two games are linked if the same competitor played in both of them. The same can be told on boxers and football teams. This construction is known as a line graph [1–3]. Each graph can be converted to its line graph. Under this transformation links become nodes, and two nodes of the line graph are linked if the respective links in the original graph share a node. The mathematical representation of a network by its line graph can be of interest in the science of complex networks [4]; for some applications of line graphs see [5–11].

Our concern in line graphs is due to their specific topology. Recently we shown that line graphs formed from the Erdös–Rényi networks, the growing exponential networks and the Barabási–Albert scale-free networks are highly clustered, with the clustering coefficient C higher than 0.5 [12]. This makes the line graphs to be potentially attractive for modeling of social networks,

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which are also highly clustered [13]. Here we focus on the degree–degree correlation in the line graphs, formed from the three kinds of networks listed above. Once this correlation is positive, nodes of high degree are more frequently linked to nodes of high degree; such networks are termed to show assortative mixing [14]. If the degree–degree correlation is negative, the mixing is termed disassortative.

In our former calculations [12], theoretical calculations of the clustering coefficient C in the line graphs were based on the assumption, that there is no degree–degree correlations in the initial networks. The accordance of the theoretical results with the simulations was quite reasonable at least for well connected networks, with mean degree larger than 10. Still, some differences could be observed for the exponential networks (Fig. 4 in [12]). Our aim here is (i) to compare the degree–degree correlations in the Erdös–Rényi networks, the exponential networks and the scale-free networks, (ii) to calculate these correlations in the line graphs, obtained from the three kinds of networks. The results are presented in the form of $\langle k'(k)\rangle$, where $k'(k)$ is the degree of a neighbour of a node of degree k.

Next section is devoted to the numerical calculations of the degree–degree correlations in the initial networks. In Section 3, analytical calculations of $\langle k'(k)\rangle$ for the line graphs are presented. In Section 4 we show the correlations in the line graphs, obtained numerically. Last section is devoted to conclusions.

2. Numerical calculations for the initial networks

The original Erdös–Rényi network is generated from $N = 10^4$ nodes; a link is placed between two nodes with the probability p. For the exponential and scale-free networks the algorithm starts from a fully connected cluster of M nodes. In a series of steps new nodes are added, each with M edges. Each edge of this node is connected to a randomly chosen node. For the scale-free network we have to use the preferential attachment; nodes are selected proportionally to their degree. The size of the initial exponential and scale-free networks is again $N = 10^4$ nodes.

To evaluate the degree–degree correlation we check how the average degree k' of the nearest neighbours of nodes with degree k depends on k. Numerical calculation begins with a search for nodes with degree k. Then, the average degree is calculated of all nearest neighbours of these nodes. Those steps are repeated for subsequent values of k.

In Fig. 1 we show exemplary results of the degree–degree correlations in the initial networks of three kinds. As it is shown there, the slope of obtained curve for the Erdös–Rényi network is close to zero. This means, that these networks show no degree–degree correlations, *i.e.* no assortativity

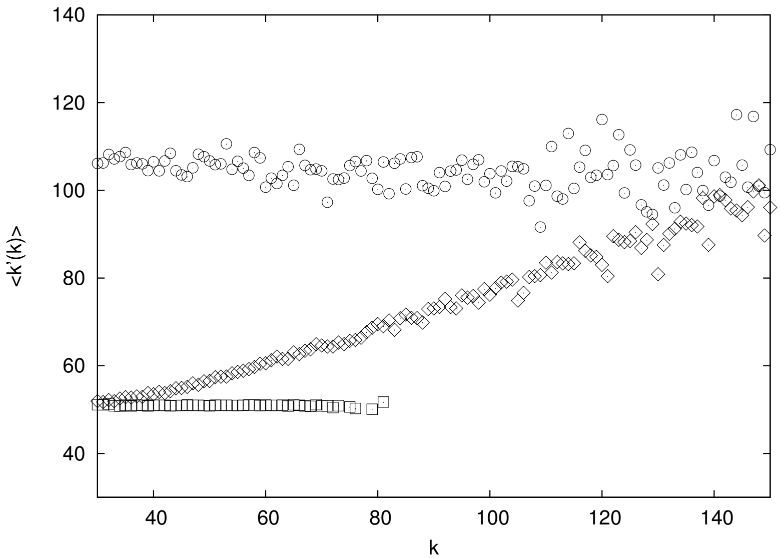

Fig. 1. Degree–degree correlations in the the Erdös–Rényi network (squares), the Barabási–Albert network (circles) and exponential network (rhombs) for $\langle k \rangle = 50$, measured by the curves $\langle k'(k) \rangle$, where k' is the degree of a neighbour of a node of degree k.

at all. This result is a natural consequence of the construction of this kind of networks. On the contrary, the results for the exponential networks indicate that the degree–degree correlations are positive: more connected nodes are nearest neighbours of also more connected ones. The result is in accordance with analytical [15, 16] calculations. The results for the Barabási–Albert networks are more fuzzy. Still, except perhaps the case of small ks, the correlations are not observed. This observation coincides with the conclusion of [14], obtained from analytical method.

3. Analytical calculations for the line graphs

The assortativity of the line graph is to be investigated by the calculation of the mean degree of a node, converted from a link, which is a neighbour of another node of degree k, converted also from a link. These two links shared a node in the initial graph. The notation is as follows: the first link joined nodes of degrees k_1 and k_2, and the second link joined nodes of degrees k_2 and k_3. Now these links are nodes, with degrees $k_1 + k_2 - 2$ and $k_2 + k_3 - 2$, respectively. We assume that there is no degree–degree correlations in the initial graph. Then the mean degree $\langle k'(k) \rangle$ of a neighbour of a node of degree k in the line graph can be found as

$$\langle k'(k)\rangle = \frac{\sum_{k_1,k_2,k_3} k_1 P(k_1) k_2 P(k_2) k_3 P(k_3)(k_1+k_2-2)\delta_{k,k_2+k_3-2}}{\sum_{k_1,k_2,k_3} k_1 P(k_1) k_2 P(k_2) k_3 P(k_3)\delta_{k,k_2+k_3-2}}\,, \quad (1)$$

where $P(k)$ is the degree distribution for the initial graph. We use the Kronecker delta to eliminate the sums over k_2. Then, the sums over k_1 are from one to infinity, and the sums over k_3 from one to k.

For the Erdös–Rényi networks $P(k)$ is Poissonian; let us denote $\langle k\rangle = \lambda$. We get

$$\langle k'(k)\rangle = \lambda + 1 + k - \frac{2^{k-1}(2+k)-1-k}{2^k-1} \quad (2)$$

what is close to $\lambda + k/2$ for large k.

For the exponential networks with the minimal degree M the degree distribution is $P(k) \propto c^k$, what gives $\langle k\rangle = 2M$, $c = M/(1+M)$ and

$$\langle k'(k)\rangle = \frac{2k+5\langle k\rangle - 2}{4}\,. \quad (3)$$

In this case the sums in Eq. (1) start from $k_i = M$, $i = 1,2,3$. After eliminating the sum over k_2, the sum over k_3 ends at $k_3 = k - M + 2$.

For the scale-free networks $P(k) \propto k^{-3}$ and the obtained series does not converge. For finite networks we can use Eq. (1) with an upper cut-off of k_1, determined by the system size [18]. The obtained plot is practically the same for the cut-off between 10^3 and 10^4. The limits of summations are the same as for the exponential networks.

4. Numerical calculations for the line graphs

The line graphs are constructed from the initial networks as follows. In the connectivity matrix of the initial network, the number of units above the main diagonal are substituted by their consecutive numbers. The maximal number is equal to the number of nodes in the line graph. In the connectivity matrix of the line graph, two nodes i and j are linked if the numbers i and j are in the same row or the same column in the renumbered connectivity matrix of the initial network. The same algorithm of construction of the line graphs was applied in [12].

The size of the initial network is equal 10^4. The calculations are performed for the line graphs of size dependent on the size, type and connectivity of the initial network. Then, the line graphs constructed from the Erdös–Rényi networks of the mean degree $\langle k\rangle = 5, 10, 20$ and 50 are of size of $25, 50, 100$ and 250 thousands, respectively. For the initial exponential and the Barabási–Albert networks of degree $\langle k\rangle = 4, 10, 20$ and 50 the sizes of the line graphs are respectively $20, 50, 100$ and 250 thousands. The degree distribution of the obtained line graphs was described in details in [12]; briefly, the line graphs retain the degree distributions of the initial networks.

The degree–degree correlations in the line graphs, obtained numerically, are shown in Figs. 2, 3 and 4 for the initial networks of three kinds: the Erdös–Rényi networks, the exponential networks and the Barabási–Albert networks, respectively. In the same graphs the theoretical curves are shown, derived from Eq. (1) with an assumption, that there are no degree–degree correlations in the initial networks. However, as we see in Fig. 1, this assumption is perfectly true only in the case of the Erdös–Rényi networks. Then it is not surprising, that the numerical results on the degree–degree correlations agree perfectly with theory only for this kind of networks (Fig. 2). As the exponential networks show assortativity (Fig. 1), the degree–degree correlations in the line graphs formed from the exponential networks differ from the theoretical data (Fig. 3). Finally, the noisy character of the correlations in the initial scale-free networks, observed in Fig. 1, has some counterpart in Fig. 4. Moreover, in the latter case the numerical curves show some systematic deviation from theory till some value of the degree k. One of possible explanations of the observed deviations could be the influence of hubs. We checked that this part of data differ from one generated graph to another.

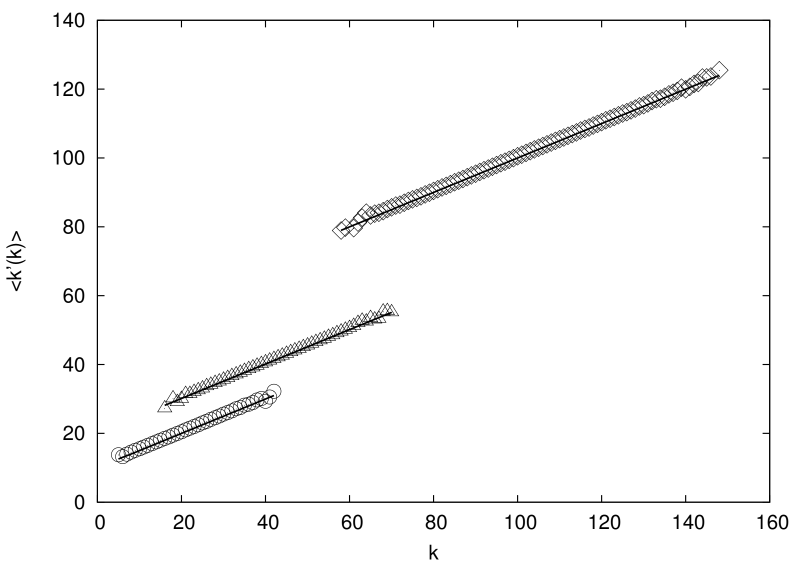

Fig. 2. Degree–degree correlations in the line graphs constructed from the Erdös–Rényi networks. The data shown are obtained for $\langle k \rangle = 10, 20$ and 50 (circles, triangles and rhombs, respectively). Lines are obtained from Eq. (2).

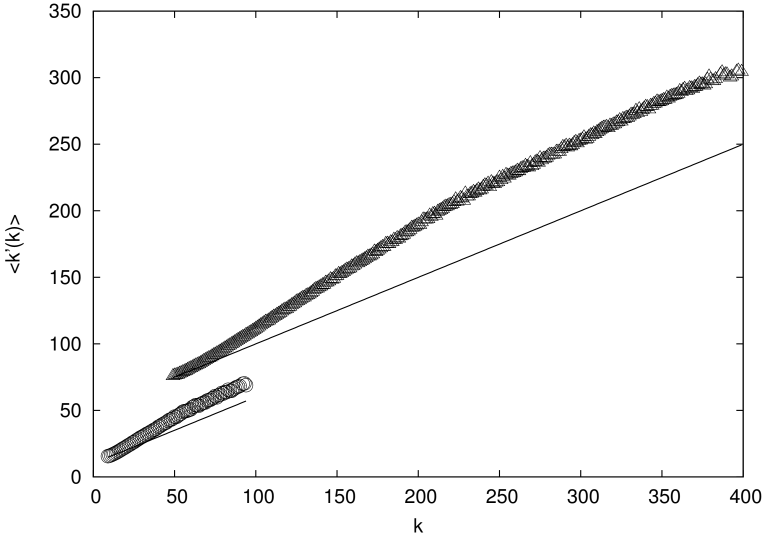

Fig. 3. Degree–degree correlations in the line graphs constructed from the growing exponential networks. The data shown are obtained for $\langle k \rangle = 10$ and 50 (circles and triangles, respectively). Lines are obtained from Eq. (3).

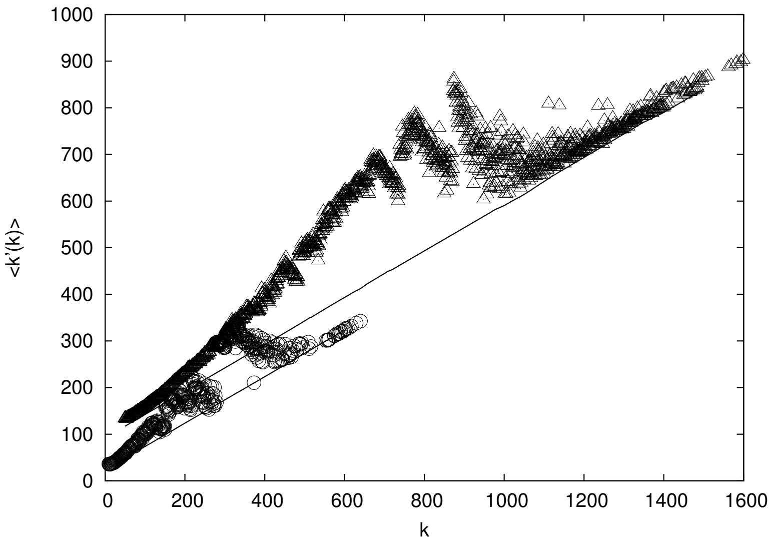

Fig. 4. Degree–degree correlations in the line graphs constructed from the growing Barabási–Albert networks. The data shown are obtained for $\langle k \rangle = 10$ and 50 (circles and triangles, respectively). Lines are obtained from Eq. (1).

5. Conclusions

Our numerical results on the $\langle k'(k)\rangle$ for the original networks indicate that the degree–degree correlations are remarkable for the exponential networks, but they are negligible for the Erdös–Rényi networks and the Barabási –Albert scale-free networks as long as the mean degree is large enough. These results coincide with the former calculations of the clustering coefficient C [12], where the largest difference between theoretical and numerical results were found for the exponential networks. These results agree also with analytical calculations of other authors [14–16]. Similar numerical calculations for the growing networks were performed with the same results [17]. A simple explanation of the positive degree–degree correlations in the exponential networks could be that the node degree increases with its age, and the nodes most old are connected to each other.

The degree–degree correlations in the exponential networks allow to interpret also the results on the $\langle k'(k)\rangle$ dependence in the line graphs. As before, the theoretical calculations are performed with the assumption that the correlations are absent in the initial networks. We know that this assumption is not true for the exponential networks. As a result, the theoretical curves $\langle k'(k)\rangle$ for the line graphs formed from the exponential networks differ from the same curves obtained from the numerical simulations. On the contrary, the accordance is quite good for the Erdös–Rényi networks, where the degree correlations are absent. For the scale-free networks of finite size, theory gives a linear plot $\langle k'(k)\rangle$. The simulation for these networks gives a broad distribution of points, and therefore the accordance is only qualitative.

Summarizing, all the investigated line graphs are assortative. These degree–degree correlations can be understood as a consequence of the fact that the neighboring nodes in the line graphs are formed from links sharing a common node in the initial graph. The degree of this common node contributes to the degree of both neighboring nodes in the line graph.

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REFERENCES

[1] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA 1969.

[2] V.K. Balakrishnan, *Schaum's Outline of Graph Theory: Including Hundreds of Solved Problems*, McGraw-Hill, New York 1997.

[3] `http://en.wikipedia.org/wiki/`

[4] A.-L. Barabási, *Linked: The New Science of Networks*, Perseus Publishing, Cambridge, Massachusetts 2002.

[5] M. Morris, M. Kretzschmar, *Social Networks* **17**, 299 (1995).

[6] F. Liljeros, C.R. Edling, L.A. Nunes Amaral, *Microbes and Infection* **5**, 189 (2003).

[7] T.S. Evans, R. Lambiotte, *Phys. Rev.* **E80**, 016105 (2009).

[8] J.C. Nacher, N. Ueda, T. Yamada, M. Kanehisa, T. Akutsu, *BMC Bioinformatics* **24**, 207 (2004).

[9] J.C. Nacher, T. Yamada, S. Goto, M. Kanehisa, T. Akutsu, *Physica A* **349**, 349 (2005).

[10] S. Zhang, H.-W. Liu, X.-M. Ning, X.-S. Zhang, *Int. J. of Data Mining and Bioinformatics* **3**, 68 (2009).

[11] D. Ucar, S. Parthasarathy, S. Asur, C. Wang, Bioinformatics and Bioengineering, Proc. of the Fifth Symp. on Bioinformatics and Bioengineering, 2005, p. 129.

[12] A. Mańka-Krasoń, A. Mwijage, K. Kułakowski, *Comp. Phys. Commun.* **181**, 118 (2010).

[13] M.E.J. Newman, *Phys. Rev. Lett.* **103**, 058701 (2009).

[14] M.E.J. Newman *Phys. Rev. Lett.* **89**, 208701 (2002).

[15] D.S. Callaway, J.E. Hopcroft, J.M. Kleinberg, M.E.J. Newman, S.H. Strogatz, *Phys. Rev.* **E64**, 041902 (2001).

[16] Y. Qi, Z. Zhang, B. Ding, S. Zhou, J. Guan, *J. Phys. A: Math. Theor.* **42**, 165103 (2009).

[17] Jing-zhou Liu, Yi-fa Tang, *Chin. Phys.* **14**, 643 (2005).

[18] M. Boguná, R. Pastor-Satorras, A. Vespignani, *Eur. Phys. J.* **B38**, 205 (2004).