CRITICAL PHENOMENA IN CELLULAR AUTOMATA: PERTURBING THE UPDATE, THE TRANSITIONS, THE TOPOLOGY*

Nazim Fatès

INRIA Nancy — Grand Est, Loria, France

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We survey the effect of perturbing the regular structure of a cellular automaton (CA). We are interested in critical phenomena, *i.e.*, when a continuous variation in the local rules of a cellular automaton triggers a qualitative change of its global behaviour. We focus on three types of perturbations: (a) when the updating is made asynchronous, (b) when the transition rule is made stochastic, (c) when topological defects are introduced. It is shown that although these perturbations have various effects on CA models, they are generally identified as first-order or second-order phase transitions. We present open questions related to this topic and discuss some implications on the use of CA to model natural phenomena.

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1. Introduction

1.1. Position of the problem

The thesis of this paper is that cellular automata (CA) should not only be studied with various initial conditions but also with various perturbations to assess how they resist modifications of their structure. Why do we need to examine the effect of perturbations on cellular automata? When considering a cellular automaton, why should we make efforts to estimate its robustness to various perturbations? There are different reasons for asking these questions and it is convenient to answer them depending on the context in which we are using a cellular automaton. A useful dichotomy is to separate the CA models that mimic natural phenomena and those designed to study massively parallel computations.

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Firstly, if the CA model is meant to mimic a natural phenomenon, it is important to know which of the implicit hypotheses of the model are necessary to re-produce the phenomenon. By "implicit" hypotheses we mean the mathematical regularities which define the "structure" of CA, in opposition to the local rules, which vary from one model to another. For instance, let us consider an epidemic modelling with a simple contamination rule: a cell gets infected if one of its neighbours is infected. Clearly, the evolution of the "contamination pattern" depends on the type of neighbourhood chosen. It also strongly depends on the type of updating. How do we choose the right updating scheme? Since there is no clock in nature that synchronises transitions, we are free to choose the updating procedure as we wish. For some authors, this freedom casts some doubts on the relevance of using cellular automata as a modelling tool [11, 13]. On the other hand, if we observe this system from a higher viewpoint, we may also declare the behaviour robust since we always observe that the whole lattice is progressively contaminated. Is this a more appropriate definition of robustness? It all depends on how we observe the system. It is thus important to note that there is no unique way of defining robustness. This implies that our definition of robustness will strongly depend on the "level of granularity" we use to observe the system. Since CA are mainly used for developing qualitative predictions, testing the robustness to various perturbations is necessary to know which part of the "structure" is involved in the production of a given behaviour.

Secondly, we examine the case where CA are models of massively parallel computation. In this context, we are in the domain of mathematics and we do not ask whether or not a hypothesis is "realistic" since we may choose our hypotheses freely, as mathematicians do when they build new worlds. The question here might more be seen as a challenge: how can we build massively parallel computation tools that resist various perturbations? Can we imitate living organisms or natural societies to achieve this goal? Answering these questions would pave the way for important innovations. For example, it is well known that the existence of a central clock to synchronise components brings important limitations to the design of novel devices. How can we overcome this constraint? Can we build asynchronous or partially synchronous computing devices? (see *e.g.*, [16]). These issues illustrate some of the scientific motivations to explore CA robustness.

Although the idea of perturbing cellular automata is not new, it is surprising to see that only a few researchers have given their attention to it. To date, defining the robustness of a system is more of an art than a science; it mainly consists in finding the right balance between a too restrictive definition, which renders the system sensitive to any perturbation, and a too loose definition, which would result in considering the system as robust to any perturbation.

1.2. Critical phenomena

In this paper, we choose to tackle the robustness of CA models from the angle of critical phenomena. By *critical phenomena*, we mean all the cases where a continuous variation in the local rules of a cellular automaton triggers a *qualitative* change of its global behaviour. Why is this viewpoint interesting? Its main advantage is that it allows us to build a bridge between CA and statistical physics as the abrupt changes of behaviour can be studied as *phase transitions*. These phenomena follow well-structured scaling laws; in particular, near the critical point, the system's behaviour obeys power laws. The exponents of these laws are called the *critical exponents*. They are not arbitrary but are generally found in well-known classes. It has been observed that different models with very dissimilar local rules exhibit the same critical exponents, we say that they belong to the same *universality class*. For computer scientists, this other type of universality brings out an interesting method of classification, which, as we will see later, is largely uncorrelated to the traditional classifications. Grouping cellular automata according to their critical behaviour might give us new insights on how complex global behaviours may emerge from simple local interactions.

1.3. Methodology

To present the effect of various perturbations on different CA models, we may either proceed by grouping our analysis by types of models or by examining the effect of each perturbation on one model or by examining the effect of each perturbation separately. We adopt the latter choice and we divide our survey by considering perturbations one type after the other. This presentation is meant to emphasize that different models, which do not share much in common, sometimes react identically to a given perturbation. For the sake of brevity, we select a few models, mainly chosen from our own research intrest. Whenever possible, we broaden our point of view by linking our observations to wider questions on cellular automata.

The structure of the paper follows the three main properties found in the structure of "regular" CA: (a) synchronous updating, (b) determinism of the transitions, and (c) regular arrangement of cells. In Section 2, we survey the existing works which examine how the asynchronous updating may trigger phase transitions. We then examine in Section 3 the case where the local transition rule is made stochastic. The effect of topology perturbations is surveyed in Section 4. Each section is devoted to a perturbation type and in each section, we mainly concentrate on simple models such as the Game of Life, Elementary Cellular Automata or the Greenberg–Hastings model.

2. Asynchronous updating

2.1. The Game of Life

The asynchronous version of Game of Life, or simply `Life`, provides us with a sound introduction to the topic of robustness. Let us recall the rules which define the "game". A cell takes one of the two values: occupied or empty. The new state of a cell depends on its current state and on the sum σ of occupied sites among the 8-nearest neighbours. An empty cell becomes occupied if and only if $\sigma = 3$; an occupied cell remains occupied if and only if $\sigma \in \{2, 3\}$. The rule was invented by Conway in 1970, it has been shown to have a global behaviour rich enough to embed a universal Turing machine [2]. Now what happens to the system if the cells are no longer updated synchronously at each time step? There are various methods for defining asynchronous systems and we restrict our scope to the two most intuitive ways: (a) We call *α-asynchronous updating* the procedure which consists in updating a cell with probability α at each time step, otherwise keeping its state constant. We call α the *synchrony rate*. (b) We call *fully asynchronous updating* the procedure where n cells are updated sequentially and randomly in one step (n is the lattice size) — the cells are chosen uniformly and independently without any memory.

Fig. 1 displays three evolutions of `Life` with different updating schemes and cyclic boundary conditions. The asynchronous updating has overwhelmed the global behaviour in such a way that the system evolves into a steady state which consists of "labyrinth" patterns. This strange modification was first reported by Bersini and Detours [3] for the fully asynchronous case and was studied quantitatively by Blok and Bergersen for the α-asynchronous updating [1]. Their results indicate that the systems obey

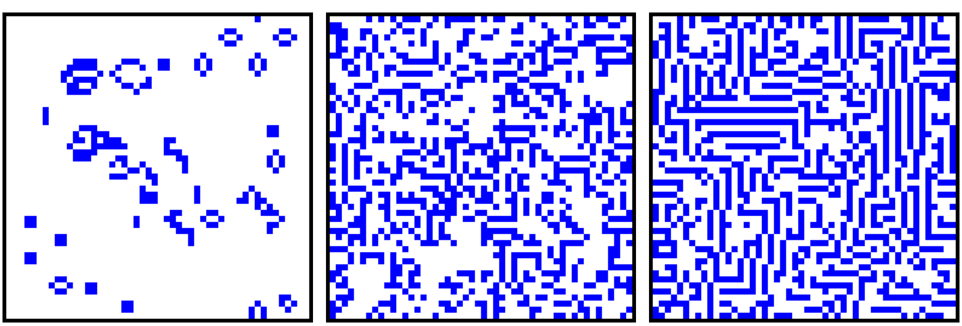


Fig. 1. Three snapshots showing evolutions of `Life` for a random initial configuration: (left) synchronous updating $t = 200$, (middle) α-asynchronous updating with $\alpha = 0.5$, $t = 400$, (right) fully asynchronous updating $t = 200$. Note that the three cases use the same number of updates on average.

phase transition with a critical threshold located at $\alpha \sim 0.910$. The transition between the two phases belongs to the *directed percolation* (DP) universality class (see [19]). Above the threshold, the system's behaviour is similar to the synchronous case, although the periodic stable structures disappear. Below this threshold, the "labyrinth" phase develops, its development on the grid increases as the synchrony rate α decreases. A first study on how labyrinth phase develops on the grid was proposed by Fatès and Morvan but a detailed description of the properties of this phase is still needed [8].

Many questions emerge for these observations. What is the effect of asynchronous updating on "complex" CA such as `Life`? Bersini and Detours suggested that it was a "stabilisation" of the system. In our view, it is not so clear whether the asynchronous system is more stable as this stability strongly depends on the system's size. For a small-sized system, *i.e.*, for a grid length smaller or equal to 20, it is indeed possible to observe the system reach a fixed point. However, as soon as the grid becomes large enough, for instance for $L = 30$, one no longer sees the system freeze on a fixed point, even for very long simulation times. So the state of the labyrinth phase is a meta-stable phase: it evolves rather erratically until by chance, it "touches" a fixed point. In mathematical terms we say that the system is a metastable state and it is an open problem to prove this property. More generally, how is the existence of a phase transition related to the Turing-universality of `Life`? How frequent asynchrony-induced phase transitions are in the space of CA? In the next paragraph, we start to gain insights on these issues by shifting our attention from a single rule to a small set of rules, namely the Elementary Cellular Automata (ECA).

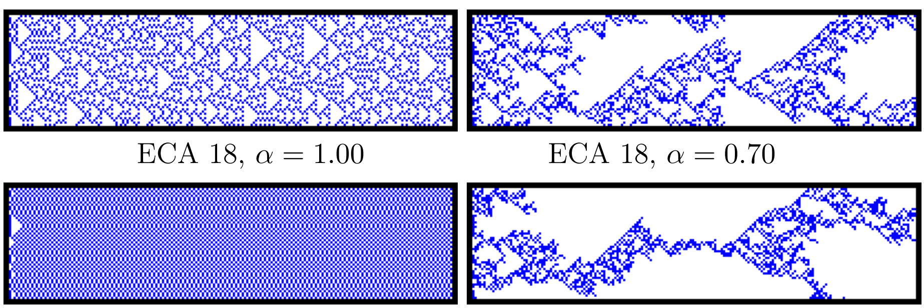
2.2. Elementary cellular automata

Ingerson and Buvel were among the first authors to raise the question about the importance of synchronous updating in the emergence of structures in cellular automata [13]. According to their own words,

> it is necessary to "*estimate how much of the interesting behaviour of cellular automata comes from synchronous modelling and how much is intrinsic to the iteration process*".

Although their study consisted mainly in describing qualitatively the effect of asynchronous updating, they proposed a first panorama of the effects observed on the 256 ECA rules (binary 1D nearest-neighbour rules). A quantitative assessment of the effect of asynchrony on ECA was conducted by Fatès and Morvan [9]. Results showed that quantifying the behaviour with the density produced a discrimination between the robust and non-robust rules. Four classes were proposed; depending on whether the perturbation of behaviour occurred for $\alpha = 1$ and for $\alpha < 1$. More specifically, a first set of rules was identified as having a non-trivial critical phenomenon. Seven of these rules were characterised as belonging to the DP universality class [4] and another rule, ECA 178, was identified in another universality class specific to the case where the states 0 and 1 are symmetrical [5].

The main outcome of the exploration of asynchronous ECA is that is no direct relationship between the "traditional" CA classifications as dynamical systems and the classifications which concern robustness. For example, ECA 18 and ECA 50 are respectively class-3 (chaotic) and class-2 cellular automata according to Wolfram's informal classification but they are both found to display DP behaviour (see Fig. 2). This implies that two dissimilar rules may exhibit a similar global behaviour when simulated near their critical threshold. Rouquier and Morvan also observed similar behaviour with coalescing automata, *i.e.*, asynchronous evolutions of CA with a unique source of randomness [20]. It is an open problem to determine which are the "ingredients" that makes a rule belong to the DP universality class.



ECA 18, $\alpha = 1.00$ ECA 18, $\alpha = 0.70$

ECA 50, $\alpha = 1.00$ ECA 50, $\alpha = 0.60$

Fig. 2. Four space-time diagrams showing the evolution of two ECA with (left) synchronous updating and (right) with α-asynchronous updating near the critical point (similar evolutions). Time goes from left to right, rings are made of 50 cells (see Ref. [5] for more details).

2.3. Other models

Concerning other two-dimensional binary CA, we point out the recent work by Fatès and Gerin [7] and by Regnault *et al.* [21] where the tools from the stochastic process theory are used to analyse the convergence properties of some well-selected rules — see references in these two papers for a wider scope. Apart from the two-state models mentioned so far, we must acknowledge that there are not many models whose robustness have been estimated with different types of asynchronous updating, and still less where critical phenomena have been observed.

An interesting exception lies in the debate set out by Huberman and Glance, who illustrated how in the spatial version of the Prisoner's Dilemma a change from synchronous to fully asynchronous updating turned the system to a different attractor where all players are defecting [11]. These results were later re-examined independently by Mukherji *et al.* [18] and by Grilo and Correia [10] who showed that the α-asynchronous updating leads to different attractors, and that the transition from one attractor to another could be abrupt. Recently, Saif and Gade discovered that the asynchronous version of the spatial Prisoner's dilemma triggers *first-order* phase transitions [22]. This behaviour is rather unusual as the phase transitions found in similar situations are generally second-order, *i.e.*, continuous. It would be interesting to explore whether other cases of discontinuous behaviour are present in other systems.

3. Perturbations of the transition rule

We now broaden our perspective by considering that the outcome of a local transition is no longer deterministic. This defines the *probabilistic CA* or *stochastic CA*, a class which contains the α-asynchronous CA (but not the fully asynchronous CA). An abundant literature is available on this topic. For the sake of conciseness, we only mention a few stochastic CA which are obtained as *perturbations* of a well-known deterministic model. For instance, Ising or Domany–Kinzel models will be out of consideration since they are "intrinsically" stochastic — the deterministic behaviour obtained with some settings are more degenerate cases than common cases.

3.1. The Game of Life

The idea to test how `Life` resists noise dates back to as early as 1978 with paper by Schulman and Seiden [23]. They examined how the introduction of a stochastic element in the local evolution rule would perturb the long term evolution of the system. They replaced the deterministic transitions $p_{k,s}$ — which are equal to 0 or 1 if a cell in state k with s living neighbours dies or lives — by $p_{k,s}(T)$ which are the probabilities to be in state 1 at the next iteration. They take $p_{k,s}(T) = (p_{k,s} + d.T)/(1 + T)$, where d is the current density and T is the stochastic parameter, named "temperature" by analogy with the physical parameter. Interestingly, this parameter is chosen such as not to influence the evolution towards increasing or decreasing the density — this property is also valid for the α parameter. One problem with this approach is that the calculus of the probabilities is not local: the density should first be computed using the state of the whole grid before the transitions are determined. Despite this difficulty, the authors performed first-order and improved mean-field analysis. Their findings can be summed up as thus: mean-field analysis fails to predict the evolution of the classical deterministic Life but succeeds with high-temperature models. More surprisingly, their analysis shows that a phase transition exists which would separate two regimes where the system is attracted to the null density and to an attractor of density 0.37. However, no characterisation of the phase transition was made. It is very likely that the effect of the "temperature" is similar to that of the asynchrony.

3.2. Other stochastic CA

The existence of phase transitions were also observed by Kaneko and Akutsu for 4-neighbour-outer-totalistic cellular automata with the 0/1 symmetry [15]. Noting that "noise plays an important role for the formation of patterns in non-equilibrium systems", their perturbation consisted in flipping cells randomly. They observed the existence of first-order transitions and classified the various types of patterns observed (labyrinth, turbulence, glassy, ferro, anti-ferro, *etc.*). A quantitative study of these phenomena still remains to be performed.

4. Adding topology perturbations

Let us now complete our panorama of the perturbations by considering topological modifications. Here again there are so many ways of modifying the topology that it is difficult to be exhaustive. We choose to mainly focus on the cases where locality of the connections between cells is conserved. We point out that generalising the CA model of computation to any topology is a rather novel field, interested readers may refer to the work of Marr and Hürr for recent developments [17].

4.1. The Game of Life

In all the preceding situations, phase transitions appeared when the local rule was modified. Is it possible to obtain similar effects by changing only the links between cells? Huang *et al.* answered the question positively in a study where they perturbed the Game of Life by (arbitrarily) rewiring links between cells [12]. They found similar effects to those of asynchrony, *i.e.*, that `Life` exhibits a phase transition from an "inactive-sparse" phase to an "active-dense" phase. It was surprising that this phase transition belongs to the DP universality class in spite of the use of non-local links (see the hypotheses of the Janssen–Grassberger's conjecture [14] on the occurrence of DP, *e.g.*, in the reviews by Hinrichsen [19]).

What happens if asynchronous updating and topology perturbations are added? This combination was examined by Fatès and Morvan [8] by removing links randomly and independently in the neighbourhood of the cells.

They observed that the critical synchrony rate was lowered by the topological perturbation and that if more than 10% of the links were destroyed, no phase transition was observed. Figure 3 illustrates this progressive modification of the phase transition. A first explanation was proposed for this phenomenon: as some links were missing, it became more difficult for the labyrinth (or active-dense) phase to spread. However, things are not as trivial as they appear. We may note that in the `Life` CA, a condition for the survival of a cell is to avoid overcrowding, *i.e.*, to have less than four occupied neighbours. The removal of links should thus not only favour death of cells but also their survival. The question remains today as to how asynchrony-induced phase transitions are affected by topology perturbations. In particular, what would happen if links were added or simply rewired? Would the effects be similar?

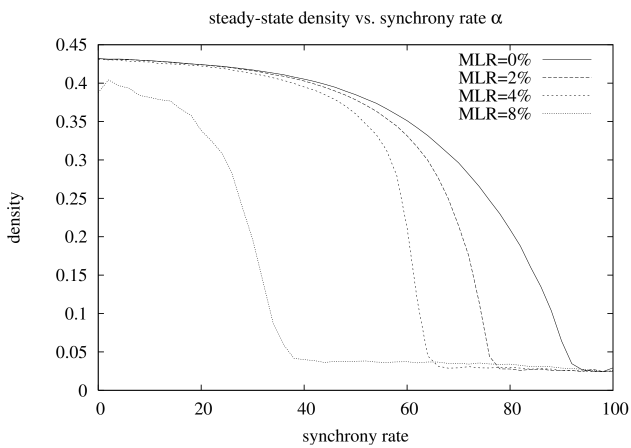


Fig. 3. Modification of the phase transition in `Life`. MLR (missing link rate) is the probability that a cell does not "see" one of its neighbour (the modification is definitive). Grid size is $L = 100$; the steady-state is approximated with a transient time of 5000 steps and an average computed on the next 1000 steps (see [8]).

4.2. The Greenberg–Hastings model

Finally, how do the topological perturbations combine with the stochastic perturbations of a CA rule? The subject is once again very wide and could be tackled in various ways. To close our tour of critical phenomena, we report on a recent study of the stochastic Greenberg–Hastings model [6]. The local rule is as follows: a cell can be either neutral, excited or refractory. A neutral cell stays neutral unless it has at least one excited neighbour, it then becomes excited with a given probability (the *excitability*). An excited cell becomes refractory in one step, and then it stays refractory during M steps until it becomes again neutral.

For the regular grid, varying the excitability triggers a second-order phase transition from an "active" phase, where the waves survive an arbitrary long time, and an "extinct" phase where the waves disappear. With no surprise, the phase transition was found in the DP universality class for various neighbourhoods tested. More interesting was the analysis of the position of the critical threshold. The simulation results showed that as links were removed, the critical excitability was progressively increased and became more difficult to measure ("blurring effect"). However, contrary to `Life`, the limit here was attained for about 40% of topological defects. These unexpected phenomena were interpreted as a result of the combination of the effects of classical percolation and directed percolation. Although the prediction of the position of the critical threshold is a difficult problem in general, we could verify that this position varied as the inverse of the neighbourhood size. It is an open question to determine whether this law applies for other systems.

5. Discussion

We presented a brief (and partial) survey on critical phenomena in cellular automata. We aimed to illustrate how cellular automata, despite their simplicity of definition, may exhibit unexpected changes of behaviours when their structure is only slightly perturbed. We focused on three perturbations: from synchronous to asynchronous updating, from deterministic to stochastic transitions, from regular to irregular grids.

Each of these perturbations could trigger first-order or second-order transitions, and when combined, the perturbations could lead to strong modifications of the phase transitions. In our view, keeping in mind these observations is primordial for a better use of cellular automata in the modelling field. As the science of complex systems is in an early stage, we should be careful when using CA models to mimic real world phenomena. It has been a long-lasting debate to know whether CA are a too "simplistic" modelling tool and we believe that a careful assessment of the robustness of CA may plead in favour of their use in a modelling context. Our review aimed at showing that the use of CA as modelling tools often has *limits*. When submitted to structural perturbations, their behaviour may be robust, or it might be immediately overwhelmed, or it might resist only up to a certain extent of perturbations. As we see it, gaining insights on critical phenomena in CA now requires a wide "cartography" project to explore a large panel of models from the robustness viewpoint.

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