LOWEST NEGATIVE PARITY BARYONS IN THE $1/N_c$ EXPANSION*

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We review a recently proposed approach to study the lowest negative parity baryons within the $1/N_c$ expansion. The method is based on the derivation of the matrix elements of SU($2N_f$) generators for mixed symmetric $[N_c-1,1]$ flavor-spin states. Presently it is applied to the $N=1$ band and a comparison is made with a former method based on the decoupling of the system into a symmetric core of N_c-1 quarks and an excited one. We prove that the decoupling is not necessary and moreover, it misses some important physical consequences.

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1. Introduction

In the energy regime of the hadron spectroscopy, QCD does not admit a perturbative expansion in terms of the coupling constant of the theory. An interesting solution to the problem is to generalize QCD to N_c colors, $1/N_c$ becoming the expansion parameter of the theory [1, 2]. In the large N_c limit, QCD possesses an exact contracted SU$(2N_f)_c$ symmetry [3, 4]. For large but finite N_c, the contracted SU$(2N_f)_c$ becomes the SU($2N_f$) symmetry group of the constituent quark model. The ground states baryons are degenerate in the $N_c \to \infty$ limit and the mass splitting starts at order $1/N_c$ [5, 6].

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Following the quark model classification of excited baryons, it is generally assumed that in the large N_c limit, the $\mathrm{SU}(2N_f)\times\mathrm{O}(3)$ symmetry is satisfied. For the $N = 1$ band, at large but finite N_c, it has been shown that this symmetry is broken at order N_c^0 by several operators, in particular by the spin-orbit operator, so that the mass splitting starts at order N_c^0.

This review is devoted to the description of the lowest negative parity baryons, belonging to the $[\mathbf{70}, 1^-]$ multiplet (the $N = 1$ band). The wave function of these baryons is mixed symmetric in the orbital and the spin-flavor parts. As a consequence, the calculation of the spectrum becomes more complicated than for symmetric states. During the last four years, this became a controversial subject. The traditional procedure was based on a Hartree approximation, the baryon being described by an orbitally excited ($\ell = 1$) quark moving in the collective potential generated by a ground state core composed of $N_c - 1$ quarks. Accordingly, only the $s^{N_c-1}p$ configuration was considered in the orbital part. The other components of the wave function needed to obtain a mixed symmetric orbital and spin-flavor states were neglected [7]. Accordingly, the generators of $\mathrm{SU}(2N_f)$ were written as the sum of two terms, one acting on the core and the other on the excited quark. In this way the number of linearly independent terms in the mass formula becomes tremendously large which creates difficulties in choosing the dominant ones.

One can go beyond this approximation and treat the wave function of the baryons exactly in order to avoid the decoupling of the system into a core and an excited quark [8]. In this case the wave function has the correct symmetry but it requires the knowledge of the matrix elements of the spin-flavor generators for the mixed symmetric states. These matrix elements have been derived for SU(4) in the sixties [9] but only recently for SU(6) [10].

For a comparison, one can include the neglected terms of the wave function in the traditional decoupling picture, but others group theoretical difficulties appear related to the knowledge of isoscalar factors of the permutation group S_{N_c}. This problem has already been solved for nonstrange baryons [11]. The strange baryons are underway and will be presented in a future publication [12].

In the following we shall briefly describe the two procedures.

2. The wave function

To satisfy the Pauli principle, the wave function describing a baryon must be antisymmetric. This wave function is usually the product of its orbital (O), spin (S), flavor (F) and color (C) parts. As in nature the baryons are colorless, the color part is always antisymmetric, the orbital-spin-flavor is then symmetric. As the color plays no role in the mass analysis, one integrates it out and considers only the orbital-spin-flavor part only.

Here we are interested in the multiplet $[\mathbf{70}, 1^-]$. Its orbital and spin-flavor parts must have both a mixed symmetry which for arbitrary N_c has the partition $[N_c-1, 1]$. In terms of inner products of the permutation group S_{N_c}, the wave function takes the form

$$|[N_c]1\rangle = \frac{1}{\sqrt{N_c-1}} \sum_Y |[N_c-1,1]Y\rangle_{\mathrm{O}} |[N_c-1,1]Y\rangle_{\mathrm{FS}}\,, \tag{1}$$

where Y is the corresponding Young tableau. Here we sum over the N_c-1 possible standard Young tableaux. The factor $1/\sqrt{N_c-1}$ represents the Clebsch–Gordan coefficient (CG) of S_{N_c} needed to construct a symmetric wave function $[N_c]$ from its mixed symmetric parts.

Let us first introduce the global wave function, without the decoupling of the excited quark from the ground state core. In that case, as the matrix elements of the operators are identical for all Y due to Weyl's duality between a linear group and a symmetric group in a given tensor space[1], one does not need to specify Y. Then the explicit form of a wave function of total angular momentum $\vec{J} = \vec{\ell} + \vec{S}$ and isospin I is

$$\begin{aligned} &|\ell S J J_3; I I_3\rangle = \\ &\sum_{m_\ell, m_s} \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & m_s & J_3 \end{array} \right) |[N_c-1,1]\ell m_\ell\rangle |\,[N_c-1,1] S m_s I I_3\rangle, \end{aligned} \tag{2}$$

each term containing an SU(2) CG coefficient, an orbital part $|[N_c-1,1]\ell m_\ell\rangle$ an a spin-flavor part $|[N_c-1,1]Sm_sII_3\rangle$.

The exact but decoupled wave function reads [11]

$$\begin{aligned} &|\ell S J J_3; I I_3\rangle = \\ &\sum_{\substack{p,p',p'',\ell_c,\ell_q,m_\ell,m_q,\\ m_c,m_s,m_1,m_2,i_1,i_2}} a(p,\ell_c,\ell_q) \left(\begin{array}{cc|c} \ell_c & \ell_q & \ell \\ m_c & m_q & m_\ell \end{array} \right) \left(\begin{array}{cc|c} \ell & S & J \\ m_\ell & m_s & J_3 \end{array} \right) \\ &\times K([f']p'[f'']p''|[N_c-1,1]p) \left(\begin{array}{cc|c} S_c & \frac{1}{2} & S \\ m_1 & m_2 & m_s \end{array} \right) \left(\begin{array}{cc|c} I_c & \frac{1}{2} & I \\ i_1 & i_2 & I_3 \end{array} \right) \\ &\times |\ell_c m_c\rangle |S_c m_1\rangle |I_c i_1\rangle |\ell_q m_q\rangle |1/2 m_2\rangle |1/2 i_2\rangle, \end{aligned} \tag{3}$$

where ℓ_c and ℓ_q represent the angular momenta of the core and of the decoupled quark respectively and where $a(p,\ell_c,\ell_q)$ are the one-body fractional parentage coefficients to decouple the N_cth quark from the rest in the orbital part. These are given by [11]:

[1] See Ref. [13], Sec 4.5.

$$a(2,\ell_c=0,\ell_q=1) \;=\; \sqrt{\frac{N_c-1}{N_c}}\,, \tag{4}$$

$$a(2,\ell_c=1,\ell_q=0) \;=\; -\sqrt{\frac{1}{N_c}}\,, \tag{5}$$

$$a(1,\ell_c=1,\ell_q=0) \;=\; 1\,. \tag{6}$$

The isoscalar factors $K([f']p'[f'']p''|[N_c-1,1]p)$ used in Eq. (3) are also given in [11].

3. The mass operator

The mass operator M is defined as a linear combination of independent operators O_i

$$M=\sum_i c_i O_i\,, \tag{7}$$

where the coefficients c_i are reduced matrix elements that encode the QCD dynamics and are determined from a fit to the existing experimental data. The building blocks of the operators O_i are the SU($2N_f$) generators S_i, T_a and G_{ia} and the SO(3) generators ℓ_i. Their general form is

$$O_i=\frac{1}{N_c^{n-1}}\,O_\ell^{(k)}\cdot O_{\rm SF}^{(k)}\,, \tag{8}$$

where $O_\ell^{(k)}$ is a k-rank tensor in SO(3) and $O_{\rm SF}^{(k)}$ a k-rank tensor in SU(2)-spin, but invariant in SU(N_f). O_i, with i a numbering index, is then rotational invariant. For the ground state one has $k=0$. The excited states also require $k=1$ and $k=2$ terms.

4. Results

4.1. The exact wave function

In Table I we show results obtained with the exact wave function (2). The first 6 operators are linearly independent. In SU(3), O_7 is also linearly independent, but not in SU(4). In the latter case it can be written as a linear combination of O_1, O_3 and O_4 [14]

$$O_7=-\frac{3(4N_c-9)}{16N_c^3}\;O_1+\frac{3(N_c-1)}{8N_c}\;(O_3+O_4)\,. \tag{9}$$

We did however include it in Table I but made two distinct fits, either with O_3 and O_4 as independent operators (Fit 1 and Fit 2) or with O_7 alone (Fit 3), ignoring O_3 and O_4, as they are contained in O_7.

TABLE I

List of operators contributing to the mass (7) up to order $1/N_c$ included, and their coefficients resulting from numerical fits using the global wave function (2). The values of c_i are indicated under the headings Fit n, in each case.

Operator	Fit 1 [MeV]	Fit 2 [MeV]	Fit 3 [MeV]
$O_1 = N_c\mathbb{1}$	481 ± 5	484 ± 4	495 ± 3
$O_2 = \ell^i s^i$	-31 ± 26	3 ± 15	-30 ± 25
$O_3 = 1/N_c\ S^i S^i$	161 ± 16	150 ± 11	—
$O_4 = 1/N_c\ T^a T^a$	169 ± 36	139 ± 27	—
$O_5 = 15/N_c\ \ell^{(2)ij} G^{ia} G^{ja}$	-29 ± 31	—	-32 ± 29
$O_6 = 3/N_c\ \ell^i T^a G^{ia}$	32 ± 26	—	28 ± 20
$O_7 = 3/N_c^2\ S^i T^a G^{ia}$	—	—	649 ± 61
$\chi^2_{\rm dof}$	0.43	1.04	0.24

It turns out that $\chi^2_{\rm dof}$ is the best in the latter case. However, the fit with independently varying coefficient for O_3 and O_4 is interesting for physical reasons. Namely, in Table II, one can see that the isospin term O_4 becomes as dominant in Δ as the spin term O_3 in ${}^4N$ resonances, bringing in each case a contribution of about 200 MeV to the mass. The fit of Table III, with O_7 alone, shows the same pattern, with a similar contribution. This gives more physical insight into O_7 defined in Eq. (9).

TABLE II

The partial contribution and the total mass M (MeV), predicted by the $1/N_c$ expansion, Eq. (7), using Fit 1 and the global wave function (2). The last two columns give the empirically known resonance masses, name and status.

	Part. contrib. [MeV]						Total [MeV]	Exp. [MeV]	Name, status
	c_1O_1	c_2O_2	c_3O_3	c_4O_4	c_5O_5	c_6O_6			
${}^2N_{\frac{1}{2}}$	1444	10	40	42	0	−8	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
${}^4N_{\frac{1}{2}}$	1444	26	201	42	−31	−20	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
${}^2N_{\frac{3}{2}}$	1444	−5	40	42	0	4	1525 ± 8	1523 ± 8	$D_{13}(1520)$****
${}^4N_{\frac{3}{2}}$	1444	10	201	42	25	−8	1714 ± 45	1700 ± 50	$D_{13}(1700)$***
${}^4N_{\frac{5}{2}}$	1444	−16	201	42	−6	12	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
${}^2\Delta_{\frac{1}{2}}$	1444	−10	40	211	0	−40	1645 ± 30	1645 ± 30	$S_{31}(1620)$****
${}^2\Delta_{\frac{3}{2}}$	1444	5	40	211	0	20	1720 ± 50	1720 ± 50	$D_{33}(1700)$****

TABLE III

The same as Table II but for Fit 3 of Table I.

	Part. contrib. [MeV]					Total [MeV]	Exp. [MeV]	Name, status
	c_1O_1	c_2O_2	c_5O_5	c_6O_6	c_7O_7			
$^2N_{\frac{1}{2}}$	1486	10	0	−7	41	1529 ± 11	1538 ± 18	$S_{11}(1535)$****
$^4N_{\frac{1}{2}}$	1486	25	−33	−18	203	1663 ± 20	1660 ± 20	$S_{11}(1650)$****
$^2N_{\frac{3}{2}}$	1486	−5	0	4	41	1525 ± 7	1523 ± 8	$D_{13}(1520)$****
$^4N_{\frac{3}{2}}$	1486	10	26	−7	203	1718 ± 41	1700 ± 50	$D_{13}(1700)$***
$^4N_{\frac{5}{2}}$	1486	−15	7	11	203	1677 ± 8	1678 ± 8	$D_{15}(1675)$****
$^2\Delta_{\frac{1}{2}}$	1486	−10	0	−35	203	1643 ± 29	1645 ± 30	$S_{31}(1620)$****
$^2\Delta_{\frac{3}{2}}$	1486	5	0	18	203	1711 ± 24	1720 ± 50	$D_{33}(1700)$****

4.2. Discussion of the decoupling scheme

Here we examine the differences in the results obtained, on the one hand, with the exact wave function (2) and, on the other hand, with the approximate wave function of Ref. [7]. The latter corresponds to the term with $p = 2$ in Eq. (3). In the large N_c limit the amplitude (5) can be neglected and (4) can safely be taken equal to 1.

In the decoupling scheme one has

$$S^i = s^i + S^i_c\,, \qquad T^a = t^a + T^a_c\,, \qquad G^{ia} = g^{ia} + G^{ia}_c\,, \tag{10}$$

where the operators with a lower index c act on the core and the lower case operators act on the decoupled quark. Then in the SU(2)-spin Casimir operator $S^iS^i = S^i_cS^i_c + 2s^iS^i_c + s^is^i$ the first two terms are considered linearly independent. A similar treatment is made for T^aT^a. In addition, because the matrix elements of $S^i_cS^i_c$ and $T^a_cT^a_c$ are identical for SU(4) symmetric spin-flavor states, $T^a_cT^a_c$ is ignored. The s^is^i and t^at^a are constants which can be also be ignored. The linearly independent operators remain $S^i_cS^i_c$, $s^iS^i_c$ and $t^aT^a_c$. In Table IV we show a fit which includes these operators plus O_1 and O_2. One can see that the fit is poor and that the coefficients c_3, c_4 and c_5 (the latter for the approximate wave function) obtain abnormally large values with alternating signs. This is a clear indication that the linearly independence assumption is not correct.

In Ref. [8] we have considered the effect of S^iS^i instead of its separate parts. The fit is much better, the $\chi^2_{\rm dof}$ lowers to 1.04 and the values of the corresponding c_i coefficients have natural sizes, as defined, for example, in Ref. [15]. A similar behaviour is observed in the analysis of the isospin

TABLE IV

List of operators O_i and their coefficients c_i obtained in the numerical fit, Ref. [8], to the 7 known experimental masses of the lowest negative parity resonances.

O_i	c_i [MeV] with approx. w.f.	c_i [MeV] with w.f. Eq. (3)
$O_1 = N_c \mathbb{1}$	211 ± 23	299 ± 20
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O_3 = 1/N_c \; s^i S_c^i$	-1486 ± 141	-1096 ± 125
$O_4 = 1/N_c \; S_c^i S_c^i$	1182 ± 74	1545 ± 122
$O_5 = 1/N_c \; t^a T_c^a$	-1508 ± 149	417 ± 79
$\chi^2_{\rm dof}$	1.56	1.56

operators. Table V shows the result, which implies that the coefficients are identical irrespective of the wave function, the exact (2), or the approximate one from Ref. [7]. This is natural, because these operators are symmetric under any interchange of particles and it does not matter if the spin-flavor state is fully symmetrized or not.

TABLE V

Fit with the SU(2)-spin and SU(2)-isospin Casimir operators acting on the whole system.

O_i	c_i [MeV] with approx. w.f.	c_i [MeV] with w.f. Eq. (3)
$O_1 = N_c \mathbb{1}$	484 ± 4	484 ± 4
$O_2 = \ell^i s^i$	3 ± 15	3 ± 15
$O_3' = \frac{1}{N_c}\left(2 s^i S_c^i + S_c^i S_c^i + \frac{3}{4}\right)$	150 ± 11	150 ± 11
$O_5' = \frac{1}{N_c}\left(2 t^a T_c^a + T_c^a T_c^a + \frac{3}{4}\right)$	139 ± 27	139 ± 27
$\chi^2_{\rm dof}$	1.04	1.04

The above study helped us to better understand the previous choice of operators in the scheme based on the separation of the system into a symmetric core of $N_c - 1$ quarks and an excited quark. There, as mentioned above, the SU(2)-isospin Casimir operator was written as $T^2 = T_c^2 + 2t\,T_c + 3/4$ and decomposed into three independent pieces, corresponding to the terms in the above decomposition. As mentioned, in SU(4) T_c^2 and S_c^2 have identical matrix elements because the spin and isospin states of a symmetric core are identical, so that T_c^2 was ignored. But $t\,T_c$ has different matrix elements from $s\,S_c$ as one can clearly see from Table II of Ref. [7]. Then in the decoupling scheme the isospin can be introduced only through $t\,T_c$. In Table IV we have shown that the introduction of the operators $\frac{1}{N_c} t\,T_c$ together

with $\frac{1}{N_c}S_c^2$ and $\frac{1}{N_c}s\,S_c$ separately deteriorates the fit. This may explain why $\frac{1}{N_c}t\,T_c$ has been avoided in previous numerical fits both in SU(4) [7] and in SU(6) [16].

5. Conclusions

We plan to extend our study to SU(6), to look for differences between results obtained with the exact wave function on the one hand, and the approximate one, used in the decoupling scheme, on the other hand. The analytic procedure is described in Ref. [12]. The SU(3)-flavor Casimir operator is expected to bring new information with respect to SU(2)-isospin. This will complement another future work on the spectrum of both non-strange and strange baryons in the $1/N_c$ expansion, based on the use of the matrix elements of SU(6) generators derived in Ref. [10], needed for baryons in mixed symmetric states spin-flavor $[N_c-1,1]$.

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