QED THERMODYNAMICS AT INTERMEDIATE COUPLING*

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We discuss reorganizing finite temperature perturbation theory using hard-thermal-loop (HTL) perturbation theory in order to improve the convergence of successive perturbative approximations to the free energy of a gauge theory. We briefly review the history of the technique and present new results for the three-loop HTL-improved approximation for the free energy of QED. We show that the hard-thermal-loop perturbation reorganization improves the convergence of the successive approximations to the QED free energy at intermediate coupling, $e \sim 2$. The reorganization is gauge invariant by construction, and due to cancellation among various contributions, one can obtain a completely analytic result for the resummed thermodynamic potential at three loops.

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1. Introduction

In the early 1990s the free energy of a massless scalar field theory was calculated to order g^4 in Refs. [1, 2]. This was quickly followed by similar calculations in QED [3] and QCD [2]. The scalar, QED, and QCD free energies to order g^5 were then obtained in Refs. [4, 5], Refs. [6, 7] and Refs. [8,9], respectively. Recent results have extended the calculation of the

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QCD free energy by determining the coefficient of the $g^6 \log(g)$ contribution [10]. For massless scalar ϕ^4 the perturbative free energy is now known to order g^6 [11] and $g^8 \log(g)$ [12].

However, the resulting weak-coupling approximations, truncated order-by-order in the coupling constant, are poorly convergent unless the coupling constant is extremely small. For example, simply comparing the magnitude of low-order contributions to the $N_f = 3$ QCD free energy one finds that the g_s^3 contribution is smaller than the g_s^2 contribution only for $g_s \lesssim 0.9$ ($\alpha_s \lesssim 0.07$). This is a troubling situation since at phenomenologically accessible temperatures near the critical temperature for the QCD deconfinement phase transition, the strong coupling constant is on the order of $g_s \sim 2$.

The poor convergence of finite-temperature perturbative expansions of the free energy is not limited to QCD. The same behavior can be seen in weak-coupling expansions in scalar field theory [13, 14] and QED [6]. In Fig. 1 we show the successive perturbative approximations to the QED free energy. As can be seen from this figure, at couplings larger than $e \sim 1$ the QED weak-coupling approximations also exhibit poor convergence. For this reason a concerted effort has been put forth to find a reorganization of finite-temperature perturbation theory which converges at phenomenologically relevant couplings.

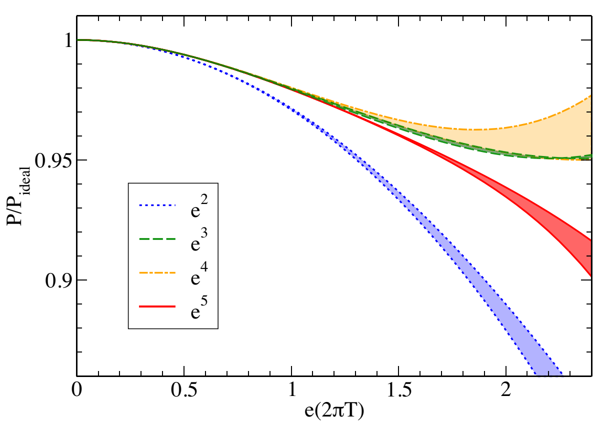

Fig. 1. Successive perturbative approximations to the QED pressure (negative of the free energy). Each band corresponds to a truncated weak-coupling expansion accurate to order e^2, e^3, e^4, and e^5, respectively. Shaded bands correspond to variation of the renormalization scale μ between πT and $4\pi T$.

There are several ways of systematically reorganizing the perturbative expansion to improve its convergence and the various approaches have been reviewed in Refs. [15–17]. Here we will describe recent advances in the application of hard-thermal-loop perturbation theory (HTLpt) [18–22]. The HTLpt method is inspired by variational perturbation theory [23–26] and is a gauge-invariant extension of scalar screened perturbation theory (SPT) [13, 14, 27, 28]. The basic idea of the technique is to add and subtract an effective mass term from the bare Lagrangian and to associate the added piece with the free Lagrangian and the subtracted piece with the interactions. However, in gauge theories, one cannot simply add and subtract a local mass term since this would violate gauge invariance. Instead one adds and subtracts a HTL improvement term which modifies the propagators and vertices in such a way that the framework is manifestly gauge-invariant. The free part of the Lagrangian then includes the HTL self-energies and the remaining terms are treated as perturbations.

In this brief proceedings review we present results of a calculations of the QED free energy (pressure) to three-loop order in HTLpt based on the work detailed in Ref. [29]. As we will show, the next-to-leading order (NLO) and next-to-next-to-leading order (NNLO) HTLpt resummed QED free energy give approximations which show improved convergence for couplings as large as $e \sim 2.5$ (see Fig. 2). In addition, we compare our results to those obtained using the 2PI Φ-derivable approach [30] and show that at three loops the agreement between the HTLpt and Φ-derivable approaches is quite good.

2. Formalism

The Lagrangian density for massless QED in Minkowski space is

$$\mathcal{L}_{\rm QED} \;=\; -\tfrac{1}{4}\, F_{\mu\nu}F^{\mu\nu} + i\bar\psi\gamma^\mu D_\mu\psi + \mathcal{L}_{\rm gf} + \mathcal{L}_{\rm gh} + \Delta\mathcal{L}_{\rm QED}\,.$$

Here the field strength is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and the covariant derivative is $D^\mu = \partial^\mu + ieA^\mu$. The ghost term $\mathcal{L}_{\rm gh}$ depends on the gauge-fixing term $\mathcal{L}_{\rm gf}$. We will use dimensional regularization with a renormalization scale μ and covariant gauge fixing such that the ghost terms decouple.

Hard-thermal-loop perturbation theory is a reorganization of the perturbation series for thermal gauge theories. In the case of QED, the Lagrangian density is written as

$$\mathcal{L} = \left(\mathcal{L}_{\rm QED} + \mathcal{L}_{\rm HTL}\right)\Big|_{e\to\sqrt{\delta}e} + \Delta\mathcal{L}_{\rm HTL}\,. \tag{2.1}$$

The HTL improvement term is

$$\begin{aligned}\mathcal{L}_{\rm HTL} \;=\; & -\tfrac{1}{2}\,(1-\delta)m_{\rm D}^2 F_{\mu\alpha}\left\langle \frac{y^\alpha y^\beta}{(y\cdot\partial)^2}\right\rangle_y F^\mu_{\;\beta} \\ & +(1-\delta)\, i m_{\rm f}^2\bar\psi\gamma^\mu\left\langle\frac{y^\mu}{y\cdot D}\right\rangle_y\psi\,,\end{aligned} \tag{2.2}$$

where $y^\mu = (1, \hat{\boldsymbol{y}})$ is a light-like four-vector, and $\langle \ldots \rangle_y$ represents an average over the directions of $\hat{\boldsymbol{y}}$. The term (2.2) has the form of the effective Lagrangian that would be induced by a rotationally-invariant ensemble of charged sources with infinitely high momentum. The parameter $m_{\rm D}$ can be identified with the Debye screening mass and the parameter $m_{\rm f}$ can be identified as the induced finite-temperature electron mass. HTLpt is defined by treating δ as a formal expansion parameter and expanding order by order in δ around $\delta = 1$. This generates loops with fully dressed propagators and vertices and also automatically generates the counterterms necessary to remove the dressing as one proceeds to higher loop orders.

If the expansion in δ could be calculated to all orders, the final result would not depend on $m_{\rm D}$ or $m_{\rm f}$. However, any truncation of the expansion in δ produces results that depend on $m_{\rm D}$ and $m_{\rm f}$. Some prescription is required to determine $m_{\rm D}$ and $m_{\rm f}$ as a function of T and e. For example, one can choose to treat both as variational parameters that should be determined by minimizing the free energy or one can fix $m_{\rm D}$ and $m_{\rm f}$ using a perturbative prescription. We will compare both methods. We will obtain the thermodynamic potential $\Omega(T, e, m_{\rm D}, m_{\rm f}, \mu, \delta = 1)$ which is a function of the mass parameters $m_{\rm D}$ and $m_{\rm f}$. The free energy $\mathcal{F}$ is obtained by evaluating the thermodynamic potential at the appropriate values of the thermal masses. Other thermodynamic functions can then be obtained by taking appropriate derivatives of $\mathcal{F}$ with respect to T.

3. Results

In this section we present the final renormalized thermodynamic potential explicitly through order δ^2, also known as NNLO, as obtained by us in Ref. [29]. The final NNLO expression is completely analytic; however, there are some numerically determined constants which remain in the final expressions at NLO.

3.1. Next-to-leading order

The renormalized NLO thermodynamic potential is

$$\begin{aligned}
\Omega_{\rm NLO} = -\frac{\pi^2 T^4}{45}\Bigg\{ & 1 + \frac{7}{4}N_f - 15\hat{m}_{\rm D}^3 - \frac{45}{4}\left(\log\frac{\hat{\mu}}{2} - \frac{7}{2} + \gamma + \frac{\pi^2}{3}\right)\hat{m}_{\rm D}^4 \\
& + 60N_f\left(\pi^2 - 6\right)\hat{m}_{\rm f}^4 + N_f\frac{\alpha}{\pi}\bigg[-\frac{25}{8} + 15\hat{m}_{\rm D} + 5\left(\log\frac{\hat{\mu}}{2} - 2.33452\right)\hat{m}_{\rm D}^2 \\
& -45\left(\log\frac{\hat{\mu}}{2} + 2.19581\right)\hat{m}_{\rm f}^2 - 30\left(\log\frac{\hat{\mu}}{2} - \frac{1}{2} + \gamma + 2\log 2\right)\hat{m}_{\rm D}^3 + 180\hat{m}_{\rm D}\hat{m}_{\rm f}^2\bigg]\Bigg\}, \qquad (3.1)
\end{aligned}$$

where we have introduced the dimensionless parameters $\hat{m}_{\rm D} = m_{\rm D}/(2\pi T)$, $\hat{m}_{\rm f} = m_{\rm f}/(2\pi T)$, and $\hat{\mu} = \mu/(2\pi T)$.

3.2. Next-to-next-to-leading order

The resulting NNLO thermodynamic potential is

$$\begin{aligned}\Omega_{\text{NNLO}} = -\frac{\pi^2 T^4}{45}\Bigg\{&1+\frac{7}{4}N_f-\frac{15}{4}\,\hat{m}_{\text{D}}^3\\
&+N_f\frac{\alpha}{\pi}\left[-\frac{25}{8}+\frac{15}{2}\hat{m}_{\text{D}}+15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_{\text{D}}^3-90\hat{m}_{\text{D}}\hat{m}_{\text{f}}^2\right]\\
&+N_f\left(\frac{\alpha}{\pi}\right)^2\left[\frac{15}{64}(35-32\log 2)-\frac{45}{2}\hat{m}_{\text{D}}\right]\\
&+N_f^2\left(\frac{\alpha}{\pi}\right)^2\Bigg[\frac{25}{12}\left(\log\frac{\hat{\mu}}{2}+\frac{1}{20}+\frac{3}{5}\gamma-\frac{66}{25}\log 2+\frac{4}{5}\frac{\zeta'(-1)}{\zeta(-1)}-\frac{2}{5}\frac{\zeta'(-3)}{\zeta(-3)}\right)\\
&+\frac{5}{4}\frac{1}{\hat{m}_{\text{D}}}-15\left(\log\frac{\hat{\mu}}{2}-\frac{1}{2}+\gamma+2\log 2\right)\hat{m}_{\text{D}}+30\,\frac{\hat{m}_{\text{f}}^2}{\hat{m}_{\text{D}}}\Bigg]\Bigg\}\,. \qquad (3.2)\end{aligned}$$

3.3. Free energy

The mass parameters m_{D} and m_{f} in hard-thermal-loop perturbation theory are in principle completely arbitrary. To complete a calculation, it is necessary to specify m_{D} and m_{f} as functions of e and T. In Ref. [29] we considered two possible mass prescriptions in order to see how much the results vary given the two different assumptions. First we considered the variational solutions for the thermal masses and second we considered using the e^5 perturbative expansion of the Debye mass [7,31] and the e^3 perturbative expan-

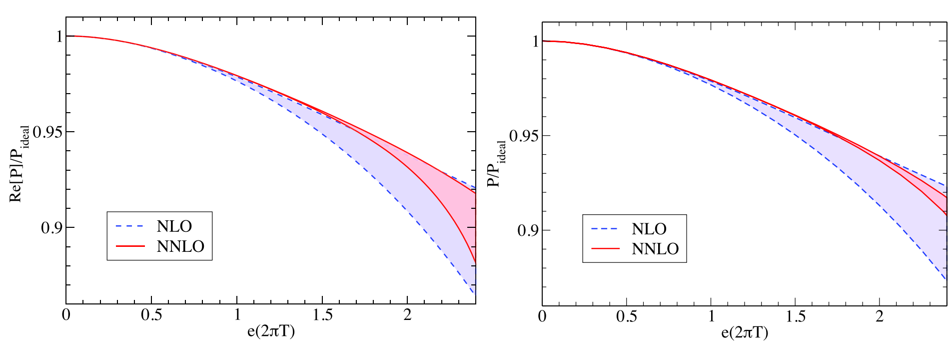

Fig. 2. A comparison of the renormalization scale variations between NLO and NNLO HTLpt predictions for the free energy of QED with $N_f = 1$ and the variational Debye mass (left) and using the perturbative thermal masses (right). The bands correspond to varying the renormalization scale μ by a factor of 2 around $\mu = 2\pi T$.

sion of the fermion mass [32]. The resulting predictions for the free energy are shown in Fig. 2. As can be seen from these figures both the variational and perturbative mass prescriptions seem to be consistent when going from NLO to NNLO. As a further check of our results in Fig. 3 we show a comparison of our NNLO HTLpt results with a three-loop calculation obtained previously using a truncated three-loop Φ-derivable approximation [30]. As can be seen from this figure, there is very good agreement between the NNLO Φ-derivable and HTLpt approaches out to large coupling.

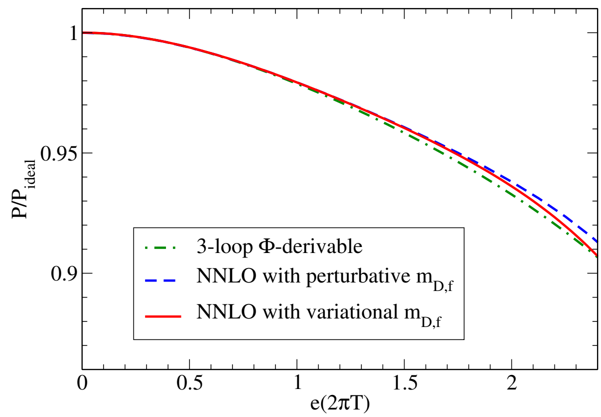

Fig. 3. A comparison of the predictions for the free energy of QED with $N_f = 1$ between three-loop Φ-derivable approximation [30] and NNLO HTLpt at $\mu = 2\pi T$.

3.4. Conclusions and outlook

In this paper we discussed reorganizing finite temperature perturbation theory using HTLpt in order to improve the convergence of successive perturbative approximations to the free energy of QED. We presented results of a recent three-loop HTLpt calculation of the pressure in QED [29] and showed that the HTLpt reorganization improves the convergence of the successive approximations to the QED free energy at intermediate coupling, $e \sim 2$. We studied two different mass prescriptions and showed that the results for the free energy using both prescriptions were the same to an accuracy of 0.6% at $e = 2.4$. We also compared the HTLpt three-loop result with a three-loop Φ-derivable approach [30] and found agreement at the sub-percentage level. In closing, we mention that the HTLpt reorganization is gauge invariant by construction we were able to obtain a completely analytic result for the resummed QED thermodynamic potential at three loops. This gives us confidence to apply the method also to full QCD.

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