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HIGH ORDER TRANSPORT COEFFICIENTS FROM AdS/CFT*

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We consider the energy momentum tensor of (linearized) relativistic hydrodynamics to all orders in fluid velocity gradient expansion. We apply the AdS/CFT duality for $\mathcal{N}=4$ SUSY in order to compute the retarded correlators of the energy-momentum tensor. From these correlators we determine a large set of transport coefficients of third- and fourth-order hydrodynamics and propose a new all order resummation model.

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1. Introduction

Refs. [1] pioneered the study of transport coefficients via dual description. For a static plasma the ratio of shear viscosity to entropy is independent of the coupling and is remarkably small

$$\frac{\eta_0}{s} = \frac{1}{4\,\pi}\,. \tag{1}$$

In this presentation we will discuss higher order gradients, which will provide certain corrections to the first order viscosity term when gradients grow[1]. These corrections could be relevant for propagation of smaller size objects in quark gluon plasma or for early time of hydro evolution. The high order gradient expansion generically includes two types of terms: *(i)* non-linear terms in the velocity field (like $(\nabla u)^2$) and *(ii)* linear terms with multiple gradient operators acting on a single velocity field (like $\nabla\nabla u$). These two types of terms are controlled by two different parameters. The non-linearities are important when the field amplitude is large. However, even for small amplitude waves, one can get large contributions from the linear terms when the momenta associated with the wave are large.

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[1] Second order terms have been studied in Ref. [2, 3].

In this paper we report on our study in [4] of all order gradient expansion in the linear approximation. Instead of introducing new transport coefficients at each new order, we are thinking of viscosity and other transport coefficients as frequency and momentum dependent functions. We are working in the framework of $\mathcal{N}=4$ SUSY. We set all dimensionfull units to be related with the temperature, $2\pi T=1$.

More generally, we will find that higher order terms have tendency to cancel (or reduce) the effect of shear viscosity. In particular, in our earlier paper [5], Shuryak and myself argued that the extremely low viscosity observed at the RHIC may essentially be some "effective viscosity", which includes these high order gradient terms. The real systems probed in RHIC collisions have finite gradients and the inclusion of their effects may demand going beyond Navier Stokes approximation. In [5] we attempted to extract a momentum-dependent viscosity from the imaginary part of the sound dispersion curve. Our main observation was that the effective viscosity as probed at finite momenta turns out to be smaller compared to the value at the origin.

Here we report on our attempt to put the idea of a momentum-dependent viscosity on a more solid ground compared to the naive treatment in [5]. Our focus here is on the retarded correlators of the stress tensor. The complete information on the correlators is equivalent to the knowledge of the energy momentum tensor in the linearized approximation. Our strategy is to first write a most generic hydro-like representation of the energy momentum tensor $T^{\mu\nu}$, in terms of the fluid velocity field u. We find that, generically, there are four structures (or operators involving derivatives of u or the metric g) which can occur in $T^{\mu\nu}$ and are consistent with all symmetries. Each structure enters with a coefficient which is momentum-dependent. These are the generalized transport coefficient we are looking for. One of them is associated with the shear viscosity, while the remaining three encode responses of the system to external ($4d$) gravity perturbations. We call them gravitational susceptibilities of the fluid (GSF). The operators which are multiplied by the GSFs involve the Weyl tensor of the metric and vanish in the flat Minkowski space.

We use this hydro-like representation of $T^{\mu\nu}$ in order to compute its retarded correlators defined as follows

$$G^{\mu\nu\alpha\beta}(k,\omega)=-i\int\limits_0^\infty dt\int d^3x\, e^{-i\omega t+ikx}\left\langle\left[T^{\mu\nu}(x,t),T^{\alpha\beta}(0)\right]\right\rangle\,. \qquad (2)$$

Here the average is over the equilibrated thermal bath. For conformally invariant plasma with traceless $T^{\mu\nu}$, there are only three independent correlators $G^T\equiv G^{xyxy}$ (tensor), $G^D\equiv G^{txtx}$ (shear), and $G^S\equiv G^{tztz}$ (sound) with the vector k pointing in the z-direction.

We then attempt to determine the momentum-dependent transport coefficients from the matching to the functions G^S, G^D, and G^T computed, via AdS/CFT correspondence, directly from the bulk gravity side. Our program runs into a problem, which we were not able to resolve completely: there are in fact four independent transport functions to be extracted from three equations. Despite the fact that we could not determine the entire functions, we were able to get them to quite high order in the perturbative expansion at small momenta.

The conceptual problem mentioned above, prevented us from computing shear viscosity in the whole kinematic region of arbitrary frequency and momentum. Instead, we build a model which utilizes the information about the new transport coefficients and preserves the causality condition. We propose this model for phenomenological studies of hydrodynamics at the RHIC.

2. Correlators from the gravity

In this section we closely follow the setup and results of [6]. From the bulk gravity side, in order to compute the retarded correlators at non-zero temperature one has to solve certain wave equations (one for each symmetry channel). These equations describe propagation of the corresponding metric perturbations in the AdS–Schwartschild BH background of the dual description. The differential equations are of the form

$$\frac{d^2}{dr^2}Z_a(r) + p_a(r)\frac{d}{dr}Z_a(r) + q_a(r)Z_a(r) = 0\,, \tag{3}$$

where the coefficients $p_a(r)$, $q_a(r)$ depend on the frequency ω and momentum k, and $a = T, D, S$ labels the three symmetry channels. The coefficient functions are taken from [6]. The fifth dimension coordinate r ranges from 0 to 1, where $r = 0$ corresponds to the boundary of the asymptotically AdS space, and $r = 1$ corresponds to the event horizon of the background metric.

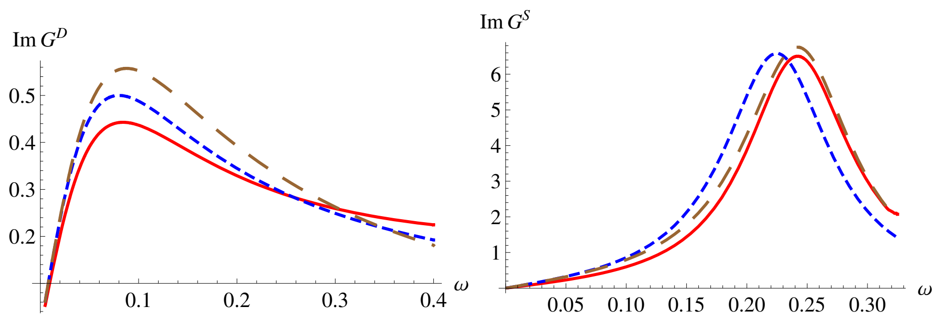

Fig. 1. Shear and sound channels at $k=0.4$. Solid line corresponds to the AdS/CFT correlator. Short dashes display the NS hydrodynamics while long dashes show the IS hydrodynamics.

The information about the retarded correlation functions is encoded in the solutions to Eq. (3), which satisfy the incoming wave condition at the horizon $Z_a(r \to 1) \sim \exp[-i\omega/2]$. The results of calculations are illustrated in Fig. (1). In the figures we compared AdS/CFT results with the Navier Stokes curves as well as with second order Israel Stewart hydro.

3. All order (linearized) hydrodynamics

The hydro representation of the energy-momentum tensor is

$$\langle T^{\mu\nu}\rangle^{\rm h}_{\rm cl} = (\epsilon + P)u^\mu u^\nu + Pg^{\mu\nu} + \Pi^{\langle\mu\nu\rangle}\,. \tag{4}$$

Here for any tensor $\Pi^{\mu\nu}$ we define its traceless and symmetric component

$$\Pi^{\langle\mu\nu\rangle} = \tfrac{1}{2}\,\Delta^{\mu\alpha}\Delta^{\mu\beta}(\Pi_{\alpha\beta} + \Pi_{\beta\alpha}) - \tfrac{1}{3}\Delta^{\mu\nu}\Delta^{\alpha\beta}\,\Pi_{\alpha\beta} \tag{5}$$

with the projector

$$\Delta^{\mu\nu} = g^{\mu\nu} + u^\mu\, u^\nu\,. \tag{6}$$

The energy-momentum conservation leads to equations of motion for the fluid:

$$\nabla_\mu \langle T^{\mu\nu}\rangle = 0\,. \tag{7}$$

Here ∇_μ stands for covariant derivative with respect to the metric g. The tensor $\Pi^{\mu\nu}$ is considered to have all order gradient expansion. Within the linearized approximation discussed above, and constrained by the Lorentz and Weyl symmetries, there are four independent structures (operators)

$$\Pi^{\mu\nu} = -2\eta\nabla^\mu u^\nu + 2\kappa u_\alpha u_\beta C^{\mu\alpha\nu\beta} + \rho(u_\alpha\nabla_\beta + u_\beta\nabla_\alpha)C^{\mu\alpha\nu\beta} + \xi\nabla_\alpha\nabla_\beta C^{\mu\alpha\nu\beta}\,. \tag{8}$$

By representing $\Pi^{\mu\nu}$ in the form (8) we essentially postulate a constitutive relation between $\langle T^{ij}\rangle$ and v^i.

Each of the four transport coefficient functions η, κ, ρ and ξ is considered to be a function of the Lorentz scalar operators ∇^2 and $(u\,\nabla)$

$$\eta = \eta[\nabla^2, (u\nabla)]\,, \quad \kappa = \kappa[\nabla^2, (u\nabla)]\,, \quad \rho = \rho[\nabla^2, (u\nabla)]\,, \quad \xi = \xi[\nabla^2, (u\nabla)]\,.$$

In momentum space representation (adequate for our framework of linear approximation) these functions will depend on $i\,\omega$ and k^2: $\nabla^2 \to \omega^2 - k^2$ and $(u\,\nabla) \to -i\,\omega$. The hydro ansatz (8) can be probed by small gravity perturbations. Using linear response theory we can then compute the retarded correlators in the three symmetry channels

- The scalar:

$$G^T(k,w) = -i\,\omega\eta - \kappa\frac{1}{2}\left(w^2 + k^2\right) - \rho\frac{i\,\omega}{2}\left(w^2 - k^2\right) + \xi\frac{1}{4}\left(\omega^2 - k^2\right)^2\,, \tag{9}$$

- The shear:

$$G^D(k,w) = (\epsilon + P) \times \frac{\bar{\eta}k^2 - i\bar{\kappa}\omega k^2/2 - \bar{\rho}k^2(k^2-2\omega^2)/4 + i\bar{\xi}\omega k^2(\omega^2-k^2)/4}{-i\omega + \bar{\eta}k^2}\,, \qquad (10)$$

- The sound:

$$G^S(k,w) = (\epsilon + P) \times \frac{k^2 - 4i\bar{\eta}\omega k^2 - 2\bar{\kappa}\omega^2 k^2 - 2i\bar{\rho}\omega^3 k^2 + \bar{\xi}\omega^4 k^2}{k^2 - 3\omega^2 - 4i\bar{\eta}\omega\, k^2}\,, \qquad (11)$$

$$\bar{\eta} \equiv \eta/(\epsilon+P)\,, \quad \bar{\kappa} \equiv \kappa/(\epsilon+P)\,, \quad \bar{\rho} \equiv \rho/(\epsilon+P)\,, \quad \bar{\xi} \equiv \xi/(\epsilon+P)\,.$$

4. Matching the bulk with the boundary

There should be one to one correspondence between linearized $T^{\mu\nu}$ and the full set of its correlators. Our program is to equate the expressions (9)–(11) for the correlators to the correlators computed from the bulk gravity. The goal is to invert these equations in order to determine the four transport coefficient functions. We have got an apparent problem as we end up having only three equations for four unknown functions. Despite our failure to simultaneously determine all transport coefficient functions, we are able to extract them perturbatively in the near long-wave limit approximation.

$$\begin{aligned}
\eta &= \eta_0(1 + i\eta_{0,1}\omega + \eta_{2,0}k^2 + \eta_{0,2}w^2 + i\eta_{2,1}\omega\, k^2 + i\eta_{0,3}\omega^3 + \eta_{4,0}k^4 \\
&\quad + \eta_{2,2}\,\omega^2 k^2 + \eta_{0,4}\omega^4 + \ldots)\,, \\
\kappa &= \kappa_0(1 + i\kappa_{0,1}\omega + \kappa_{2,0}k^2 + \kappa_{0,2}w^2 + i\kappa_{2,1}\omega k^2 + i\kappa_{0,3}\omega^3 + \ldots)\,, \\
\rho &= \rho_0(1 + i\rho_{0,1}\omega + \rho_{2,0}k^2 + \rho_{0,2}w^2 + \ldots)\,, \\
\xi &= \xi_0(1 + i\xi_{0,1}\,\omega + \ldots)\,. \qquad (12)
\end{aligned}$$

Here we explicitly list all terms up to fifth order. The third order coefficients are determined (practically all) analytically. The other coefficients are extracted numerically. 1st and 2nd order hydro

$$\eta_0 = \tfrac{1}{2}\,, \qquad \tau_R \equiv \eta_{0,1} = 2 - \ln 2\,, \qquad \kappa_0 = 2\,\eta_0\,,$$

3rd order hydro

$$\begin{aligned}
\lambda &\equiv \eta_{2,0} = -\tfrac{1}{2}\,, \qquad \eta_{0,2} \simeq -1.379 \pm 0.001 \simeq -\tfrac{3}{2} + \tfrac{1}{4}\ln^2 2\,, \\
\kappa_{0,1} &= \tfrac{5}{2} - 2\,\ln 2\,, \qquad \rho_0 = 4\,\eta_0\,,
\end{aligned}$$

4th order hydro

$$\eta_{2,1} = -2.275 \pm 0.005\,, \qquad \eta_{0,3} = -0.082 \pm 0.003\,. \tag{13}$$

Based on our findings we proposed the following improved causal hydrodynamics model for the viscosity function

$$\eta_{\,\mathrm{model}} = \frac{\eta_0}{1 - \lambda k^2 - i\omega\tau_R}\,.$$

5. Conclusions

We have initiated study of all order (linearized) hydrodynamics. We have encountered the "4 *versus* 3" puzzle, which we were not able to resolve. We, nevertheless, succeeded in determining a few new transport coefficients. These lead us to propose an improved phenomenological model, which is well consistent with our high order results. The effective viscosity is a decreasing function both of frequency and momentum. This behavior might be the reason behind the low viscosity observed at the RHIC. It may also explain the exceptionally good survival of various hydrodynamic flows, particularly the sound waves.

This talk is based on the results obtained in collaboration with Edward Shuryak.

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