η–η' MIXING FROM THE CHIRAL LAGRANGIAN*

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The mass matrix for η–η' is derived in the flavor basis at $\mathcal{O}(p^4)$ of the chiral Lagrangian using the large N approximation. Under certain assumptions, the mixing angle $\phi = 41.4°$ and the decay constants ratio $f_K/f_\pi = 1.15$ are calculated and in agreement with the data. It appears that the FKS scheme arises as a special limit of the chiral Lagrangian.

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1. Introduction

The complexity of the η–η' mixing was investigated in many theoretical works [1–4]. Guided by symmetry, the chiral Lagrangian is one of the most powerful tool to study the interaction of Goldstone bosons. The η' is not a Goldstone bosons because of the U(1) axial anomaly, but its inclusion in the chiral Lagrangian can be done using large N arguments. A convention for approximating the expansion could be to keep just the leading order in $1/N$ term for any given order in momenta. Arguing that dynamics and large N are independent, such an expansion was proposed [1] and leads to a good description of η–η' properties in terms of a few low-energy constants [1].

I summarize here the development proposed in [4] (to be consulted for details), in which we rewrote the mass matrix in terms of physical quantities instead of low-energy constants. We also identified the relevant flavor basis well suited to deal with the mixing problem. On this basis, every unknown can be expressed (approximately) in terms of four masses M_π, M_K, M_η, $M_{\eta'}$, which become our only inputs.

The derivation of the mass matrix is presented in Section 2. In Section 3, we compare our approach based on the chiral Lagrangian to Feldmann–Kroll–Stech (FKS) formalism [2] and show how the assumption of Ref. [2] influences the mass matrix. Section 4 is devoted to the conclusion.

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2. The chiral Lagrangian

Let us briefly review what is known from the chiral Lagrangian. In the large N limit, the relevant degrees of freedom are the nine Goldstone bosons of the symmetry breaking $\mathrm{U}(3)_\mathrm{L} \otimes \mathrm{U}(3)_\mathrm{R} \to \mathrm{U}(3)_\mathrm{V}$. We collect the Goldstone mesons in a nonlinear parametrization $U = \exp\left(i\sqrt{2}\pi/f\right)$ transforming according to $U \xrightarrow{G} LUR^\dagger$, $L \in \mathrm{U}(3)_\mathrm{L}$, $R \in \mathrm{U}(3)_\mathrm{R}$, with $\pi = \pi^a\lambda_a$ ($\lambda_0 \equiv \mathbf{1}_3\sqrt{2/3}$).

The chiral Lagrangian is constructed *via* an expansion in the momenta. At each order in p^2 we only retain the dominant term in the $1/N$ expansion [1]. To lowest order in derivatives, the chiral Lagrangian encodes the symmetry breaking terms and reads

$$\begin{aligned}\mathcal{L}^{(p^2)} &= \frac{f^2}{8}\left\langle \partial_\mu U^\dagger \partial^\mu U + B\left(\mathcal{M}U^\dagger + U\mathcal{M}^\dagger\right)\right\rangle + \frac{\alpha_0}{2N}\left[\frac{f}{4}\left\langle \ln\left(\frac{\det U}{\det U^\dagger}\right)\right\rangle\right]^2 \\ &= \frac{1}{2}\partial_\mu\pi^a\partial^\mu\pi^b\delta_{ab} - \frac{1}{2}B\pi^a\pi^b\langle\lambda_a\lambda_b\mathcal{M}\rangle - \frac{1}{2}\alpha_0\eta_0^2\,. \qquad (1)\end{aligned}$$

$\mathcal{M}$ being the mass matrix transforming like U. We use isospin SU(2) symmetry and therefore $\mathcal{M} = \mathrm{diag}(\tilde{m}, \tilde{m}, m_s)$. The physical masses (squared) are $m_\pi^2 = B\tilde{m}$ and $m_K^2 = B(\tilde{m} + m_s)/2$.

The diagonal mass matrix induces a privileged basis. For this, it is useful to work in the flavor basis. For this purpose, we use the representation

$$\pi = \sqrt{2}\ \mathrm{diag}\ \left(u\bar{u}, d\bar{d}, s\bar{s}\right)\,. \qquad (2)$$

We dropped the noninteresting nondiagonal fields. We next introduce the fields $\eta_s = s\bar{s}$ and $\eta_q = (u\bar{u} + d\bar{d})/\sqrt{2}$ allowing to rewrite the mass matrix as

$$\mathcal{M}_{qs}^2 = \begin{pmatrix} m_\pi^2 + 2(\frac{\alpha_0}{N}) & (\frac{\alpha_0}{N})\sqrt{2} \\ (\frac{\alpha_0}{N})\sqrt{2} & 2m_K^2 - m_\pi^2 + (\frac{\alpha_0}{N}) \end{pmatrix}\,. \qquad (3)$$

The mixing is provided only by the anomaly as expected from the ideal mixing between the vector particles ω and ϕ. We have only one parameter to reproduce two physical states, this is clearly not enough [1, 4].

We next explore the chiral Lagrangian at $\mathcal{O}(p^4)$. For our purposes, we are only interested in terms contributing to the kinetic part, the mass matrix and the decay constants. We restrict ourselves to terms involving only zero or two derivatives, and we only keep the leading order in $1/N$. There are three terms of interest at $\mathcal{O}(p^4)$ and they involve a single trace over flavor [1]

$$\frac{f^2}{8}\left[-\frac{B}{\Lambda^2}\left\langle \mathcal{M}\partial_\mu\partial^\mu U^\dagger\right\rangle + \frac{B^2}{2\Lambda_1^2}\left\langle \mathcal{M}U^\dagger\mathcal{M}U^\dagger\right\rangle + \frac{B}{2\Lambda_2^2}\left\langle \mathcal{M}U^\dagger\partial_\mu U\partial^\mu U^\dagger\right\rangle\right] + \mathrm{h.c.}$$

The three low-energy constants enter the observables without a clear physical meaning. Λ and Λ_2 induce a splitting between the π and K decay constants. Λ_1 and Λ_2 enter in the corrections to the mass matrix. Fitting their values on observables leads to a consistent η–η' scheme at $\mathcal{O}(p^4)$ [1].

We are interested in a more physical interpretation of the low-energy constants and we aim at an analytical resolution of the mass matrix. We, therefore, would like to reduce the number of parameters since a two-by-two matrix gives us only two independent equations. To this aim, we rotate away the Λ term by a chiral transformation at $\mathcal{O}(p^2)$ [1]

$$U \longrightarrow U' = U - \frac{B}{2\Lambda^2}\left(\mathcal{M} - U\mathcal{M}^\dagger U\right) . \tag{4}$$

Such a redefinition preserves the unitarity of U up to $\mathcal{O}(p^4)$ in the Lagrangian, but does not preserve the anomalous term. The Lagrangian after rotation reads (see [4] for the expressions of K_1 and K_2)

$$\begin{aligned}\mathcal{L}^{(p^4)} \;=\; & \mathcal{L}^{(p^2)} + B\frac{f^2}{8}\left[K_1\left\langle \mathcal{M}U^\dagger\mathcal{M}U^\dagger\right\rangle + BK_2\left\langle \mathcal{M}U^\dagger\partial_\mu U\partial^\mu U^\dagger\right\rangle\right] + \text{h.c.} \\ & -\frac{\alpha_0}{2N}\frac{B}{\Lambda^2}\frac{f^2}{8}\left\langle \mathcal{M}^\dagger U - U^\dagger\mathcal{M}\right\rangle\left\langle \ln\left(\frac{\det U}{\det U^\dagger}\right)\right\rangle . \end{aligned} \tag{5}$$

The physics described by the Lagrangian (5) is exactly the same as that of the original Lagrangian. But now, the low-energy constants appear in a more convenient way. Indeed, to derive the mass matrix, one should always bring the kinetic terms into its canonical form by means of a redefinition of the fields. With our particular Lagrangian (5), the wave-function renormalizations are simply proportional to the matrix of the decay constants. This matrix is proportional to $\langle\lambda_a\lambda_b\mathcal{M}\rangle$. Renormalizing the fields is easy in the case of a diagonal matrix since it only amounts to renomalizing the fields by a simple rescaling without any rotation. This is not the case in the U(3) basis, where the rotation is mandatory [5] to bring us in the flavor basis where $\langle\lambda_a\lambda_b\mathcal{M}\rangle$ becomes diagonal. It is, then, more advantageous to start directly in the flavor basis (2). In the flavor basis the kinetic terms

$$\frac{1}{2}\left(\frac{f_q}{f}\right)^2\partial_\mu\eta_q\partial^\mu\eta_q + \frac{1}{2}\left(\frac{f_s}{f}\right)^2\partial_\mu\eta_s\partial^\mu\eta_s \tag{6}$$

does not present any mixing term. Moreover, since the kinetic energy and the decay constants follow the same pattern, the matrix of the decay constants is also diagonal. This avoids unwanted (and sometimes overlooked [6]) transition elements $J^8_\mu|\eta_0\rangle$. The particular structure of the mass matrix imposes then a privileged basis as claimed below and used in the FKS scheme [2].

The flavor decay constants can be easily expressed in terms of the physical ones (in the U(3) basis):

$$f_q^2 = f_\pi^2\,, \qquad f_s^2 = 2f_K^2 - f_\pi^2\,. \tag{7}$$

The mass matrix then involves those physical decay constants. Introducing the parameter $y = f_q/f_s$, we can derive from the Lagrangian (5) the mass matrix in a more convenient form

$$\mathcal{M}_{qs}^2 = \begin{pmatrix} M_{qq}^2 + 2\alpha & \alpha y\sqrt{2} \\ \alpha y\sqrt{2} & M_{ss}^2 + \alpha y^2 \end{pmatrix} + \mathcal{O}\left(\frac{\alpha}{\Lambda^2}\right)\,. \tag{8}$$

For convenience we have defined $\alpha = (\alpha_0/N)(f/f_q)^2$.

$\mathcal{O}\left(\alpha/\Lambda^2\right)$ stands from the contribution of the last term in (5). Assuming a negligible term $\mathcal{O}\left(\alpha/\Lambda^2\right)$ provides us with analytical formulas, whereas the inclusion of this term leads to a unavoidable numerical procedure.

From (8) and neglecting the term $\mathcal{O}\left(\alpha/\Lambda^2\right)$ (or equivalently taking the limit $\Lambda \to \infty$), it is straightforward to extract the value of the parameters y and α in function of the masses. Equating the determinant and the trace of (8) with the mass matrix of the physical states, $\mathrm{diag}(M_\eta^2, M_{\eta'}^2)$, we obtain [4] (with ϕ the mixing angle in the flavor basis)

$$y^2 = 2\frac{M_\eta^2 M_{\eta'}^2 - M_{ss}^2(M_\eta^2 + M_{\eta'}^2 - M_{ss}^2)}{M_\pi^2(M_\eta^2 + M_{\eta'}^2 - M_\pi^2) - M_\eta^2 M_{\eta'}^2}\,, \tag{9}$$

$$\alpha = \frac{M_\eta^2 + M_{\eta'}^2 - M_\pi^2 - M_{ss}^2}{2 + y^2}\,, \tag{10}$$

$$\sin 2\phi = \frac{2\sqrt{2}\alpha y}{M_{\eta'}^2 - M_\eta^2}\,. \tag{11}$$

In the mass matrix (8), M_{qq} and M_{ss} are the unknown masses of the pseudoscalar $q\bar{q}$ and $s\bar{s}$ states. They can be related to physical masses by expressing them in the $U(3)$ basis in analogy with (7). We will use $M_{qq}^2 = M_\pi^2$ and $M_{ss}^2 = 2M_K^2 - M_\pi^2$. The latter expression is only valid at leading order and received an extra contribution, around 5%, at $\mathcal{O}(p^4)$ [1]. This approximation coincides with the value used in the FKS paper [2].

We then obtain the value for the two parameters y and α and a prediction for the physical quantities

$$\frac{f_K}{f_\pi} = 1.146\,, \qquad \phi = 41.40^\circ\,. \tag{12}$$

Our calculated values (12) lie in the usual range of $[40°, 45°]$ [7] and are in agreement with the data on η and η' decays. The phenomenological value for the ratio $f_K/f_\pi = 1.193(0.009)$ [8] is also close to our predicted value.

The approximation $M_{ss}^2 = 2M_K^2 - M_\pi^2$ is only valid at leading order and receives a correction at order $\mathcal{O}(p^4)$ [1]. The extra contribution reduces the value of the mixing angle and f_K/f_π as shown in [4].

3. Comparison with FKS formalism

We next compare the prediction of the chiral Lagrangian with the FKS results [2]. The basic hypothesis of the FKS formalism was the assumption that decay constants in the flavor basis follow the same mixing pattern of the states

$$\begin{pmatrix} f_\eta^q & f_\eta^s \\ f_{\eta'}^q & f_{\eta'}^s \end{pmatrix} = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} f_q & 0 \\ 0 & f_s \end{pmatrix} . \tag{13}$$

Under this assumption, the mass matrix is derived from current algebra and reads

$$\mathcal{M}_{\rm FKS}^2 = \begin{pmatrix} M_{qq}^2 + \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_q\rangle & \frac{1}{f_s}\langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_q\rangle \\ \frac{\sqrt{2}}{f_q}\langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_s\rangle & M_{ss}^2 + \frac{1}{f_s}\langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_s\rangle \end{pmatrix} .$$

Let us see how it arises from the chiral Lagrangian.

At the effective level, we find [4]

$$\langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_q\rangle = \alpha\sqrt{2}f_q\,, \qquad \langle 0|\frac{\alpha_{\rm s}}{4\pi}G\tilde{G}|\eta_s\rangle = \alpha y^2 f_s\,. \tag{14}$$

Rewriting $\mathcal{M}_{\rm FKS}^2$ in our notation, we get the mass matrix (8), but without the $\mathcal{O}(\alpha/\Lambda^2)$ term. The mass matrix is then symmetric and we do not need to impose the equality of the off-diagonal terms by hand as in Ref. [2].

The important point is that we did not use any assumptions of the mixing scheme for the decay constants in the derivation of (8). The hypothesis (13) is not mandatory from our effective field theory point of view and is not physically justified on general grounds. By comparing both approaches, we notice that the hypothesis on the decay constants pattern is not consistent with the chiral Lagrangian at next-to-leading order in the large N limit since it amounts to neglecting of the first term in (2).

4. Conclusion

Keeping only the leading term in $1/N$ at each order in the momentum expansion, we derived at $\mathcal{O}(p^4)$ a mass matrix in the flavor basis. The quantities are expressed in terms of the masses and the ratio of decay constants y,

instead of the low-energy constants, allowing a clear understanding. Such a form allows the extraction of two parameters ϕ and y from mass inputs. Their values are in agreement with the data.

The choice of the flavor basis is motivated by the simple expression of the kinetic term and the diagonal matrix for the decay constants. Such features, not apparent in the U(3) basis, make the calculation tedious in that basis and render the physics more obscure.

We also relate the FKS formalism [2] with the chiral Lagrangian approach, where no assumption on the mixing scheme for the decay constants is required. It is shown that the FKS Ansazt (13) is not compatible with the chiral Lagrangian. The difference stands in the parameter Λ. Taking the limit $\Lambda \to \infty$ renders the FKS scheme whereas its value, extracted from the data, is of the same magnitude as those of the other low-energy constants [1].

The predicted values (12) are in agreement with the data, but leave a room for improvement since assumptions were invoked. Moreover, the glue content in the η' wave-function should be properly considered at $\mathcal{O}(p^4)$. To this aim, the inclusion of the pseudoscalar glueball in the chiral Lagrangian in under construction [3].

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