Vol. 4 (2011) *Acta Physica Polonica B Proceedings Supplement* No 3

INFINITE RANDOM GRAPHS AS STATISTICAL MECHANICAL MODELS*

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(Received April 4, 2011)

We discuss two examples of infinite random graphs obtained as limits of finite statistical mechanical systems: a model of two-dimensional discretized quantum gravity defined in terms of causal triangulated surfaces, and the Ising model on generic random trees. For the former model we describe a relation to the so-called uniform infinite tree and results on the Hausdorff and spectral dimension of two-dimensional space-time obtained in B. Durhuus, T. Jonsson, J.F. Wheater, *J. Stat. Phys.* **139**, 859 (2010) are briefly outlined. For the latter we discuss results on the absence of spontaneous magnetization and argue that, in the generic case, the values of the Hausdorff and spectral dimension of the underlying infinite trees are not influenced by the coupling to an Ising model in a constant magnetic field (B. Durhuus, G.M. Napolitano, in preparation).

DOI:10.5506/APhysPolBSupp.4.287
PACS numbers: 02.10.Ox, 04.60.Nc, 04.60.Pp, 05.40.Fb

1. Introduction

Recall that a *graph* G is specified by its *vertex* set $V(G)$ and its *edge* set $E(G)$. Vertices will be denoted by v or v_i *etc.* An edge is then an unordered pair (v,v') of different vertices. Both finite and infinite graphs will be considered, *i.e.* $V(G)$ may be finite or infinite, and all graphs will be assumed to be locally finite, *i.e.* the number σ_v of edges containing a vertex v, called the *degree* of v, is finite for all $v \in V(G)$. By the *size* of G we shall mean the number of edges in G and denote it by $|G|$, *i.e.* $|G| = \sharp E(G)$, where $\sharp M$ is used to denote the number of elements in a set M.

A *path* in G is a sequence of different edges $(v_0,v_1),(v_1,v_2),\ldots,(v_{k-1},v_k)$ where v_0 and v_k are called the *end vertices.* If $v_0 = v_k$ the path is called a *circuit* originating at v_0. The graph G is called *connected* if any two vertices

* Presented at the Conference "Geometry and Physics in Cracow", Poland, September 21–25, 2010.

v and v' of G can be connected by a path, *i.e.* they are end vertices of a path. The *graph distance* between v and v' is then defined as the minimal number of edges in a path connecting them. A connected graph is called a *tree* if it has no circuits.

A *planar graph* is a graph together with an embedding $\phi : V(G) \to \mathbb{R}^2$ and an association to each edge $(v, v') \in E(G)$ of an arc $\psi(v, v')$ in $\mathbb{R}^2$ connecting $\phi(v)$ and $\phi(v')$ such that arcs corresponding to different edges are disjoint except possibly for endpoints. Two planar graphs are considered identical if one can be continuously deformed into the other in $\mathbb{R}^2$.

A *planar tree* is a planar connected graph without circuits.

The statistical mechanical models considered in this paper are defined in terms of planar graphs as follows. Let $\mathcal{G}_N$ be a subset of the set of planar graphs of size N, and let us assume that the graphs in $\mathcal{G}_N$ are rooted, *i.e.* they contain a distinguished oriented edge $e = (r, r')$, called the *root edge*, and whose initial vertex r is called the *root vertex*. To each graph $G \in \mathcal{G}_N$ we attribute a weight $w(G) \geq 0$, and we define a partition function Z_N, for each $N \geq 1$, by

$$Z_N = \sum_{G \in \mathcal{G}_N} w(G)\,. \tag{1}$$

Here, $w(G)$ may be given in terms of graph data alone or in terms of some additional data on the graph. In particular, $w(G)$ may be given as the partition function of a statistical mechanical system on G. In Sec. 3 below we consider specifically the case, where $w(G)$ is the partition function of an Ising model on G.

On the basis of (1) a probability distribution $\mu_N(G)$ is defined on $\mathcal{G}_N$ by setting

$$\mu_N(G) = Z_N^{-1}\, w(G)\,, \qquad G \in \mathcal{G}_N\,,$$

and our goal is to study the limiting distribution for $N \to \infty$. This will be a probability measure $d\mu$ on an appropriate space $\mathcal{G}_\infty$ of infinite graphs. In the cases considered below this will be a metric space with the distance $d(G, G')$ between two graphs G and G' being defined by

$$d\left(G, G'\right) = \inf\left\{\frac{1}{R+1} \;\middle|\; B_R(G) = B_R\left(G'\right)\right\}\,, \tag{2}$$

where $B_R(G)$ denotes the ball in G of radius R and centered at the root r, *i.e.* $B_R(G)$ is the subgraph of G spanned by the vertices at graph distance $\leq R$ from r. We then say that $\mu_N \to \mu$ for $N \to \infty$ if

$$\sum_{G \in \mathcal{G}_N} f(G)\, \mu_N(G) \xrightarrow{N\to\infty} \int f(G)\, d\mu(G) \tag{3}$$

for all bounded continuous functions f on $\mathcal{G}_\infty$, which are also naturally defined on $\bigcup_{N=1}^{\infty} \mathcal{G}_N$ (see *e.g.* [4]). As will be seen, the limiting distributions in the cases considered below can be expressed quite explicitly in such a way that a number of their characteristics, such as the Hausdorff and the spectral dimension, can be analyzed in some detail.

To conclude this section, let us define these two notions of dimension.

1.1. Hausdorff dimension

Given a connected graph G and $R \geq 0$ and $v \in V(G)$ we denote by $B_R(G, v)$ the closed ball of radius R centered at v. If G is connected and the limit

$$d_{\mathrm{h}} = \lim_{R \to \infty} \frac{\ln |B_R(G, v)|}{\ln R} \tag{4}$$

exists, we call d_{h} the *Hausdorff dimension* of G. If G is a finite graph we clearly have $d_{\mathrm{h}} = 0$, a case we leave out of consideration in the following. It is easily seen that the existence of the limit as well as its value do not depend on the vertex v.

For an ensemble of graphs $(\mathcal{G}_\infty, \mu)$, as described above, we define the *annealed* Hausdorff dimension by

$$\bar{d}_{\mathrm{h}} = \lim_{R \to \infty} \frac{\ln \langle\, |B_R(G)|\, \rangle_\mu}{\ln R}\,, \tag{5}$$

provided the limit exists, where $\langle \cdot \rangle_\mu$ denotes the expectation value w.r.t. μ. If there exists a subset $\mathcal{G}_0$ of $\mathcal{G}_\infty$ such that $\mu(\mathcal{G}_0) = 1$ and such that every $G \in \mathcal{G}_0$ has Hausdorff dimension d_{h} we say that the Hausdorff dimension of $(\mathcal{G}_\infty, \mu)$ is almost surely d_{h}.

1.2. Spectral dimension

A *walk* on a graph G is a sequence $(v_0, v_1), (v_1, v_2), \ldots, (v_{k-1}, v_k)$ of (not necessarily different) edges in G. We shall denote such a walk by $\omega : v_0 \to v_k$ and call v_0 the *origin* and v_k the *end* of the walk. Moreover, the number k of edges in ω will be denoted by $|\omega|$. To each such walk ω we associate a weight

$$p_G(\omega) = \prod_{i=0}^{|\omega|-1} \sigma_{\omega(i)}^{-1}\,,$$

where $\omega(i)$ is the i-th vertex in ω. Denoting by $\Pi_n(G, v_0)$ the set of walks of length n originating at vertex v_0 we have

$$\sum_{\omega\in\Pi_n(G,v_0)} p_G(\omega) = 1\,,$$

i.e. p_G defines a probability distribution on $\Pi_n(G, v_0)$. We call p_G the *simple random walk* on G.

For a connected graph G and $v \in V(G)$ we denote by $p_t(G, v)$ the *return probability* of the simple random walk to v at time t, that is

$$p_t(G,v) = \sum_{\substack{\omega:v\to v\\ |\omega|=t}} p_G(\omega)\,.$$

One can in a standard manner relate this quantity to the discrete heat kernel on G, but we shall not need this interpretation in the following. If the limit

$$d_{\mathrm{s}} = -2 \lim_{t\to\infty} \frac{\ln p_t(G,v)}{\ln t} \tag{6}$$

exists, we call d_{s} the *spectral dimension* of G. Again in this case, the existence and value of the limit are independent of v. Moreover, $d_{\mathrm{s}} = 0$ if G is finite since $p_t(G, v) \to (\sharp V(G))^{-1}$ for $t \to \infty$. If G is infinite one has

$$d_{\mathrm{h}} \geq 1 \qquad \text{and} \qquad d_{\mathrm{s}} \geq 1\,.$$

If G is the hyper-cubic lattice $\mathbb{Z}^d$ it is clear that $d_{\mathrm{h}} = d$ and by Fourier analysis it is straight-forward to see that also $d_{\mathrm{s}} = d$. However, examples of graphs with $d_{\mathrm{h}} \neq d_{\mathrm{s}}$ are abundant, see *e.g.* [9].

The *annealed* spectral dimension of an ensemble $(\mathcal{G}_\infty, \mu)$ of rooted graphs is defined as

$$\bar{d}_{\mathrm{s}} = -2 \lim_{t\to\infty} \frac{\ln \langle\, p_t(G,r)\,\rangle_\mu}{\ln t} \tag{7}$$

provided the limit exists. As above, we say that the spectral dimension of $(\mathcal{G}_\infty, \mu)$ is almost surely d_{s}, if the set of graphs with spectral dimension different from d_{s} has vanishing μ-measure.

2. The uniform infinite causal triangulation

The so-called uniform infinite planar tree is obtained by letting $\mathcal{G}_N \equiv \mathcal{T}_N$ be the set of planar rooted trees with N edges and root r of order 1, and setting $w(T) = 1$ for $T \in \mathcal{T}_N$. The existence of the limiting distribution ν was demonstrated in [8]. An important feature of ν is that it is concentrated on trees with a single infinite path starting at r, called the *spine*, and attached to each spine vertex u_i, $i = 1, 2, \dots,$ is a finite number n_i of finite trees, called *branches* as illustrated in Fig. 1.

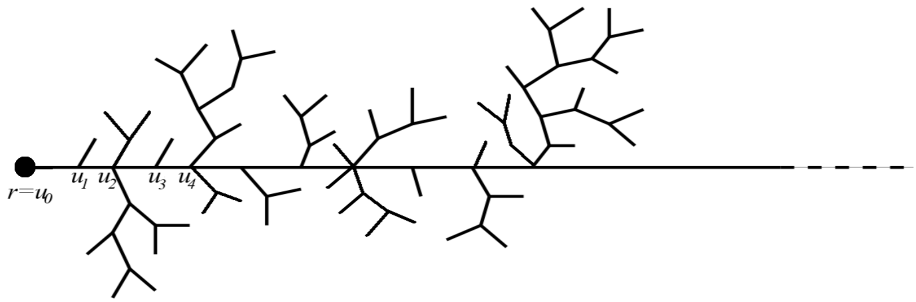

Fig. 1. Example of an infinite tree, consisting of a spine and left and right branches.

The individual branches T are independently distributed according to

$$\rho(T) = \prod_{v \in T \setminus r} 2^{-\sigma_v}\,, \tag{8}$$

where the product is over vertices in T except the root (which is identical to the vertex on the spine at which T is attached). Moreover, the orders $n_i + 2$ of spine vertices u_i, $i \geq 1$, are also independent random variables with probability distribution

$$p_n = \sum_{\substack{k'+k''=n \\ k',k'' \geq 0}} p_{k',k''} = 2^{-(n+1)}\,, \tag{9}$$

where

$$p_{k',k''} = 2^{-(k'+k''+2)} \tag{10}$$

is the probability of having k' left branches and k'' right branches (as seen from the root r) attached at u_i.

Properties of the uniform infinite tree and, more generally, of so-called generic trees, were analyzed in [10]. In particular, it was found that $\bar{d}_{\mathrm{h}} = 2$ and $\bar{d}_{\mathrm{s}} = \frac{4}{3}$. We shall not discuss these cases further in this article.

In the remainder of this section, we give a brief description of some results from [11], where the uniform infinite tree is exploited in the context of the causal dynamical triangulation (CDT) model of quantum gravity originally proposed in [2]. In order to define this model, we let $\mathcal{G}_N \equiv \mathcal{C}_N$ denote the set of *sliced* triangulations of the disc with N vertices. Here, a triangulation S of the disc is said to be sliced if the subgraph of S spanned by vertices at distance n and $n+1$ from the root r, $n = 1, \ldots, M$, is an annulus S_n such that every triangle in S_n has all vertices in the boundary and not all in the same boundary component of S_n. For $n = 0$ we require that $B_1(S)$ is a disc (see Fig. 2). Moreover, M denotes the maximal distance of vertices in S

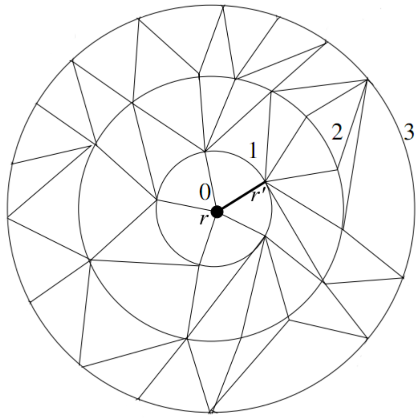

Fig. 2. A sliced triangulation S of the disc with circles containing vertices at distance 1, 2 and 3 from the root. Here S consists of two annuli S_1, S_2 and the disc $B_1(S)$. The bold edge indicates the root edge.

from r. In particular, if the boundary components of S_n contain l_n and l_{n+1} edges, respectively, then the total number of vertices, edges and triangles in S are

$$|V(S)| = 1 + \sum_{n=1}^{M} l_n\,, \quad |E(S)| = 3\sum_{n=1}^{M} l_n - l_M\,, \quad |S| = 2\sum_{n=1}^{M} l_n - l_M\,, \tag{11}$$

respectively. Here we have assumed $M < \infty$. However, the definition of a sliced surface is also valid for infinite triangulations of the plane, corresponding to $M = \infty$.

We then define μ_N to be the uniform distribution on $\mathcal{C}_N$, *i.e.* we set $w(S) = 1$ for $S \in \mathcal{C}_N$. Thus, in this case

$$Z_N = \sharp\mathcal{C}_N\,. \tag{12}$$

We claim that

$$\sharp\mathcal{C}_N = \sharp\mathcal{T}_N\,. \tag{13}$$

To see this, pick an orientation of the plane and consider $S \in \mathcal{C}_N$. For any vertex v at distance $n \geq 1$ from r, order the edges in $S_n \setminus \partial S_n$ emerging from v from left to right in accordance with the orientation of the plane. Next, delete from S all edges in $\bigcup_{n=1}^{M} \partial S_n$ as well as the rightmost edge emerging from v into S_n for each v as above. Finally, attach a new edge (r_0, r) to the root vertex r. Then, the resulting graph is a tree $\beta(S)$ with a unique embedding into the plane such that the root edge (r, r') in S becomes the rightmost edge emerging from r in $\beta(S)$ (see Fig. 3).

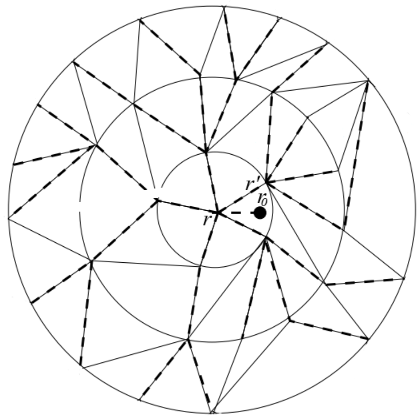

Fig. 3. The dashed lines indicate the edges of the tree $\beta(S)$ constructed from the triangulation in Fig. 2.

It is a fact, as the reader may easily verify, that $\beta : \mathcal{C}_N \to \mathcal{T}_N$ is a bijection which proves (13). In fact, β is a particular case of the so-called Schaeffer bijection applicable for labeled trees [17].

It is easy to check that β extends to the case $M = \infty$ corresponding to infinite sliced triangulations:

$$\beta : \mathcal{C}_\infty \to \mathcal{T}_\infty .$$

Taking this fact into account the following result is an immediate consequence of [8].

Theorem 2.1. *The distributions (μ_N) defined by*

$$\mu_N(S) = (\sharp\mathcal{C}_N)^{-1} , \qquad S \in \mathcal{C}_N ,$$

converge to a probability distribution μ on $\mathcal{C}_\infty$, which is given by

$$\mu(A) = \nu(\beta(A))$$

for measurable sets $A \subseteq \mathcal{C}_\infty$, where ν denotes the distribution of the uniform infinite tree.

We call the ensemble $(\mathcal{C}_\infty, \mu)$ the *uniform infinite causal triangulation* (UICT) [11, 14].

Except for the root $r_0 \in \beta(S)$, the vertices in S and $\beta(S)$ are the same and β preserves the distance from r to $v \in S$. It follows that the Hausdorff dimensions of the two ensembles $(\mathcal{T}_\infty, \nu)$ and $(\mathcal{C}_\infty, \mu)$ are identical.

Theorem 2.2. *For the uniform infinite causal triangulation we have*

$$\bar{d}_{\mathrm{h}} = 2$$

and

$$d_{\mathrm{h}} = 2 \quad \text{a.s.}$$

Proof. That $\bar{d}_{\mathrm{h}} = 2$ follows from the remarks above and [8], while the second statement follows from the corresponding result for generic trees established in [11]. □

Obviously, there is no canonical bijective correspondence between walks in S and in $\beta(S)$ and hence results on the spectral dimension for the uniform tree cannot be carried over to the UICT. A result by Benjamini and Schramm [3] states that under rather general circumstances a planar random graph is recurrent, which means that the simple random walk starting at r will return to r with probability 1. It is well known that this is the case if and only if $d_{\mathrm{s}} \leq 2$. Since the result of [3] presupposes a fixed upper bound on vertex degrees for the graphs in question it cannot be applied to the UICT. However, it was shown in [11], by combining the so-called Nash-Williams criterion for recurrency of graphs [16] with the known structure of the distribution ν described above, that the UICT is recurrent with probability 1. Thus we have

Theorem 2.3. *For the UICT the spectral dimension fulfills $d_{\mathrm{s}} \leq 2$ almost surely.*

It is generally believed that $d_{\mathrm{s}} = 2$ almost surely. A proof of this is still missing. To our knowledge the best known lower bound is

$$d_{\mathrm{s}} \geq \frac{4}{3} \quad \text{a.s.}$$

which is obtained by applying the inequality [5]

$$d_{\mathrm{s}} \geq \frac{d_{\mathrm{h}}}{d_{\mathrm{h}} + 1}$$

to the present situation using Theorem 2.2.

This finishes our discussion of the UICT. For more details the reader should consult [11].

3. The Ising model on a generic infinite tree

In this section, we consider an interacting spin system (Ising model) on an infinite generic tree.

Ising models on tree graphs have been considered previously by several authors. The most well known example is perhaps the Ising model on a Cayley tree [7]. This model is exactly solvable and was shown to exhibit spontaneous magnetization of a fixed central spin in [6], see also [15].

A grand canonical ensemble of Ising models on finite trees was considered in [1], where it was argued that the model has no spontaneous mean magnetization.

Our model may be considered as a thermodynamic limit of the model of [1], in the sense that trees have infinite size. To be specific we give a description of measures on the set of spin configurations on infinite trees, obtained as limits of Ising models on finite trees. This allows a detailed study of the magnetization properties of the system and of the Hausdorff and spectral dimension.

Here we give an overview of the main results and outline some of the arguments leading to them. For details we refer the reader to [12].

3.1. The partition functions

Let Λ_N be the set of planar rooted trees of size N decorated with Ising spin configurations,

$$\Lambda_N = \left\{\tau_s = (\tau, s) \;\middle|\; \tau \in \mathcal{T}_N,\, s \in S_\tau\right\}\,, \tag{14}$$

where $S_\tau = \{\pm 1\}^{V(\tau)}$. Decomposing S_τ into $S_\tau^\pm = \{s \in S_\tau \mid s(r) = \pm 1\}$, we have corresponding decompositions

$$\Lambda_N = \Lambda_{N+} \cup \Lambda_{N-}$$

and

$$\Lambda = \left(\bigcup_{N=1}^{\infty} \Lambda_N\right) \cup \Lambda_\infty = \Lambda_+ \cup \Lambda_-\,.$$

Let Z_N be the partition function of the Ising model on trees in $\mathcal{T}_N$ given by

$$Z_N(\beta,h) \;=\; \sum_{\tau\in\mathcal{T}_N} \mathcal{Z}(\beta,h,\tau)\rho(\tau)\,, \tag{15}$$

$$\mathcal{Z}(\beta,h,\tau) \;=\; \sum_{s\in S_\tau} e^{-H_\tau(s)}\,, \tag{16}$$

with
$$H_\tau(s) = -\beta \sum_{\langle ij\rangle\in\tau} s_i s_j - h \sum_{i\in\tau\setminus r} s_i\,,$$
where the notation $\sum\limits_{\langle ij\rangle\in\tau}$ is used for the sum over pairs of neighbouring spins, and
$$\rho(\tau) = \prod_{v\in\tau\setminus r} p_{\sigma_v - 1}\,. \tag{17}$$

Here, $(p_n)_{n\geq 0}$ is a sequence of non-negative numbers such that $p_0 \neq 0$ and $p_n > 0$ for some $n \geq 2$.

For the generating function
$$Z(\beta,h,g) = \sum_{N=1}^{\infty} Z_N(\beta,h)\, g^N$$
we then have
$$Z(\beta,h,g) = Z_+(\beta,h,g) + Z_-(\beta,h,g)\,,$$
where the generating functions $Z_\pm(\beta,h,g)$ are defined by restricting the sum in (16) to $S_\tau^\pm$.

Setting
$$\varphi(z) = \sum_{n=0}^{\infty} p_n z^n\,, \tag{18}$$
the functions $Z_\pm$ are determined by
$$\begin{cases} Z_+ = g(a\,\varphi(Z_+) + a^{-1}\,\varphi(Z_-))\,, \\ Z_- = g(b\,\varphi(Z_+) + b^{-1}\,\varphi(Z_-))\,, \end{cases} \tag{19}$$
where
$$a = e^{\beta+h}\,, \qquad b = e^{-\beta+h}\,. \tag{20}$$
Indeed, let R denote the radius of convergence of (18) and define $F : \{|z| < R\}^2 \times \mathbb{C} \to \mathbb{C}^2$ by
$$F(Z_+,Z_-,g) = g\begin{pmatrix} a\,\varphi(Z_+) + a^{-1}\,\varphi(Z_-) \\ b\,\varphi(Z_+) + b^{-1}\,\varphi(Z_-)\end{pmatrix} \equiv g\,\Phi(Z_+,Z_-)\,. \tag{21}$$
Assuming $R > 0$, we have
$$\frac{\partial F}{\partial Z} = g\,\frac{\partial \Phi}{\partial Z} = g\begin{pmatrix} a\,\varphi'(Z_+) & a^{-1}\,\varphi'(Z_-) \\ b\,\varphi'(Z_+) & b^{-1}\,\varphi'(Z_-)\end{pmatrix}\,,$$

and in particular, $F(0,0,0) = 0$ and $\frac{\partial F}{\partial Z}(0,0,0) = 0$. The holomorphic implicit function theorem (see *e.g.* [13], Appendix B.5 and references therein) implies that the fixpoint equation (19) has a unique holomorphic solution $Z_\pm(g)$ in a neighborhood of $g = 0$. Let g_0 be the radius of convergence of the Taylor series of $Z_+(g)$. Since the Taylor coefficients of Z_+ are non-negative, g_0 is the singularity of Z_+ closest to 0. Setting

$$Z_+(g_0) = \lim_{g \nearrow g_0} Z_+(g)$$

it is easy to see that $Z_+(g_0) < +\infty$ and that $g_0 < \infty$ also equals the radius of convergence for the Taylor series of $Z_-(g)$.

In the following, we make the *genericity assumption*

$$Z_\pm(g_0) < R\,. \tag{22}$$

It should be noted that, for $h = 0$, the value of $Z_+(g_0) = Z_-(g_0)$ is independent of β and (22) reduces to the genericity assumption of [10] for the underlying random tree.

Assuming (22), the implicit function theorem gives

$$\det\left(\mathbb{1} - g_0\, \Phi_0'\right) = 0\,, \tag{23}$$

where

$$\Phi_0' = \Phi'\left(Z_+^0, Z_-^0\right) = \begin{pmatrix} a\,\varphi'\left(Z_+^0\right) & a^{-1}\,\varphi'\left(Z_-^0\right) \\ b\,\varphi'\left(Z_+^0\right) & b^{-1}\,\varphi'\left(Z_-^0\right) \end{pmatrix},$$

with $Z_\pm^0 = Z_\pm(g_0)$. From the expansion of (19) around $Z_\pm^0$ we get

$$\left(\mathbb{1} - g_0\, \Phi_0'\right)\begin{pmatrix} \Delta Z_+ \\ \Delta Z_- \end{pmatrix} = \Delta g\, \Phi_0 + \frac{g}{2}\,\Delta Z\, \Phi_0''\, \Delta Z + O\left(\Delta Z^3, \Delta g\, \Delta Z\right)\,, \tag{24}$$

where $\Delta Z_\pm = Z_\pm - Z_\pm^0\,, \Delta g = g - g_0$. By (23), we have

$$\begin{pmatrix} c_1 & c_2 \end{pmatrix}\left(\mathbb{1} - g_0\, \Phi_0'\right) = 0\,,$$

where

$$c_1 = g_0\, b\, \varphi'\left(Z_+^0\right)\,, \qquad c_2 = 1 - g_0\, a\, \varphi'\left(Z_+^0\right)\,. \tag{25}$$

Together with the first equation in (19), this gives

$$(\Delta Z_\pm)^2 = -K_\pm\, \Delta g + o(\Delta g)\,, \tag{26}$$

where the constants $K_\pm$ (depending only on β and h) are given by

$$K_+ = \alpha^2\, \frac{2}{g_0}\, \frac{\alpha\, a\, \varphi\left(Z_+^0\right) + b^{-1}\, \varphi\left(Z_-^0\right)}{\alpha^3\, a\varphi''\left(Z_+^0\right) + b^{-1}\, \varphi''\left(Z_-^0\right)} \tag{27}$$

and

$$K_- = \frac{2}{g_0} \frac{\alpha\, a\, \varphi\left(Z_+^0\right) + b^{-1}\, \varphi\left(Z_-^0\right)}{\alpha^3\, a\varphi''\left(Z_+^0\right) + b^{-1}\, \varphi''\left(Z_-^0\right)}\,, \tag{28}$$

where

$$\alpha = \frac{g_0\, a^{-1}\, \varphi'\left(Z_-^0\right)}{1 - a\, g_0\, \varphi'(Z_+^0)}\,.$$

This proves that $Z_\pm(g)$ has a square root branch point at $g = g_0$ in the disc $\{g \mid |g| \leq g_0\}$. Making further use of the implicit function theorem it can be shown that $Z_\pm(g)$ have extensions to a so-called Δ-domain, as described by the following proposition.

Proposition 3.1. *Suppose the greatest common divisor of $\{n \mid p_n > 0\}$ is 1. Then the functions $Z_\pm(g)$ can be analytically extended to a domain*

$$D_{\epsilon,\vartheta} = \{|z| < g_0 + \epsilon,\ z \neq g_0,\ |\arg(z - g_0)| > \vartheta\} \tag{29}$$

and (26) holds in $D_{\epsilon,\vartheta}$, for some $\epsilon, \vartheta > 0$.

This result allows us to use a standard transfer theorem [13] to determine the asymptotic behaviour of $Z_{N\pm}(\beta, h)$ for $N \to \infty$. We state it as follows.

Corollary 3.2. *We have*

$$Z_{N\pm}(\beta, h) = \frac{1}{2}\sqrt{\frac{g_0 K_\pm}{\pi}}\, g_0^{-N}\, N^{-3/2}(1 + o(1)) \tag{30}$$

for $N \to \infty$, where g_0, $K_\pm > 0$ are determined by (19), (23), (27) and (28).

3.2. The measure on the set of infinite trees

For $1 \leq N < \infty$ and fixed β, $h \in \mathbb{R}$ we define the probability measures μ_N, $\mu_{N\pm}$ on Λ_N, $\Lambda_{N\pm} \subset \Lambda$ by

$$\mu_N(\tau_s) \;=\; Z_N(\beta, h)^{-1} e^{\beta \sum\limits_{\langle ij \rangle} s_i s_j + h \sum\limits_{i \in V(\tau)\setminus r} s_i} \rho(\tau)\,, \tag{31}$$

$$\mu_{N\pm}(\tau_s) \;=\; Z_{N\pm}(\beta, h)^{-1} e^{\beta \sum\limits_{\langle ij \rangle} s_i s_j + h \sum\limits_{i \in V(\tau)\setminus r} s_i} \rho(\tau)\,, \tag{32}$$

such that

$$\mu_N = \frac{Z_{N+}}{Z_N}\mu_{N+} + \frac{Z_{N-}}{Z_N}\mu_{N-}\,.$$

Here, $\rho(\tau)$ is given by (17).

As a generalization of (2) we introduce on Λ the metric d by

$$d\left(\tau_s, \tau'_{s'}\right) = \inf\left\{\frac{1}{R+1} \mid B_R(\tau) = B_R(\tau'),\ s|_{B_R(\tau)} = s'|_{B_R(\tau')},\ R \geq 0\right\}. \tag{33}$$

Then (Λ, d) is a separable metric space. Using a general argument concerning weak convergence of probability measures on metric spaces (see *e.g.* [4]), together with some combinatorics, one can prove the following result.

Theorem 3.3. *Assume (22) holds and that the greatest common divisor of* $\{n \mid p_n > 0\}$ *is 1. Then, for all* $(\beta, h) \in \mathbb{R}^2$*, the limits*

$$\mu_\pm = \lim_{N\to\infty} \mu_{N\pm} \qquad \text{and} \qquad \mu = \lim_{N\to\infty} \mu_N \tag{34}$$

exist as probability measures on Λ *and*

$$\mu = \frac{\sqrt{K_+}}{\sqrt{K_+} + \sqrt{K_-}}\,\mu_+ + \frac{\sqrt{K_-}}{\sqrt{K_+} + \sqrt{K_-}}\,\mu_- \,. \tag{35}$$

In particular, introducing the notation

$$A(\tau_0, s_0) = \{\tau_s \mid B_R(\tau) = \tau_0,\ s|_{\tau_0} = s_0\},$$

where τ_0 is a finite tree of height R with spin configuration s_0, we find that the volume of this set is given by

$$\mu_\pm(A(\tau_0, s_0)) = \frac{g_0^{|\tau_0|}}{\sqrt{K_\pm}}\, e^{-H_{\tau_0}(s_0)} \sum_{i=1}^{M} \sqrt{K_{s_0(v_i)}}\varphi'\left(Z^0_{s_0(v_i)}\right) \prod_{j\neq i} \varphi\left(Z^0_{s_0(v_j)}\right)$$

if $s_0(r) = \pm 1$ and where $v_1, \ldots, v_M$ are the vertices at maximal distance from the root in τ_0.

The following corollary provides a complete description of the limiting measures $\mu_\pm$.

Corollary 3.4. *The measures* $\mu_\pm$ *are concentrated on the sets*

$$\bar{\Lambda}_\pm = \{\tau_s \in \Lambda_\pm \mid \tau \text{ has a single spine}\},$$

respectively, and can be described as follows:

1. *The probability that the spine vertices* $u_0 = r, u_1, u_2, \ldots, u_N$ *have* $k'_1, \ldots, k'_N$ *left branches and* $k''_1, \ldots, k''_N$ *right branches and spin values* $s_0 = \pm 1, s_1, s_2, \ldots, s_N$*, respectively, equals*

$$\begin{aligned}&\rho^{\pm}_{k'_1,\ldots,k'_N,k''_1,\ldots,k''_N}(s_0, \ldots, s_N)\\ &= g_0^N e^{\beta \sum_{i=1}^N s_{i-i}s_i + h\sum_{i=1}^N s_i} \left(\prod_{i=1}^N \left(Z^0_{s_i}\right)^{k'_i+k''_i} p_{k'_i+k''_i+1}\right)\sqrt{\frac{K_{s_N}}{K_\pm}}\,. \end{aligned} \tag{36}$$

2. *The conditional probability distribution ν_{s_i} of any finite branch τ_s at a fixed u_i, $1 \leq i \leq N$, given $k'_1, \ldots, k'_N$, $k''_1, \ldots, k''_N$,$s_0, \ldots, s_N$ as above, is given by*

$$\nu_{s_i}(\tau_s) = \left(Z^0_{s_i}\right)^{-1} g_0^{|\tau|}\, e^{-H_\tau(s)} \prod_{v\in\tau\setminus u_i} p_{\sigma_v - 1} \tag{37}$$

for $s(u_i) = s_i$, and 0 otherwise.

3. *The conditional distribution of the infinite branch at u_N, given $k_1, \ldots,$ k_N, $k''_1, \ldots, k''_N$, $s_0, \ldots, s_N$, equals μ_{s_N}.*

3.3. Absence of spontaneous magnetization

Write $\mu^{(\beta,h)}_\pm$, $\mu^{(\beta,h)}$ for $\mu_\pm$, μ and $K_\pm(\beta,h)$ for $K_\pm$.

Theorem 3.5. *Under the assumptions of Theorem 3.3 the probability*

$$\mu^{(\beta,h)}(\{s_0 = +1\}) = \frac{\sqrt{K_+(\beta,h)}}{\sqrt{K_+(\beta,h)} + \sqrt{K_-(\beta,h)}} \tag{38}$$

is a smooth function of β, h. In particular, there is no spontaneous magnetization in the sense that

$$\lim_{h\to 0} \mu^{(\beta,h)}(\{s_0 = +1\}) = \tfrac{1}{2}\,. \tag{39}$$

Sketch of proof. The identity (38) follows from (35).

From Eqs. (27) and (28) it follows that it is sufficient to show that $Z^0_\pm$ are smooth functions of β, h. This can be established by analyzing the system of equations

$$\begin{cases} \left(Z^0_+, Z^0_-\right) = g_0\, \Phi\left(Z^0_+, Z^0_-\right)\,, \\ \det\left(\mathbb{1} - g_0 \Phi'\left(Z^0_+, Z^0_-\right)\right) = 0 \end{cases} \tag{40}$$

determining (Z^0_+, Z^0_-, g_0) implicitly as functions of (β, h).

To verify (39) first note that for $h = 0$ we have $Z^0_+ = Z^0_-$ and $a = b^{-1}$. Hence, from Eq. (23) one finds $\alpha = 1$ and $K_+ = K_-$. Then continuity of (38) obviously implies (39). □

More generally, from (36) it is seen that the distribution of spin variables $s_0, \ldots, s_N$ on the spine can be rewritten as

$$\rho(s_0, \ldots, s_N) = e^{\beta \sum\limits_{i=1}^{N} s_{i-1}s_i + h' \sum\limits_{i=1}^{N} s_i} \left(g_0^2\, \varphi'\left(Z^0_+\right) \varphi'\left(Z^0_-\right)\right)^{N/2} \frac{\sqrt{K_{s_N}}}{\sqrt{K_+} + \sqrt{K_-}}\,,$$

where

$$h' = h + \tfrac{1}{2}\,\ln \frac{\varphi'\left(Z^0_+\right)}{\varphi'\left(Z^0_-\right)}\,.$$

Since $\rho(s_0, \ldots, s_N)$ is normalized, this shows that the expectation value of a function $f(s_0, \ldots, s_{N-1})$ w.r.t. μ equals that of the Ising model on $[0, N]$ with Hamiltonian

$$H_N(s_0, \ldots, s_N) = -\beta \sum_{i=1}^{N} s_{i-1}s_i - h' \sum_{i=1}^{N} s_i - \left(\tfrac{1}{2}\,\ln \alpha\right) s_N\,. \qquad (41)$$

Letting $N \to \infty$ we conclude that μ restricted to functions of the spin variables on the spine equals the Ising model on $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ at inverse temperature β and magnetic field h'. In particular, the mean magnetization vanishes as $h \to 0$ since h' is a smooth function of h by the proof of Theorem 3.5 (see [12]) and since $h' = 0$ for $h = 0$

$$\lim_{h\to 0}\lim_{N\to\infty}\left\langle \frac{s_0 + \cdots + s_{N-1}}{N}\right\rangle_{\mu_{\beta,h}} = 0\,.$$

For the mean magnetization on the full infinite tree we have the following result, which requires some additional estimates in combination with Theorem 3.5. First define the mean magnetization in the ball of radius R around the root by

$$M_R(\beta, h) = \langle |B_R(\tau)|\rangle^{-1}_{\mu_{\beta,h}} \left\langle \sum_{v\in B_R(\tau)} s_v \right\rangle_{\mu_{\beta,h}}$$

and set

$$M(\beta, h) = \limsup_{R\to\infty} M_R(\beta, h)\,.$$

Then the following holds true.

Theorem 3.6. *Under the assumptions of Theorem 3.3 the mean magnetization vanishes for $h \to 0$, i.e.*

$$\lim_{h\to 0} M(\beta, h) = 0\,, \qquad \beta \in \mathbb{R}\,.$$

3.4. Hausdorff and spectral dimension

Next, we give an account of some results on the Hausdorff and spectral dimensions of the ensemble of trees $(\mathcal{T}, \bar{\mu})$ determined by (Λ, μ), where

$$\bar{\mu}(A) = \mu(\{\tau_s \mid \tau \in A\})$$

for $A \subseteq \mathcal{T}$ measurable. Note that the mapping $\tau_s \to \tau$ from $\bar{\Lambda}$ to $\mathcal{T}$ is a contraction w.r.t. the metrics (33) and (2).

The following result is surprisingly easy to establish.

Theorem 3.7. *Under the assumptions of Theorem 3.3 the annealed Hausdorff dimension of $\bar{\mu}$ is 2 for all β, h:*

$$\bar{d}_{\mathrm{h}} = \lim_{R\to\infty} \frac{\ln \langle |B_R| \rangle_\mu}{\ln R} = 2\,.$$

Proof. Consider the probability distribution $\nu_\pm$ on $\{\tau_s \mid \tau \text{ is finite}\}$ given by (37) and denote by $D_R(\tau)$ the number of vertices at distance R from the root in τ. Setting

$$f_R^\pm = \langle D_R \rangle_{\nu_\pm} \, Z_\pm^0$$

one finds that

$$\begin{cases} f_R^+ = g_0 \, \left(a\,\varphi'\left(Z_+^0\right) \, f_{R-1}^+ + a^{-1}\,\varphi'\left(Z_-^0\right) \, f_{R-1}^- \right)\,, \\ f_R^- = g_0 \, \left(b\,\varphi'\left(Z_+^0\right) \, f_{R-1}^+ + b^{-1}\,\varphi'\left(Z_-^0\right) \, f_{R-1}^- \right)\,. \end{cases}$$

In particular,

$$f_R \equiv c_1 \, f_R^+ + c_2 \, f_R^- = f_{R-1} = \cdots = f_1 = c_1 \, Z_+^0 + c_2 \, Z_-^0\,,$$

where c is given by (25), and we conclude that

$$d_1 \leq f_R^\pm \leq d_2\,, \qquad R \geq 1\,,$$

where d_1, d_2 are positive constants (depending on β, h). Using this result and (36) we obtain

$$d_1 \, R \leq \langle |B_R| \rangle_{\nu_\pm} \leq d_2 \, R$$

and

$$\tfrac{1}{2} \, d_1 \, R^2 \leq \langle |B_R| \rangle_\mu \leq \tfrac{1}{2} \, d_2 \, R^2\,.$$

□

A more elaborate argument using ideas from [10] is required to prove the following result for the annealed spectral dimension (see [12]).

Theorem 3.8. *Under the assumptions of Theorem 3.3 the annealed spectral dimension of $(\mathcal{T}, \bar{\mu})$ is*

$$\bar{d}_{\mathrm{s}} = \tfrac{4}{3}\,.$$

4. Conclusions

We have in this article considered two models of random graphs: a model of triangulated sliced planar surfaces and the Ising model in a constant magnetic field on planar random trees. For the former model the spectral dimension is shown to be at most 2. A goal of future work is to obtain effective lower bounds on the spectral dimension for this model as well as for more general models of planar random surfaces.

For the second model we have considered the generic case characterized by the condition (22), for which we have shown, in particular, the absence of spontaneous magnetization. This should be compared with the result of [14] that the Ising model on the UICT exhibits a phase transition. It is an interesting topic for future work to investigate models possessing critical points (β, h) at which the genericity condition (22) is violated and to study features of the corresponding transition.

This work is supported by the Danish National Research Foundation (DNRF) through the Centre for Symmetry and Deformation and by the Danish Agency for Science, Technology and Innovation through the Geometry and Mathematical Physics School (GEOMAPS). One of us (B.D.) thanks Thordur Jonsson and John Wheater for collaboration on parts of the work reported here.

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