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ABOUT KÄHLER QUANTIZATION AND THE CALABI PROBLEM*

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We explain how the Calabi problem on a smooth projective complex manifold can be discussed from the point of view of quantum formalism. We derive from this approach a natural flow on the space of Kähler potentials that has an interpretation in terms of moments maps. Finally, we discuss briefly how such techniques could be adapted to the study of the J-flow.

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1. Introduction

In this paper, we are interested in the Calabi problem from the perspective of the quantum formalism. Let us recall that the Calabi problem consists in finding, given a Kähler class $[\alpha]$ and a volume form Ω with expected total volume (*i.e.* $\int \Omega = \mathrm{Vol}(\alpha)$), a smooth Kähler metric ω in the class $[\alpha]$ which represents this volume form Ω. This means that the Kähler metric ω is a solution to the complex Monge–Ampère equation

$$\frac{\omega^n}{n!} = \Omega\,.$$

It is well known that the existence of a solution to this equation is proved by a famous result of Yau [1] using a continuity method argument. Later, a result of Cao [2] gave another proof using Ricci flow. We refer to [3, 4, 5] as surveys on the proofs of this result. There are still some work in progress in that area, especially when one is considering non-smooth volume forms or non-smooth underlying manifold, motivated by some natural questions related to the minimal model program for complex algebraic manifolds (see the recent progress in [6, 7, 8] for instance).

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We will describe in this paper another flow method to solve the Calabi problem. Details of the proofs will appear in another paper [9] which contains also some extended results. This flow, called the Ω-Kähler flow is natural and comes from the quantum formalism and a natural moment map construction.

We describe now the structure of this paper. In the first section, we give a brief survey of the relationship between Bergman spaces and the space of Kähler potentials via the quantization process. Then we recall some results about balanced metrics mainly due to S.K. Donaldson and we introduce the balancing flow via a moment map approach. We explain the main steps to prove that at the quantum limit, the balancing flow converges towards the Ω-Kähler flow and we discuss the behaviour of this latter flow. Finally, we will address some open questions in the last section.

2. Quantum formalism and the space of Kähler potentials

A classical physical system can be mathematically described as a symplectic manifold M equipped with a symplectic form ω. In that case, an observable on the state space (M,ω) is just a real-valued function on M. From this point of view, quantization consists, on one hand, in associating a Hilbert space $\mathcal{H}(M,\omega)$ to (M,ω), and on another hand, in associating Hermitian operators on $\mathcal{H}(M,\omega)$ to real-valued function on M. Moreover, the quantizations should come in families parametrized by a small parameter $\hbar$ (the "Planck's constant") and in the limit $\hbar \to 0$ the classical setting should emerge from the quantum one, in a suitable sense. See for example [10, 11] for a general survey on quantization.

As shown by F. Berezin, M. Cahen, S. Gutt, J. Rawnsley and others, any positively curved metric h_L on a line bundle $L \to M$ induces a Kähler quantization with $\hbar = 1/k$, where k is a positive integer. If we set $\omega = c_1(h_L)$ the curvature of h_L which is a smooth Kähler form, the quantization (at level k) of (M,ω) is obtained by considering the finite dimensional complex vector space

$$\mathcal{H}(M,\omega) := H^0\left(M, L^{\otimes k}\right) = H^0\left(M, L^k\right)$$

of holomorphic sections of $L^k = L^{\otimes k}$, that can be equipped with the Hermitian metric $\mathrm{Hilb}(h_L^k)$

$$\mathrm{Hilb}_k(h_L)(s,\bar{s}) = \int_M |s|^2_{h_L^k} \frac{\omega^n}{n!}\,.$$

Note that other choices of volume forms in the previous definitions are possible at that stage. To any observable $f \in C^\infty(M,\mathbb{R})$, one can associate the hermitian Toeplitz operator $T_f^{(k)}$ on $H^0(M, L^{\otimes k})$ with symbol f. It can be defined by

$$T_f^{(k)}(u) = P_k(f \cdot u)\,,$$

where $P_k : \mathcal{C}^\infty(M, L^{\otimes k}) \to H^0(M, L^{\otimes k})$ is the orthogonal projection induced by the Hilbert space structure.

Most of the proofs of the results of this theory reduce to understand when $k \to +\infty$ the asymptotic properties of the Bergman kernel $K_k(x, y)$, the integral kernel of the orthogonal projection P_k. These asymptotics can be obtained using the micro-local analysis of Boutet de Monvel–Sjöstrand [12, 13, 14] but other approaches do exist. The asymptotic expansion of the pointwise norm $\rho(h_L)(x) = K_k(x, x)$ given by restricting the kernel to the diagonal (also called the "distortion function" or "Bergman function") is given by

$$\rho(h_L)(x) = k^n \left(1 + k^{-1} b_1(x) + k^{-2} b_2(x) + \dots\right) \tag{1}$$

which holds in the $\mathcal{C}^\infty$-topology and where the coefficients b_i depend polynomially on h_L and its covariant derivatives. Note that one can write explicitly $\rho(h_L)$ using an orthonormal basis $(S_i) = (S_i)_{i=1,\dots,\dim H^0(L^k)}$ of $H^0(M, L^{\otimes k})$ with respect to the L^2 inner product induced by h_L^k and ω^n, as the smooth function

$$\rho_L(x) = \sum_{i=1}^{\dim H^0(L^k)} |S_i|_{h_L^k}(x)$$

(compare with Sec. 3). This asymptotics result and its generalizations have a lots of consequences and we shall quote some of them briefly. For instance, if $\mathrm{Sp}(T_f^{(k)})$ denotes the spectrum of the Toeplitz operator $T_f^{(k)}$ defined above, then, coming back to the work of Boutet de Monvel, Guillemin [12], one obtains at $k \to +\infty$

$$\frac{1}{k^n} \sum_{\lambda_i^{(k)} \in \mathrm{Sp}(T_f^{(k)})} \delta_{\lambda_i^{(k)}} \to \frac{f_*(\omega)^n}{n!}\,. \tag{2}$$

In particular, if we set $f = 1$ and integrate over $\mathbb{R}$, then appears the first terms of the asymptotic Riemann–Roch formula

$$N_k := \dim H^0\left(M, L^k\right) - 1 = k^n \int_M \frac{\omega_\phi^n}{n!} + O\left(k^{n-1}\right) \tag{3}$$

which identifies the leading asymptotics of the dimension of the quantum state space with the volume of the classical phase space.

Another consequence is a nice result of Tian [15] (see also [16] for a heat kernel approach) which shows that the Kähler metric ω in the Kähler class $2\pi c_1(L)$ can be approximated (actually in smooth topology, see [17]) by the pull-back of the Fubini–Study metrics induced by the Kodaira embeddings of $M \hookrightarrow \mathbb{P}H^0(M, L^k)^\vee$ defined by the $\mathrm{Hilb}_k(h_L)$-orthonormal sections. In other words, if one considers the Bergman space $\mathcal{B}_k$ of hermitian operators on $H^0(M, L^k)$ that can be identified with the symmetric space $\mathrm{GL}(N_k+1, \mathbb{C})/\mathrm{U}(N_k+1)$, then for any element of $\mathcal{B}_k$ corresponds a Bergman metric in $2\pi c_1(L)$ by pull-backing the Fubini–Study metric. If one denotes $\mathrm{FS}(B_k)$ the space of Bergman metrics (see Sec. 3 for definitions), then Tian's result asserts that the space of Kähler potential in $2\pi c_1(L)$ is the C^∞ closure of the union of $\mathrm{FS}(B_k)$, $k \gg 0$. This suggests that the space of Kähler potential and the Bergman spaces should enjoy similar geometric properties. This is actually the case, and different results in that directions have been proved. For instance, the geodesics of the space of Kähler potentials can be approximated uniformly by the geodesics in the Bergman space [18, 19]. The underlying motivation behind these results is explained Donaldson's work [20]. Roughly speaking, one expects to prove uniqueness, up to automorphisms, of constant scalar curvature Kähler metrics in $2\pi c_1(L)$, by connecting any given two such metrics by a geodesic segment and by considering the Mabuchi K-energy. Moreover, the geodesic distance on the space of Kähler potentials can also be approximated by the geodesic distance on $\mathcal{B}_k$ (see [21, 22]). This frame of ideas lead J. Fine to study the Calabi flow via the quantum formalism [23], in order to obtain new regularity results. His paper motivated the study of the Calabi problem that we shall present now.

3. The Ω-balanced metrics and the Ω-balancing flow

In this section, we give some definitions and a short survey about Ω-balanced metrics.

Assume that M is a smooth polarized manifold of complex dimension n and L an ample line bundle. We consider Ω a smooth volume form on M such that $\int_M \Omega = \mathrm{Vol}_L(M) := \frac{(2\pi)^n}{n!} c_1(L)^n$, the volume of M with respect to L.

In [24], Donaldson introduced a notion of Ω-balanced metric, adapted to the Calabi problem mentioned previously. His construction is natural from the Geometric Invariant Theory perspective. These Ω-balanced metrics are algebraic metrics coming from the embedding of the manifold in $\mathbb{P}H^0(L^k)^\vee$ for k sufficiently large. Let us be more precise. Given a (smooth) hermitian metric $h \in \mathrm{Met}(L^k)$, one can consider the Hilbertian map associated to a fixed smooth volume form Ω,

$$\mathrm{Hilb}_\Omega = \mathrm{Hilb}_{k,\Omega} : \mathrm{Met}\left(L^k\right) \to \mathrm{Met}\left(H^0\left(L^k\right)\right) ,$$

such that

$$\mathrm{Hilb}_\Omega(h) = \int_M h(.,.)\,\Omega$$

is the L^2 metric induced by the fibrewise h. On another hand, one can consider the Fubini–Study applications

$$\mathrm{FS} = \mathrm{FS}_k : \mathrm{Met}\left(H^0\left(L^k\right)\right) \to \mathrm{Met}\left(L^k\right)$$

such that for $H \in \mathrm{Met}(H^0(L^k))$, S_i an H-orthonormal basis of $H^0(L^k)$ and for all $p \in M$,

$$\sum_{i=1}^{\dim H^0\left(L^k\right)} |S_i(p)|^2_{\mathrm{FS}(H)} = \frac{\dim H^0\left(L^k\right)}{\mathrm{Vol}_L(M)} ,$$

thus fixing pointwise the metric $\mathrm{FS}(H) \in \mathrm{Met}(L^k)$. One of the main result of [24] asserts that the dynamical system

$$T_k = \mathrm{FS} \circ \mathrm{Hilb}_\Omega$$

has a unique attractive fixed point.

Definition 3.1 *Let (M, L) be a polarized manifold, Ω a smooth volume form. Then for any sufficiently large k, there exists a unique fixed point h_k of the map $T_k : \mathrm{Met}(L^k) \to \mathrm{Met}(L^k)$ which is called Ω-balanced. The metric $\mathrm{Hilb}_\Omega(h_k) \in \mathrm{Met}(H^0(L^k))$ and the Kähler form $c_1(h_k) \in 2\pi c_1(L)$, given by the curvature of h_k, will also be called Ω-balanced.*

When k tends to infinity, one obtains from [24] and [25, Theorem 3], the following result.

Theorem 3.2 *When $k \to \infty$, the sequence of normalized Ω-balanced metrics $(h_k)^{1/k} \in \mathrm{Met}(L)$ converges to a hermitian metric h_∞ in smooth topology and its curvature is a solution to the Calabi problem of prescribing the volume in a given Kähler class,*

$$(c_1(h_\infty))^n/n! = \Omega\,.$$

In particular, this theorem provides a way to construct numerical approximations of Calabi–Yau metrics [24].

Let us denote as before $N+1 = N_k + 1 = \dim H^0(L^k)$. Another way of presenting the notion of Ω-balanced metric is to introduce a moment map description. Firstly, let us consider

$$\mu : \mathbb{CP}^N \to i\mathfrak{u}(N+1) \tag{4}$$

the classical moment map for the $\mathrm{U}(N+1)$ action. Then, given an holomorphic embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L^k)^\vee$, we can consider the integral of μ over M with respect to the volume form

$$\mu_\Omega(\iota) = \int_M \mu(\iota(p))\Omega(p)$$

which provides a moment map for the $\mathrm{U}(N+1)$ action over the space of all bases of $H^0(L^k)$. Actually, there is a Kähler structure on that space isomorphic to $\mathrm{GL}(N+1)$, and $\mathrm{U}(N+1)$ acts isometrically with the moment map given by

$$\iota \mapsto -\sqrt{-1}\left(\mu_\Omega(\iota) - \frac{\mathrm{tr}(\mu_\Omega(\iota))}{N+1} Id_{N+1}\right).$$

Note that if one defines a hermitian metric H on $H^0(L^k)$, one can consider an orthonormal basis with respect to H and thus it also makes sense to speak of $\mu_\Omega(H)$. As we shall see, in the Bergman space $\mathcal{B}_k$, we have a preferred metric associated to the volume form Ω and the moment map we have just defined, and this is precisely an Ω-balanced metric.

We say that the embedding ι is Ω-balanced if and only if

$$\mu^0_\Omega(\iota) := \mu_\Omega(\iota) - \frac{\mathrm{tr}(\mu_\Omega(\iota))}{N+1} Id_{N+1} = 0\,.$$

An Ω-balanced embedding corresponds (up to $\mathrm{SU}(N+1)$-isomorphisms) to an Ω-balanced metric $\iota^*\omega_{\mathrm{FS}}$ by pull-back of the Fubini–Study metric from $\mathbb{P}H^0(L^k) = \mathbb{P}^N$, so our two definitions actually coincide. Note that for $H \in \mathrm{Met}(H^0(L^k))$, it also makes sense to consider $\mu_\Omega(h)$, where $h = \mathrm{FS}(H) \in \mathrm{Met}(L^k)$, *i.e.* when h belongs to the space of *Bergman* type fibrewise metric that we identify with $\mathcal{B}_k$.

On another hand, seen as a hermitian matrix, $\mu^0_\Omega(\iota)$ induces a vector field on $\mathbb{CP}^N$. Thus, inspired from [23], we study the following flow

$$\frac{d\iota(t)}{dt} = -\mu^0_\Omega(\iota(t)) \tag{5}$$

and we call this flow the Ω-balancing flow. To fix the starting point of this flow, we choose a Kähler metric $\omega = \omega(0)$ and we construct a sequence of hermitian metrics $h_k(0)$ such that $\omega_k(0) := c_1(h_k(0))$ converges smoothly to $\omega(0)$ providing a sequence of embeddings $\iota_k(0)$ for $k \gg 0$. Such a sequence of embeddings is known to exist thanks to Tian–Bouche's theorem mentionned in Sec. 2. For technical reasons coming from the asymptotics expansion, we need to rescale this flow by considering the following ODE

$$\frac{d\iota_k(t)}{dt} = -k\mu^0_\Omega(\iota_k(t)) \tag{6}$$

that we call the rescaled Ω-balancing flow. Of course, we are interested in the behavior of the sequence of Kähler metrics $\omega_k(t) = \frac{1}{k}\iota_k(t)^*(\omega_{\mathrm{FS}})$ when t and k tends to infinity. In the paper, we give a brief overview of the techniques used to prove the following result.

Theorem 3.3 *For any fixed t, the sequence $\omega_k(t)$ converges in C^∞ topology to the solution $\omega + \sqrt{-1}\partial\bar\partial\phi_t$ of the following Monge–Ampère equation*

$$\frac{\partial\phi_t}{\partial t} = 1 - \frac{\Omega}{(\omega + \sqrt{-1}\partial\bar\partial\phi_t)^n/n!} \tag{7}$$

with $\phi_0 = 0$ and $\omega = \lim_{k\to\infty}\omega_k(0)$. Furthermore, the convergence is C^1 in the variable t.

We call the flow given by Eq. (7), the Ω-Kähler flow.

Firstly, we shall identify the limit of a convergent sequence of rescaled Ω-balancing flows (Sec. 4), that we shall call the Ω-Kähler flow. Then we explain the behavior of the Ω-Kähler flow in any Kähler class (see Sec. 5). Finally, inspired from the work of [26] and especially [23] for the Calabi flow, we explain the main steps to obtain Theorem 3.3 in Sec. 6. Later, we draw some possible generalizations of this work.

4. Study of the limit of the rescaled Ω-balancing flow

In this section, we assume that the sequence $\omega_k(t)$ is convergent and we want to relate its limit to Eq. (7).

Given a matrix H in $\mathrm{Met}(H^0(L^k))$, we obtain a vector field X_H which induces a perturbation of any embedding $\iota : M \hookrightarrow \mathbb{P}H^0(L^k)^\vee$. The induced infinitesimal change in $\iota^*\omega_{\mathrm{FS}}$ is pointwisely given by the potential $\mathrm{tr}(H\mu)$, where μ is given by (4). Thus, the corresponding potential in the case of the rescaled Ω-balancing flow is $-k\mathrm{tr}(\mu^0_\Omega\mu)$. Since we are rescaling the flow in (6) and considering forms in the class $2\pi c_1(L)$, we are lead to understand the asymptotic behavior when $k \to \infty$ of the potentials

$$\beta_k = -\mathrm{tr}\left(\mu^0_\Omega\mu\right) . \tag{8}$$

We need an asymptotics expansion at that stage. The following technical result can be proved with similar arguments to Tians and Bouche's theorem [16, 15] and we refer to [27, Theorem 4.1.1 (with notation 1.4.18)]) or [28, Theorem 4.1] for a detailed proof.

Proposition 4.1 *Let (M, L) be a projective polarized manifold, $h \in \mathrm{Met}(L)$ such that its curvature $c_1(h) = \omega > 0$ is a Kähler form. Assume that Ω is a volume form then we have the following asymptotic expansion for $k \to \infty$*

$$\sum_{i=1}^{N+1} |S_i|^2_{h^k} = k^n \frac{\omega^n}{\Omega} + O\left(k^{n-1}\right), \tag{9}$$

where $(S_i) \in H^0(M, L^k)$ is an orthonormal basis with respect to $\mathrm{Hilb}_\Omega(h^k)$.

We have the following consequence.

Proposition 4.2 *Let $h_k \in \mathrm{Met}(L^k)$ be a sequence of metrics such that $\omega_k := \frac{1}{k} c_1(h_k)$ is convergent in smooth topology to the Kähler form ω. Then the potentials $\beta_k = -\mathrm{tr}(\mu^0_\Omega \mu)$ converge in smooth topology to the potential*

$$1 - \frac{\Omega}{\omega^n}.$$

Proof. Let us give a sketch of the proof. By the discussion at the beginning of Sec. 4, we can write the balancing potential $\beta_k(H_k)$ at $p \in M$. Now, the main ingredient of the proof is given by [23, Theorem 26] and [29]. Actually, we understand the asymptotic behavior of the quantification operator

$$Q_k(f)(p) = \frac{1}{k^n} \int_M \sum_{a,b} \langle S_a, S_b\rangle(q) \langle S_a, S_b\rangle(p) f(q) \omega_k^n(q), \tag{10}$$

where (S_i) is an orthonormal basis. Precisely, it is known by a result of K. Liu and X. Ma that $\|Q_k(f) - f\|_{C^m} \leq \frac{C}{k} \|f\|_{C^m}$ for an independent constant $C > 0$. Then, for $k \to \infty$, one obtains

$$\beta_k(H_k)(p) = 1 - \frac{\Omega}{\omega_k^n} Q_k \left(1 + O\left(\frac{1}{k}\right)\right).$$

The convergence of $Q_k(1+O(\frac{1}{k}))$ to $1+O(1/k)$ is proved in [23, pp. 10–11]. Remark that the previous computation shows that we need to consider the rescaled balancing flow instead of the flow defined by (5).

Here is the main result of this section which identifies the limit of the sequence of rescaled Ω-balancing flows for $k \to +\infty$. It is a simple consequence of Proposition 4.2 and a 1-parameter version of Bouche and Tian's result [16, 15].

Theorem 4.3 *Suppose that for each $t \in \mathbb{R}_+$, the metric $\omega_k(t)$ induced by Eq. (6) converges in smooth topology to a metric ω_t and, moreover, that this convergence is C^1 in $t \in \mathbb{R}_+$. Then the limit ω_t is a solution to the flow (7) starting at $\omega_0 = \lim_{k\to\infty} \omega_k(0)$.*

5. Study of the Ω-Kähler flow

5.1. Existence

We are now interested in the flow

$$\frac{\partial \phi_t}{\partial t} = 1 - \frac{\Omega}{\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t\right)^n / n!} \tag{11}$$

over a compact Kähler manifold (not necessarily in an integral Kähler class), where $\phi_0 = 0$ and ω is a Kähler form in a fixed class $[\alpha]$. Of course, this can be rewritten as

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t\right)^n = \frac{1}{1 - \frac{\partial \phi_t}{\partial t}}\, e^f \omega^n \tag{12}$$

where f is a smooth (bounded) function defined by $f = \log(\Omega/\omega^n)$. Long time existence and convergence of this flow can be studied following the ideas of [2]. Note that we have been informed that similar results were proved recently in [30] after we wrote this article and we want to thank Prof. Z. Blocki for this reference. The main tool to obtain *a priori* estimates is the maximum principle, Nash–Moser's iterations techniques (for the C^0 estimate) and Schauder regularity theory. Finally, we obtain existence for all time. To prove the convergence of the Ω-Kähler flow, one can use some results of Li and Yau for the positive solution of the heat equation on Riemannian compact manifolds [31, Sec. 2] which are still valid in that context. Then, we derive

Theorem 5.1 *Let us denote $v_t = \phi_t - \frac{1}{\mathrm{Vol}_L(M)} \int_M \phi_t \frac{\omega^n}{n!}$, where ϕ_t is solution to Eq. (12), the Ω-Kähler flow. Then, v_t converges when $t \to \infty$ to v_∞ in smooth topology and $\frac{\partial \phi_t}{\partial t}$ converges to a constant in smooth topology.*

A direct consequence of Theorem 5.1 is the convergence of the Ω-Kähler flow to the solution of the Calabi conjecture. Actually, the limit v_∞ satisfies

$$\left(\omega + \sqrt{-1}\partial\bar{\partial}v_\infty\right)^n / n! = \left(\omega + \sqrt{-1}\partial\bar{\partial}\phi_\infty\right)^n / n! = \Omega\,.$$

6. Proof of Theorem 3.3

6.1. First order approximation

We know that from any starting point $\omega = \omega_0$, there exists a solution $\omega_t = \omega + \sqrt{-1}\partial\bar{\partial}\phi_t$ to the Ω-Kähler flow from the results of Sec. 5. We can write $\omega_t = c_1(h_t)$, where h_t is a sequence of hermitian metrics on the line bundle L. Furthermore, we can construct a natural sequence of Bergman metrics $\hat{h}_k(t) = \mathrm{FS}(\mathrm{Hilb}_\Omega(h_t^k))^{1/k}$ by pulling back the Fubini–Study metric using sections which are L^2-orthonormal with respect to the inner product

$$\frac{1}{k^n}\int_M h_t(.,.)^k \Omega\,.$$

Using Proposition 4.1 we obtain the asymptotic behavior for $k \gg 0$, $\hat{h}_k(t) = \left(\frac{k^n c_1(h_t)^n/n!}{\Omega} + O\left(\frac{1}{k}\right)\right)^{1/k} h_t$. Thus, the sequence $\hat{h}_k(t)$ is convergent when k tends to infinity to h_t.

On another hand, the rescaled Ω-balancing flow provides a sequence of metrics $\omega_k(t) = c_1(h_k(t))$ solution to (6). Note that by construction, we fix $h_k(0) = \hat{h}_k(0)$ for the starting point of the rescaled Ω-balancing flow.

We wish to evaluate the distance between the two metrics $h_k(t)$ and $\hat{h}_k(t)$. Since we are dealing with algebraic metrics, we have the (rescaled) metric on Hermitian matrices given by

$$d_k(H_0, H_1) = \left(\frac{\mathrm{tr}\,(H_0 - H_1)^2}{k^2}\right)^{1/2}$$

on $\mathrm{Met}(H^0(L^k))$ which induces a metric on $\mathrm{Met}(L)$, that we denote dist_k. Using arguments similar to [23, Proposition 10] together with convergence of the balancing flow to the Ω-balanced metric [24], we derive

Proposition 6.1 *One has* $\mathrm{dist}_k(h_k(t), \hat{h}_k(t)) \leq \frac{C}{k}$, *with* $C > 0$ *independent of* k.

6.2. Higher order approximation

One can improve the result of the last section by constructing a new time-dependent function $\psi(k,t) = \phi_t + \sum_{j=1}^{m} \frac{1}{k^j}\eta_j(t)$ which is obtained by deforming the solution to the Ω-Kähler flow and which satisfies the property to be "as close" as we wish to the Ω-balancing flow. We will need to compare this metric to the Bergman metric $h_k(t)$. Thus, we introduce the Bergman metric associated to $h_0 e^{\psi(k,t)}$, *i.e.*

$$\overline{h}_k(t) = \mathrm{FS}\left(\mathrm{Hilb}_\Omega\left(h_0^k e^{k\psi(k,t)}\right)\right)^{1/k}\,.$$

We wish to minimize the quantity $\mathrm{dist}_k(\overline{h}_k(t), h_k(t))$ by showing an estimate of the form $\mathrm{dist}_k(\overline{h}_k(t), h_k(t)) < C/k^{m+1}$, with $C > 0$ a constant independent of $k \gg 0$ and t. This is the parameter version of [26, Theorem 26], and Proposition 6.1 shows that the result holds for $m = 0$. One needs to choose inductively the functions η_j and this is done by linearizing the Monge–Ampère operator. The key ingredient is that we are able to invert the second order operator

$$\frac{\Omega}{\omega_t^n}\Delta_t - \frac{\partial}{\partial t}\,.$$

Theorem 6.2 *Given ϕ_t solution to the Ω-Kähler flow (7) and $k \gg 0$, there exist functions $\eta_1, \ldots, \eta_m$, for $m \geq 1$, such that the deformation of ϕ_t given by the potential $\psi(k,t) = \phi_t + \sum_{j=1}^m \frac{1}{k^j}\eta_j(t)$ satisfies*

$$\mathrm{dist}_k\left(h_k(t), \overline{h}_k(t)\right) \leq \frac{C}{k^{m+1}}$$

for $C > 0$ is independent of (k,t).

6.3. L^2 estimates in finite dimensional set-up

We start this section by fixing some notations and giving some definitions. Let us fix a reference metric $\omega_0 \in 2\pi c_1(L)$ and denote $\tilde{\omega}_0 = k\omega_0$ the induced metric in $2\pi k c_1(L)$. We need the notion of R-bounded geometry in C^r [26, Sec. 3.2]. The purpose to work with R-bounded metrics is to avoid constants depending on k in the estimates. We say that a metric $\tilde{\omega} \in 2\pi k c_1(L)$ has R-bounded geometry in C^r if $\tilde{\omega} > \frac{1}{R}\tilde{\omega}_0$ and $\|\tilde{\omega} - \tilde{\omega}_0\|_{C^r(\tilde{\omega}_0)} < R$. We say that a basis (S_i) of $H^0(M, L^k)$ is R-bounded if the Fubini–Study metric induced by the embedding of M in $\mathbb{P}H^0(L^k)^\vee$ induced by the (S_i) has R-bounded geometry. Let us fix

$$H_A = \sum_{i,j} A_{ij}(S_i, S_j) = \mathrm{tr}(A\mu) \in C^\infty(M)\,,$$

where $A = (A_{ij})$ is a Hermitian matrix, (S_i) is a basis of $H^0(L^k)$ and $(.,.)$ denotes the fibrewise Fubini–Study inner-product induced by the basis (S_i). This function corresponds to the potential obtained by an A-deformation of the Fubini–Study metric, *i.e.* when one is moving the Fubini–Study metric in an Lie(SU($N+1$)) orbit. Moreover, we denote $\|A\|_{\mathrm{op}} = \max \frac{|A\zeta|}{|\zeta|}$ the operator norm, given by the maximum moduli of the eigenvalues of the hermitian matrix A, and the Hilbert–Schmidt norm $\|A\|^2 = \mathrm{tr}(A^2) = \mathrm{tr}(AA^*) \geq 0$. The following result is very general.

Proposition 6.3 ([26, Lemma 24], [23, Proposition 12]) *There exists $C > 0$ independent of k, such that for any basis (S_i) of $H^0(L^k)$ with R-bounded geometry in C^r and any hermitian matrix A,*

$$\|H_A\|_{C^r} \leq C\|\mu_\Omega(\iota)\|_{\mathrm{op}}\|A\|\,,$$

where ι is the embedding induced by (S_i).

A consequence is the following corollary.

Corollary 6.4 *Let us fix $r \geq 2$. Assume that for all $t \in [0, T]$, the family of basis $\{(S_i)_{i=1,..,N_k+1}\}(t)$ of $H^0(L^k)$ have R-bounded geometry. Let us define by $h(t)$ the family of Bergman metrics induced by $\{(S_i)\}(t)$. Then, the induced family of Fubini–Study metrics $\tilde{\omega}(t)$ satisfy*

$$\|\tilde{\omega}(0) - \tilde{\omega}(T)\|_{C^{r-2}} < C \sup_t \|\mu_\Omega(\iota(t))\|_{\mathrm{op}} \int_0^T \mathrm{dist}(h(s), h(0))ds\,,$$

where C is a uniform constant in k.

6.4. Projective estimates

This is the technical part of the proof and we will refer to [9] for the details. In this section, we aim to control the operator norm of the moment map in terms of the Riemannian distance in the Bergman space $\mathcal{B}_k$. The projectives estimates consists essentially in giving an upper bound of $\|H_A\|_{L^2}$ from which we derive the following result.

Proposition 6.5 *Let $b_0, b_1 \in \mathcal{B}_k$. Then,*

$$\|\mu_\Omega(b_1)\|_{\mathrm{op}} \leq e^{2\mathrm{dist}_k(b_0,b_1)}\|\mu_\Omega(b_0)\|_{\mathrm{op}}\,.$$

6.5. End of the proof

Using the results of the previous sections, we are now ready to give a sketch of the proof of Theorem 3.3, that is to show the smooth convergence of Kähler metrics $\omega_k(t)$ involved in the rescaled balancing flow (6) towards the solution ω_t to the Ω-Kähler flow. Using Theorem 6.2, for any $m > 0$, we have obtained a sequence of Kähler metrics $\omega(k;t) = c_1(h_0 e^{\psi(k,t)})$ such that $\omega(k;t)$ converges, when $k \to +\infty$ and in smooth sense, towards the solution to the Ω-Kähler flow $\omega_t = c_1(h_0 e^{\phi_t})$. Moreover, one has for k large enough and with $\overline{h}_k(t) \in \mathcal{B}_k$ the Bergman metric associated to $h_0 e^{\psi(k,t)} \in \mathrm{Met}(L)$, the estimate

$$\mathrm{dist}_k(h_k(t), \overline{h}_k(t)) \leq \frac{C}{k^{m+1}}\,, \tag{13}$$

where $h_k(t)$ is the metric induced by the rescaled Ω-balancing flow. Consequently, to get the C^0 convergence in t, all what we need to show is that

$$\left\|\omega_k(t) - c_1\left(\overline{h}_k(t)\right)\right\|_{C^r(\omega_t)} \to 0\,. \tag{14}$$

The idea is to consider the geodesic in the Bergman space between these two points. Firstly, we will get that along the geodesic from $\overline{h}_k(t)$ to $h_k(t)$ in $\mathcal{B}_k$, $\|\mu_\Omega\|_{\mathrm{op}}$ is controlled uniformly if we can apply Proposition 6.5. This requires to prove that $\overline{h}_k(t)$ is at a uniformly bounded distance of $h_k(t)$ and that $\|\mu_\Omega(\overline{h}_k(t))\|_{\mathrm{op}}$ is bounded in k. But, this comes essentially from inequality (13) and the fact that one can choose precisely $m \geq n+1$.

Secondly, one needs to show that the points along this geodesic have R-bounded geometry. This can be proved by updating [23, Lemma 9].

Thirdly, we are exactly under the conditions of the Corollary 6.4. Thus, it gives, after normalisation of the metrics and with (13), that

$$\left\|\omega_k(t) - c_1\left(\overline{h}_k(t)\right)\right\|_{C^r(\omega_t)} \;\leq\; C\left\|\mu_\Omega\left(\overline{h}_k(t)\right)\right\|_{\mathrm{op}} k^{n+2-m-1+r/2}\,,$$

where we have used that the geodesic path from 0 to 1 is just a line. Here, $C > 0$ is a constant that does not depend on k. If we choose $m > r/2 + 1 + n$, we get the expected convergence in C^r topology, *i.e.* inequality (14). Of course, this reasoning works to get the uniform C^0 convergence in t for $t \in \mathbb{R}_+$, because all the Kähler metrics ω_t that we are using are uniformly equivalent (we have convergence of the Ω-Kähler flow, thanks to Theorem 5.1).

A refinement of the ideas above allows us to prove that one has C^1 convergence in t of the flows $\omega_k(t)$, and this is actually sharp. This completes the proof of Theorem 3.3 and we refer to [9] for details.

7. Open questions

One can ask if the main results of this paper hold at least partially when one considers non ample classes or degenerate volume forms. Since a notion of balanced metric for L^p volume forms (and even more general) has been studied in details in the recent work [32, Sec. 7], we expect the long time existence and convergence of the Ω-Kähler flow when the volume form Ω is L^p ($p > 1$), and semi-positive. This is certainly related to the techniques developed by Kolodziej in his generalization of the Calabi problem [33].

We also expect that the ideas of this paper can be applied to the J-flow. Let us recall that the J-flow is a parabolic flow of Kähler potentials defined by Donaldson on manifolds where two Kähler classes have been fixed *a priori* and for which long time existence is proved and convergence is expected under some cohomological assumptions. To be more precise, let us consider as before M a smooth projective manifold, $L, \tilde{L}$ two ample line bundles, $\omega \in 2\pi c_1(L)$ a Kähler form and $\tilde{\omega} \in 2\pi c_1(\tilde{L})$ another Kähler form on M. The J-flow is the flow given by

$$\frac{\partial \phi_t}{\partial t} = \gamma - \frac{\tilde{\omega} \wedge \left(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t\right)^{n-1}}{\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi_t\right)^{n}}\,, \tag{15}$$

where γ is a topological constant, given by $\gamma = \frac{\int_M \tilde{\omega}\wedge\omega^{n-1}}{\int_M \omega^n}$. Donaldson in [34] showed that a necessary condition to have the existence of a solution of the limit of the J-flow, *i.e.* of the equation

$$\tilde{\omega} \wedge \left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^{n-1} = \gamma\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^{n} \tag{16}$$

is that, at the level of the classes, $[n\gamma\omega - \tilde{\omega}] > 0$. An important point from Donaldson's geometric construction is that if one considers $\mathcal{G}$ the group of exact ω-symplectomorphisms, it acts on the infinite dimensional manifold $\mathcal{M}$ of diffeomorphisms $f : M \to M$ homotopic to the identity. This provides, with respect to a certain symplectic form $\mathcal{M}$ depending on $(\omega, \tilde{\omega})$, a moment map in this infinite dimensional setup. The zero of this moment map corresponds precisely to the (unique) solution of Eq. (16) and the J-flow to its gradient flow.

Similarly to what we did in Sec. 3, we define the map $\mathrm{Hilb}_{\tilde{\omega}} = \mathrm{Met}(L^k) \to \mathrm{Met}(H^0(L^k))$ by

$$\mathrm{Hilb}'_{\tilde{\omega}}(h) = \frac{1}{\gamma}\int_M h(.,.)\,\tilde{\omega}\wedge c_1(h)^{n-1}\,.$$

Also, we can define a J-balanced metric as a fixed point of

$$T_{k,\tilde{\omega}} = \mathrm{FS}\circ \mathrm{Hilb}'_{\tilde{\omega}}\,.$$

It is not difficult to check with Proposition 4.1 that if a sequence of J-balanced metrics does exist for $k \gg 0$ and converges, its limit is necessarily a solution to Eq. (16). With (4), we can also define a map on the space of embeddings $\iota : M \hookrightarrow \mathbb{P}H^0(M, L^k)^\vee$

$$\mu_{\tilde{\omega}}(\iota) = \frac{1}{\gamma}\int_M \mu(\iota)\,\tilde{\omega}\wedge(\iota^*(\omega_{\mathrm{FS}}))^{n-1}$$

which is a moment map for the $\mathrm{U}(N+1)$ action. The zeros of the map

$$\sqrt{-1}\left(\mu_{\tilde{\omega}} - \frac{\mathrm{tr}(\mu_{\tilde{\omega}})}{N+1}\mathrm{Id}_{N+1}\right)$$

correspond to J-balanced embeddings.

If one considers Sym_k the normalized kth symmetric function defined on $\mathbb{R}^n$ then on the cone $\{x = (x_1, \ldots, x_n) | x_i > 0, 1 \leq i \leq n\} \subset \mathbb{R}^n$, the function

$$x \mapsto \frac{\mathrm{Sym}_n(x)}{\mathrm{Sym}_{n-1}(x)}$$

is concave [35]. In the case $L = \tilde{L}$, *i.e.* when ω and $\tilde{\omega}$ belong to the same Kähler class, this allows us to study the linearization operator of the application

$$\phi \mapsto \frac{\left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^n}{\tilde{\omega} \wedge \left(\omega + \sqrt{-1}\partial\bar{\partial}\phi\right)^{n-1}}$$

for ϕ strictly ω-plurisubharmonic and smooth. Under these assumptions, we expect that if a solution of (16) does exist, there exists a convergent sequence of J-balanced metrics for $k \gg 0$ that approximate this solution in a similar way to the main theorem of [26]. Finally, we expect that the negative gradient flow of $\mu_{\tilde{\omega}}$ converges when $k \to +\infty$ towards the J-flow (15) up to a renormalisation of the time parameter, and thus a similar result to Theorem 3.3 holds. When no condition holds on the polarisations $(L, \tilde{L})$, we expect that the algebraic notion of J-balanced metric will allow us to obtain new obstructions for the existence of solutions to Eq. (16).

REFERENCES

[1] S.T. Yau, *Comm. Pure Appl. Math.* **31**, 339 (1978).

[2] H.D. Cao, *Invent. Math.* **81**, 359 (1985).

[3] Z. Blocki, `http://gamma.im.uj.edu.pl/ blocki/publ/czest.pdf`

[4] J.-P. Bourguignon, *Lect. Notes Math.* **710**, 1 (1979).

[5] Y.T. Siu, *Lectures on Hermitian-Einstein Metrics for Stable Bundles and Kähler–Einstein Metrics*, vol. 8 of DMV Seminar, Birkhäuser Verlag, Basel 1987.

[6] J. Song, G. Tian, *Invent. Math.* **170**, 609 (2007).

[7] P. Eyssidieux, V. Guedj, A. Zeriahi, *J. Amer. Math. Soc.* **22**, 607 (2009) [`arXiv:math/0603431v2 [math.AG]`].

[8] P. Eyssidieux, V. Guedj, A. Zeriahi, `arXiv:1007.0076v1 [math.AG]`.

[9] H.D. Cao, J. Keller, `arXiv:1102.1097v1 [math.DG]`.

[10] S.T. Ali, M. Engliš, *Rev. Math. Phys.* **17**, 391 (2005).

[11] S. Gutt, *Math. Phys. Stud.* **21**, 217 (2000).

[12] L. Boutet de Monvel, J. Sjöstrand, *Astérisque* **34–35**, 123 (1976).

[13] D. Catlin, *The Bergman Kernel and a Theorem of Tian*, in *Analysis and Geom. in Several Complex Var.*, Katata, 1997, Trends Math. Birkhäuser, 1999.

[14] S. Zelditch, *Asymptotics of Holomorphic Sections of Powers of a Positive Line Bundle*, in *Séminaire sur les Équations aux Dérivées Partielles*, 1997–1998, pages Exp. No. XXII, 12. École Polytech., Palaiseau 1998.

[15] G. Tian, *J. Diff. Geom.* **32**, 99 (1990).

[16] T. Bouche, *Ann. Inst. Fourier (Grenoble)* **40**, 117 (1990).

[17] W.D. Ruan, *Comm. Anal. Geom.* **6**, 589 (1998).

[18] D.H. Phong, J. Sturm, *Invent. Math.* **166**, 125 (2006).

[19] B. Berndtsson, *J. Diff. Geom.* **81**, 457 (2009).

[20] S.K. Donaldson, *Amer. Math. Soc. Transl. Ser. 2* **196**, 13 (1999).

[21] B. Berndtsson, `arXiv:0907.1806v2 [math.DG]`.

[22] X.X. Chen, S. Sun, `arXiv:0902.4149v2 [math.DG]`.

[23] J. Fine, Calabi Flow and Projective Embeddings, to appear in *J. Diff. Geom.* (2009).

[24] S.K. Donaldson, *Pure Appl. Math. Q.* **5**, 571 (2009).

[25] J. Keller, *J. Inst. Math. Jussieu* **8**, 743 (2009).

[26] S.K. Donaldson, *J. Diff. Geom.* **59**, 479 (2001).

[27] X. Ma, G. Marinescu, *Holomorphic Morse Inequalities and Bergman Kernels*, vol. 254 of *Progress in Mathematics*, Birkhäuser Verlag, Basel 2007.

[28] J. Keller, preprint, `http://www.latp.univ-mrs.fr/~jkeller`, 2003.

[29] K. Liu, X. Ma, *Math. Res. Lett.* **14**, 165 (2007).

[30] H. Fang, M. Lai, X. Ma, `arXiv:0904.3288v2 [math.DG]`.

[31] P. Li, S.-T. Yau, *Acta Math.* **156**, 153 (1986).

[32] R. Berman, S. Boucksom, V. Guedj, A. Zeriahi, `arXiv:0907.4490v1 [math.CV]`.

[33] S. Kolodziej, *Acta Math.* **180**, 69 (1998).

[34] S.K. Donaldson, *Asian J. Math.* **3**, 1 (1999).

[35] L. Caffarelli, L. Nirenberg, J. Spruck, *Acta Math.* **155**, 261 (1985).