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THE TRUE GRAVITATIONAL DEGREES OF FREEDOM*

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More than 50 years ago it was realized that General Relativity could be expressed in Hamiltonian form. Unfortunately, just like electromagnetism and Yang–Mills theory, the Einstein equations split into evolution equations and constraints which complicates matters. The 4 constraints are expressions of the gauge freedom of the theory, general covariance. One can cleanly pose initial data for the gravitational field, but this data has to satisfy the constraints. To find the independent degrees of freedom, one needs to factor the initial data by the constraints. There are many ways of doing this. I can do so in such a way as to implement the model suggested by Poincaré for a well-posed dynamical system: Pick a configuration space and give the free initial data as a point of the configuration space and a tangent vector at the same point. Now, the evolution equations should give a unique curve in the same configuration space. This gives a natural definition of what I call the true gravitational degrees of freedom.

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1. Introduction

Following the work of Dirac [1], and Arnowitt, Deser and Misner [2], we know that general relativity can be expressed as a dynamical theory, just like the other standard theories of physics. In GR one can specify initial data and then integrate forward in time. This is more complicated than, say, particle mechanics, because of gauge freedom and of the existence of constraints. The obvious models are electromagnetism and Yang–Mills theory which share both features. The gauge freedom allows us to write

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the field equations in a very simple and compact form, while simultaneously obscuring the underlying physics. We need to find a gauge choice and a way of solving the constraints which renders everything transparent.

The ideal is to express gravity in the Poincaré form ([3], Sec. 5), *i.e.*, pick a configuration space, define the initial data as a point and a velocity in the configuration space, and have the evolution equations generate a unique curve in this configuration space. This clean pattern, which amounts to finding the true degrees of freedom of gravity, has eluded all of us until now.

The standard initial data for gravity consists of a giving a space-like 3-slice, equipped with a Riemannian 3-metric, g_{ij} and a symmetric tensor K_{ij}, which is to be the extrinsic curvature of this slice. The extrinsic curvature is the time derivative of the 3-metric. More precisely,

$$\mathcal{L}_{\hat{n}} g_{ij} = 2K_{ij}\,, \tag{1}$$

where $\mathcal{L}$ is the Lie derivative and $\hat{n}$ is the unit time-like normal to the 3-slice. This pair (g_{ij}, K_{ij}) are the analogues of the position and momentum in mechanics. A better analogy is with electromagnetism (or Yang–Mills) when one specifies the 3-vector potential and the electric field as initial data. GR is like electromagnetism in that the 10 Einstein equations split into 6 evolution equations, which propagate the 3-metric, and 4 constraints on the initial data. The constraints, which prevent free specification of g_{ij} and K_{ij}, have to be taken into account. To complicate matters, the constraints can be regarded as the generators of/generated by the gauge freedom, 4-dimensional general covariance. Therefore 'solving the constraints' is, at some level, equivalent to 'fixing the gauge'. For an up-to-date and comprehensive account of the constraints, see [4], especially Chapter VII.

The constraints are

$$^{(3)}R - K^{ij}K_{ij} + K^2 \;=\; 0\,, \tag{2}$$

$$\nabla_j K_i^j - \nabla_i K \;=\; 0\,, \tag{3}$$

known as the Hamiltonian and momentum constraints respectively; $^{(3)}R$ is the scalar curvature of g_{ij} and $K = g_{ij}K^{ij}$ is the trace of the extrinsic curvature. To identify the 'true gravitational degrees of freedom' and the configuration space of general relativity, we need to 'factor' the metric and extrinsic curvature by the constraints.

I am going to restrict my attention to vacuum solutions of the Einstein equations where the space-like 3-slices are compact, without boundary. I do not restrict the topology. I further restrict myself to those solutions which are 'CMC-sliceable', *i.e.*, I assume that each space-time has at least one space-like slice through it on which tr$K$, the trace of the extrinsic curvature, is a constant. I do not care about the value of the constant, even whether it is positive or negative. I will return to this question of the generality of this condition at the end.

The key choice in this analysis is the choice of configuration space. I claim that conformal superspace is a good choice. Conformal superspace is the space of metrics, factored both by all diffeomorphisms and by all conformal transformations.

The idea that conformal superspace (CS) is the natural configuration space for gravity goes back to Lichnerowicz [5], who used a conformal transformation to write the Hamiltonian constraint, Eq. (2), as a nice elliptic equation for the conformal factor. We take the conformal approach significantly further. This article is an extension of the key work on the subject by York [6]. I want to show that I can pick the free initial data for the gravitational field as a point and a and a tangent vector (the velocity) in conformal superspace. Combining the constraints and the Einstein evolution equations generates a vacuum space-time as a unique curve in conformal superspace. Thus, I can express gravity in Poincaré form. The key to this paper is the realization that there exists an extra symmetry in the conformal method of solving the constraints which makes everything work.

In a conformal 3-geometry $\mathcal{C}$ one can view the conformal freedom as being coded into $\sqrt{g}$, where g is the determinant of g_{ij}. I wish to regard $\sqrt{g}$ as gauge, just like the coordinates. If we take the trace of Eq. (1) we can show

$$\mathcal{L}_{\hat{n}}\sqrt{g} = K\sqrt{g}\,. \tag{4}$$

Therefore $(\sqrt{g}, K)$ are canonically conjugate variables. If $\sqrt{g}$ is gauge, so is K.

A key point of this article is that *transverse-traceless* (TT) tensors (a tensor h_{ij}^{TT} is TT if it is both *transverse* ($\nabla^j h_{ij}^{\mathrm{TT}} = 0$) and *traceless* ($g^{ij}h_{ij}^{\mathrm{TT}} = 0$)) are natural objects on CS.

First, TT tensors define the tangent space to CS. Consider two nearby metrics in Riem, g_{ij} and $g_{ij} + \epsilon h_{ij}$, with ϵ a small parameter. Any symmetric tensor, in particular h_{ij}, has a unique TT part with respect to a g_{ij} via [7]

$$h_{ij} = h_{ij}^{\mathrm{TT}} + \nabla_i\lambda_j + \nabla_j\lambda_i - \tfrac{2}{3}\nabla_k\lambda^k g_{ij} + \tfrac{1}{3}h g_{ij}\,; \tag{5}$$

$\nabla_i\lambda_j + \nabla_j\lambda_i - \frac{2}{3}\nabla_k\lambda^k g_{ij}$ is the conformal Killing form of a vector λ^j and $h = g^{ij}h_{ij}$ is the trace of h_{ij}. Now, merge the trace term in the conformal Killing form with h and rewrite the decomposition as

$$h_{ij} = h_{ij}^{\mathrm{TT}} + \nabla_i\lambda_j + \nabla_j\lambda_i + \tfrac{1}{3}\bar{h} g_{ij}\,, \tag{6}$$

where $\bar{h} = h - 2\nabla_k \lambda^k g_{ij}$. We can interpret $\nabla_i \lambda_j + \nabla_j \lambda_i$ (the Killing form) as the change in g_{ij} due to an infinitesimal change of coordinates, and $\bar{h} g_{ij}/3$ as an infinitesimal conformal transformation. These two are the generators of the symmetries we factor out in going from the space of metrics to CS, and merely change the representation in a given equivalence class corresponding to the given g_{ij}. The true perturbation in CS is h_{ij}^{TT}. The set of TT tensors defines the tangent space to CS.

Second, TT tensors are *conformally covariant* [7]. If h_{ij}^{TT} is TT w.r.t. g_{ij} and ξ is any function, then $\xi^{-2} h_{ij}^{\mathrm{TT}}$ is TT with respect to $\xi^4 g_{ij}$. This is just straightforward algebra. The exponent -2 is the power of the conformal factor for a down–down TT tensor, it is -6 for up–down and -10 for up–up. Thus $(\xi^4 g_{ij}, \xi^{-2} h_{ij}^{\mathrm{TT}})$ represents the same $\mathcal{C}$ and tangent vector as g_{ij} and h_{ij}^{TT}.

Third, asymptotically flat initial data have a well-defined total energy, the ADM energy. Brill and Deser, in [8], showed that if one made a perturbation expansion in the initial data about flat space, the first non-trivial contribution to the energy came at second order and had the form

$$16\pi \left(\tfrac{1}{2}\delta^2 E_{\mathrm{ADM}}\right) = \int d^3x \left[\tfrac{1}{4}\left(\delta g_{ij,k}^{\mathrm{TT}}\right)^2 + \left(\delta K_{ij}^{\mathrm{TT}}\right)^2\right].$$

This expression is very similar to the Poynting energy expression, $(E2+B2)$, in electromagnetism, and shows that the TT terms are the true excitations of the gravitational field in the weak-field limit. The challenge is to extend this to the strong field/no boundary case.

2. The conformal method

As stressed in the introduction, I am going to confine myself to the situation where the 3-manifold is compact without boundary, and the trace of the extrinsic curvature is a nonzero constant. The standard conformal method, [4], starts with the realization that if the extrinsic curvature has a constant trace then the momentum constraint, Eq. (3), implies that the tracefree part is transverse. Hence

$$K_{ij} = K_{ij}^{\mathrm{TT}} + \tfrac{1}{3} K g_{ij}\,. \qquad (7)$$

The conformal method, and especially the TT decomposition, (5), parallels the standard way of solving the Maxwell constraints, where the standard decomposition of a vector into a transverse part and the gradient of a scalar, $V^i = V_t^i + \nabla^i \phi$, to convert the Maxwell constraint, $\nabla_i D^i = \rho$ into the Poisson equation. We start with a freely specifiable metric and a symmetric tensor, $(\bar{g}_{ij}, \bar{F}_{ij})$, make the decomposition (5), and extract the TT part of F, with respect to the given g_{ij}, which I call $\bar{K}_{ij}^{\mathrm{TT}}$. I now consider the pair $(\bar{g}_{ij}, \bar{K}_{ij}^{\mathrm{TT}})$. I call this pair *the initial data*, considered as a point and tangent vector in CS by considering the equivalence class $(\theta 4\bar{g}_{ij}, \theta^{-2}\bar{K}_{ij}^{\mathrm{TT}})$ for all positive θs .

The standard way is to adjoin any constant (nonzero) K, and obtain the triplet $(\bar{g}_{ij}\,, \bar{K}_{ij}^{\mathrm{TT}}, K)$. Now, seek a conformal factor ϕ which maps this triplet into a new triplet satisfying the constraints via $(g_{ij}\,, K_{ij}^{\mathrm{TT}}, K) = (\phi 4\bar{g}_{ij}\,, \phi^{-2}\bar{K}_{ij}^{\mathrm{TT}}, K)$. Then K_{ij}, constructed as $K_{ij} = K_{ij}^{\mathrm{TT}} + \frac{1}{3}K g_{ij}$, following from Eq. (7), satisfies the momentum constraint, while the Hamiltonian constraint transforms into the Lichnerowicz–York (L–Y) equation

$$8\bar{\nabla}^2\phi - \bar{R}\phi + \bar{K}^{ij}_{\mathrm{TT}}\bar{K}_{ij}^{\mathrm{TT}}\phi^{-7} - \tfrac{2}{3}K^2\phi^5 = 0\,, \tag{8}$$

$\bar{R}$ is the 3-scalar curvature formed from $\bar{g}_{ij}$ (as is $\bar{\nabla}$). Equation (8) always has a unique positive solution $\phi > 0$ [9] as long as $K \neq 0$ and $\bar{K}^{\mathrm{TT}} \neq 0$.

Further, we can transform the initial data with an arbitrary positive function ξ to $(\bar{g}'_{ij}\,, \bar{K}'^{\mathrm{TT}}_{ij}, K') = (\xi 4\bar{g}_{ij}\,, \xi^{-2}\bar{K}_{ij}^{\mathrm{TT}}, K)$. The conformal covariance of the L–Y equation emerges via the fact that when these 'new' data are injected into the L–Y equation the 'new' conformal factor $\phi' = \phi/\xi$! This means that the data we construct to satisfy the constraints $(g'_{ij}\,, K'^{\mathrm{TT}}_{ij}, K') = (\phi'^4\bar{g}'_{ij}\,, \phi'^{-2}\bar{K}'^{\mathrm{TT}}_{ij}, K')$ are identical to the set we got without the transformation with ξ.

This is *almost* good enough: since making an arbitrary conformal transformation changes nothing, $\bar{g}_{ij}$ can be regarded as a point in CS and $\bar{K}_{ij}^{\mathrm{TT}}$ can be regarded as a velocity in CS at that point. However, the need to specify K as an extra initial datum complicates things. We do not have the initial data in Poincaré form. We have an extra free constant K in the initial data and, at best, we get a solution curve in superspace rather than conformal superspace. In fact, we can transform K from a 'free' to an 'auxiliary' variable and thus effectively eliminate it. It turns out that I can do this because the constraints have an extra, unexpected but simple, symmetry.

3. Rescaling freedom

Pick a (positive or negative) constant A. Let (g_{ij}, K_{ij}) solve the constraints. Now transform them as follows: $(\bar{g}_{ij}, \bar{K}_{ij}) = (A^2 g_{ij}, AK_{ij})$. The new data will also satisfy the constraints. Each term in the Hamiltonian constraint picks up a factor of A^{-2} and each term in the momentum constraint is multiplied by A^{-1}.

This symmetry also commutes with the conformal method of constructing solutions to the constraints as follows: Let us take the specified initial data and transform them as follows: pick a constant $A \neq 0$. Construct 'new' initial data (we think of these as 'rescaled' data, the terminology will become clear soon)

$$\left(\bar{g}'_{ij}\,, \bar{K}'^{\mathrm{TT}}_{ij}, K'\right) = \left(A2\bar{g}_{ij}\,, A\bar{K}^{\mathrm{TT}}_{ij}, K/A\right) . \tag{9}$$

Substitute these data into the L–Y equation. One can see that each term in the equation picks up a factor of A^{-2}. Therefore $\phi' = \phi$. Hence this rescaling commutes with the L–Y equation. We can rescale either before or after solving the L–Y equation. We get the same final (rescaled) data satisfying the constraints.

Why 'rescaling'? Take any initial data satisfying the constraints and propagate them. This gives a (patch of) space-time with a space-time 4-metric $g_{\mu\nu}$ satisfying the Einstein equations. If we use geometric units, so that the speed of light $= 1$, then we have only one dimensionful quantity (say 'meters'). Following Dicke [10] I choose to put the dimensions into the metric and consider the coordinates as pure numbers, labels of points. Let us decide to change our units from 'meters' to 'yards'. This is achieved by multiplying the space-time metric by a space-time constant A, *i.e.*, $g_{\mu\nu} \to A^2 g_{\mu\nu}$. This new metric continues to satisfy the Einstein equations. The effect of this rescaling on the $3+1$ data is $(g'_{ij}, K'_{ij}) = (A^2 g_{ij}, AK_{ij})$, or $(g'_{ij}, K'^{\mathrm{TT}}_{ij}, K') = (A^2 g_{ij}, AK^{\mathrm{TT}}_{ij}, K/A)$.

We should stress that this 'rescaling' transformation $(g'_{ij}, K'^{\mathrm{TT}}_{ij}, K') = (A^2 g_{ij}, AK^{\mathrm{TT}}_{ij}, K/A)$ is *not* a subset of the conformal transformations $(\bar{g}'_{ij}\,, \bar{K}'^{\mathrm{TT}}_{ij}, K') = (\xi 4\bar{g}_{ij}\,, \xi^{-2}\bar{K}^{\mathrm{TT}}_{ij}, K)$ mentioned earlier. In one case the solution of the constraints that emerges is rescaled, in the other case the solution is unchanged. We now show that this new extra symmetry means that K does not correspond to an extra physical initial datum in CS but merely to a choice of units in space-time.

I picked the initial data as a metric, g_{ij}, and a TT tensor $(\hat{g}_{ij}, \hat{K}^{\mathrm{TT}}_{ij})$, regarding these as a point and tangent in CS even though we have to work in Riem. Now I pick a constant K_1, which may be positive or negative, but not zero. I think of K_1 as a gauge auxiliary, necessary to implement the procedure, but which can be eliminated at the end. From these I construct 'intermediate' data

$$\left(\bar{g}_{ij}\,, \bar{K}^{\mathrm{TT}}_{ij}, \bar{K}\right) = \left(K_1^{-2}\hat{g}_{ij}\,, K_1^{-1}\hat{K}^{\mathrm{TT}}_{ij}, K_1\right) . \tag{10}$$

These intermediate data are of the standard form, *i.e.*, metric + TT tensor + constant, so I can substitute them into the L–Y equation, Eq. (8), find the solution ϕ_1 and construct data which satisfy the constraints $(g_{ij}\,, K^{\mathrm{TT}}_{ij}, K) = (\phi_1 4\bar{g}_{ij}\,, \phi_1^{-2}\bar{K}^{\mathrm{TT}}_{ij}, \bar{K}_1)$.

Let us now go back and, leaving the initial data unchanged, pick a new constant, K_2, and repeat the construction. We find new intermediate data $(\bar{g}_{ij}, \bar{K}_{ij}^{\mathrm{TT}}, \bar{K}) = (K_2^{-2}\hat{g}_{ij}, K_2^{-1}\hat{K}_{ij}^{\mathrm{TT}}, K_2)$, a new solution ϕ_2 to the L–Y equation, and new solution data satisfying the constraints. What is the relationship between the two sets of solution data? If we look at the two sets of intermediate data we can see that the mapping between them is just a rescaling transformation as introduced earlier. We have $(K_2^{-2}\hat{g}_{ij}, K_2^{-1}\hat{K}_{ij}^{\mathrm{TT}}, K_2) = (A^2 K_1^{-2}\hat{g}_{ij}, A K_1^{-1}\hat{K}_{ij}^{\mathrm{TT}}, K_1/A)$ with $A = K_1/K_2$. This means that $\phi_1 = \phi_2$, and, one of the solutions of the constraints is just a rescaling of the other. Therefore, holding the initial data fixed, and changing the value of K generates solutions of the constraints that are related by rescaling.

4. Curves in conformal superspace

We can see three routes to proceed from this point. The first, and for us least desirable, is to abandon the $3+1$ viewpoint and return to a 4-dimensional picture. Then each set of initial data, $(\hat{g}_{ij}, \hat{K}_{ij}^{\mathrm{TT}})$, will generate a family of space-times (one for each choice of K) which can be mapped into each other by constant rescalings.

The second is to maintain the $3+1$ idea, but live with many-fingered time. From a given set of initial data, we know that the evolution equations generate an infinite family of curves through superspace, each corresponding to a different slicing of the space-time. The family of curves arising from data set 1 is different from the curves from data set 2. However, when the families are mapped into conformal superspace, they coincide.

The third, and the one we favour, is to realise that we have constructed a CMC initial data slice, and that it is very natural to extend this into the space-time as a CMC foliation. Look at Eq. (4), $\mathcal{L}_{\hat{n}}\sqrt{g} = K\sqrt{g}$. This tells us that, on a CMC slice, the fractional time rate of change of the local volume is a constant. Therefore, these CMC slices are the natural 'Hubble time' slices of a cosmology. There always exists a (two-sided) CMC foliation around any given CMC slice. This, and only this, preserves the TT-ness of the extrinsic curvature. To maintain it, we solve the elliptic lapse-fixing equation

$$\nabla^2 N - K^{ij}K_{ij}N = C \tag{11}$$

for the function N, the lapse function; C is some constant, conveniently taken to be $C = -1$. Equation (11) has a unique solution. In addition, if $C < 0$, then $N > 0$ and *vice versa*.

We now evolve (g_{ij}, K_{ij}) with respect to the time label t using

$$\frac{\partial g_{ij}}{\partial t} = 2NK_{ij} + \nabla_i N_j + \nabla_j N_i\,. \tag{12}$$

This is just rewriting Eq. (1) in $3+1$ language. There is also a further equation for $\partial K_{ij}/\partial t$ that we omit; in both we may set the freely specifiable N_j to zero, but must continuously update $N(t, x^k)$ using Eq. (11). This evolution system fixes $\partial K/\partial t = C$ and generates the desired CMC foliation. Thus each choice of initial data with arbitrary, but constant, K, will generate different (rescaled) paths through superspace but will generate the *same* path through conformal superspace.

In this construction, we are locked in to constructing CMC initial data but we are *not* locked into CMC evolution. While the CMC foliation condition is a useful choice of slicing, and one where one can see clearly that one gets a unique curve in conformal superspace, *any* slicing condition which is 'rescalable' works just as well. This means that we need not worry about the fact that the CMC slicing may die before will fill out the whole space-time.

We choose the initial data to be a point and a velocity in conformal superspace, with no specification of the local scale. This local scale emerges when we solve the L–Y equation. To find initial data in Riem or in superspace we also need to specify a 'unit length'. This is why we have to pick a K. However, the solution to the Einstein equations, regarded as a curve in conformal superspace, is independent of the choice of K. We may therefore conclude that conformal superspace is the configuration space for gravity.

While rescaling invariance is a symmetry of general relativity, and therefore one does *not* require that one choose CMC data. However, only for CMC data can one use rescaling invariance to demote K from being a physical to an auxiliary variable, and thus cast the Einstein equations in Poincaré form. This seems to me to be a very attractive feature, attractive enough to restricting the set of space-times we consider to those with a CMC slice. Choosing the CMC condition does impose some restrictions. Every space-time with a single CMC slice has a CMC foliation. This eliminates all space-times with close time-like loops. I do not regret their elimination. I do know that for cosmologies, 'Hubble time = CMC slicing'. I also know that, in the space of metrics, the family of CMC slicable space-times forms an open set. I also can make a counting argument to show that the 'number of CMC initial data sets = number of vacuum space-times'. It could be that many space-times with a CMC foliation possess many different such foliations. Minkowski space is an obvious example. I do not consider this likely, but it may possibly be true. Therefore, I have to restrict my attention to the family of space-times with a CMC slice and I feel that the benefit outweighs any drawback.

I would like to draw the reader's attention to three recently posted articles [16]. These show that in GR one can trade reparametrisation invariance for conformal invariance. We use a concrete version of this trade-off in a form that mimics the classical theories of physics.

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