MODULI IN GEOMETRY AND PHYSICS*

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The concept of moduli is illustrated in several problems from geometry and physics. These problems range from complex geometry to supersymmetric gauge theories, integrable models, and string theory. Some of them are quite classical, but others have emerged only relatively recently, for example in the interplay between complex geometry and two-dimensional supergravity.

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1. Introduction

The moduli space of a smooth surface Σ is the space of complex structures on Σ. The importance of this notion was probably first recognized by Riemann. Since then, it has acquired a broader meaning, as the space of more refined geometric structures of any particular type that can be carried by a topological space. Well-known examples are the moduli space of Calabi–Yau manifolds, the moduli space of stable holomorphic vector bundles, and the moduli space of Yang–Mills connections. Other examples of more recent origin are the moduli space of 2-dimensional supergeometries, and the moduli space of complex structures with Abelian integrals with poles at given points. In general, the global function theory on a manifold will depend on its moduli. A concrete example of this is the familiar Jacobi function $\theta(z|\Omega)$ on a complex torus $X = \boldsymbol{C}/\boldsymbol{Z} + \Omega\boldsymbol{Z}$, where z is the variable on X, and Ω is the moduli parameter. So it is not surprising that moduli play a major role in geometry and function theory. What is perhaps less expected is the preponderance of geometric structures involving complex structures, even in physical problems where no complex structure was built-in at the start. In this sense, moduli theory fit most naturally in complex analysis and complex geometry.

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This article is a contribution to the proceedings of the Conference on Geometry and Physics held at the Jagiellonian University, Kraków, in September 2010. At the conference, the author was assigned the task of providing an informal and broad introduction to a selection of topics of interest in both complex geometry and theoretical physics, and to indicate some open problems. The common theme of moduli seemed particularly appropriate. The occurrence of moduli problems is quite widespread in physics, and for his lectures, the author has necessarily selected topics with which he is most familiar. The present article reflects these choices. The list of references is also necessarily very incomplete, given the vast literature on all the topics touched upon here. It is hoped that the reader can consult the many excellent review papers which are available for a more comprehensive bibliography.

2. Moduli and Riemann surfaces

The prototype of moduli is the moduli space of complex structures on a surface Σ, or moduli space of Riemann surfaces. This moduli space has an amazingly rich and deep structure. For our purposes, we need only the following facts [1, 2, 3, 4].

Let Σ be a smooth compact oriented surface of genus h. From the differential geometric viewpoint, a complex structure on Σ is an equivalence class of metrics $ds^2 = g_{km} dx^m dx^k$, modulo the combined actions of the Weyl group $ds^2 \to e^{2u(x)} ds^2$ and the diffeomorphism group $\mathrm{Diff}(\Sigma)$. The moduli space of complex structures on Σ is given by

$$\mathcal{M}_h = \{\text{metrics}\, g_{ij}\} \,/\text{Weyl} \times \mathrm{Diff}(\Sigma)\,. \tag{2.1}$$

A fundamental theorem is the uniformization theorem, one formulation of which is that any metric g_{km} is Weyl equivalent to a unique metric $\hat{g}_{km}$ of constant scalar curvature $\hat{R} = \pm 1, 0$ (with an additional normalization of area when $h = 1$). Thus we can also write

$$\mathcal{M}_h = \left\{\text{metrics}\, \hat{g}_{ij}; \hat{R} = \pm 1, 0\right\} /\mathrm{Diff}(\Sigma)\,. \tag{2.2}$$

From the complex geometric viewpoint, a complex structure on Σ is a covering of Σ by charts $\Sigma = \cup_\alpha \Sigma_\alpha$, each of which is in correspondence $\Sigma_\alpha \ni z \to z_\alpha$ with an open disk in $\boldsymbol{C}$, and $z_\beta \circ z_\alpha^{-1}$ is a holomorphic invertible map from $z_\beta(\Sigma_\alpha \cap \Sigma_\beta)$ to $z_\alpha(\Sigma_\alpha \cap \Sigma_\beta)$. The equivalence between the two definitions of complex structures follows from the existence of local isothermal coordinates z_α for any metric ds^2, with respect to which we can write $ds^2 = e^{2u_\alpha(z)} dz_\alpha \overline{dz_\alpha}$.

The global structure of the moduli space $\mathcal{M}_h$ is complicated, but its local structure is easy to understand. A complex structure can be viewed as an operator $\partial_{\bar{z}}$, given by $\partial_{\bar{z}} = \partial_{\bar{z}_\alpha}$ in local coordinates. The notion of local holomorphic functions, defined as solutions of the equation

$$\partial_{\bar{z}_\alpha} f = 0\,, \tag{2.3}$$

is independent of the choice of local coordinate systems and hence well-defined. A deformation of complex structures is a deformation of the operator $\partial_{\bar{z}}$ to an operator

$$\partial_{\bar{z}} - \mu_{\bar{z}}{}^{z}\partial_z\,. \tag{2.4}$$

The deformation term corresponds thus to a tensor $\mu = \mu_{\bar{z}}{}^{z} d\bar{z} \otimes \frac{\partial}{\partial z}$, called a Beltrami differential. Under Weyl scalings, the complex coordinate z_α and hence the operator $\partial_{\bar{z}_\alpha}$ does not change, while under local diffeomorphisms, parametrized by a vector field δv^z, it changes by $\partial_{\bar{z}}(\delta v^z)$. In this way, the tangent space $T(\mathcal{M}_h)$, which is the space of infinitesimal deformations of the complex structure defined by g_{mk}, can be identified with the following quotient vector space

$$T(\mathcal{M}_h) = \{\mu_{\bar{z}}{}^{z}\}/\left\{\partial_{\bar{z}}(\delta v^z)\right\}\,. \tag{2.5}$$

This tangent space clearly admits complex multiplication, so it inherits a natural almost-complex structure which can be verified to be integrable. This shows that $\mathcal{M}_h$ is a complex manifold (actually, $\mathcal{M}_h$ does have orbifold singularities, due to the fixed points of the diffeomorphisms not connected to the identity. But we shall ignore this important aspect in this lecture, and discuss only when necessary the closely related issue of modular invariance).

As a compact complex manifold, the Riemann surface Σ does not admit any global holomorphic function besides constants. However, there are many natural holomorphic line bundles on Σ which do admit non-trivial global sections, and these sections play a fundamental role in both geometrical and physical applications. Let $\Lambda^{p,0}(\Sigma)$ be the line bundle of $(p,0)$-forms on Σ. Then the Riemann–Roch theorem gives the dimension of its space of global holomorphic sections

$$\dim H^0\left(\Sigma, \Lambda^{1,0}\right) = h \tag{2.6}$$

and

$$\dim H^0\left(\Sigma, \Lambda^{p,0}\right) = (2p-1)(h-1) \tag{2.7}$$

for $h \geq 2$, and $\dim H^0(\Sigma, \Lambda^{p,0}) = 1$ for $h = 1$. Now, on a surface Σ of genus h, a canonical homology basis (A_I, B_I), $1 \leq I \leq h$ can be chosen, *i.e.*, $\#(A_I \cap A_J) = \#(B_I \cap B_J) = 0$, $\#(A_I \cap B_J) = \delta_{IJ}$. Such a basis determines uniquely a corresponding basis ω_I of $H^0(\Sigma, \Lambda^{1,0})$ dual to the cycles A_I. Their periods around the B_I cycles define the period matrix $\Omega = (\Omega_{IJ})$

$$\oint_{A_I} \omega_J = \delta_{IJ}\,, \qquad \oint_{B_I} \omega_J = \Omega_{IJ}\,. \tag{2.8}$$

The Torelli theorem asserts that the complex structure of Σ is determined by its period matrix. Of particular interest are also sections of $\Lambda^{2,0}$, or quadratic differentials $\phi = \phi_{zz}dz^2$. This space is dual to the space of Beltrami differentials via the canonical pairing

$$\langle \mu, \phi \rangle = \int_{\Sigma} \mu_{\bar{z}}{}^{z}\, \phi_{zz} \tag{2.9}$$

and the quotient space $T(\mathcal{M})$ is dual to $H^0(\Sigma, \Lambda^{2,0})$. It follows, in particular, that $\mathcal{M}_h$ has complex dimension $3h$–3 for $h \geq 2$ and dimension 1 when $h = 1$. In the process, we have also learnt that $\mathcal{M}_h$ comes equipped with two natural holomorphic line bundles, namely the Hodge bundle λ whose fiber λ_Σ at Σ is defined by

$$\lambda_\Sigma = \wedge^{\max} H^0\left(\Sigma, \Lambda^{1,0}\right)\,, \tag{2.10}$$

and its own canonical bundle $K(\mathcal{M}_h)$ whose fiber $K_\Sigma(\mathcal{M}_h)$ at Σ is defined by

$$K_\Sigma(\mathcal{M}_h) = \wedge^{\max} H^0\left(\Sigma, \Lambda^{2,0}\right)\,. \tag{2.11}$$

So far we have discussed only integer values of p. But the bundle $\Lambda^{1,0}(\Sigma)$ admits globally well-defined square roots, known as spin bundles. On a surface of genus h, there are actually 2^{2h} of them. By the Gauss–Bonnet theorem, the Chern class of $\Lambda^{1,0}$ is $2h-2$, and the Chern class of spin bundles is $h-1$. Now, for each k, the space $Pic_k(\Sigma)$ of line bundles of Chern class k is isomorphic with the space $Pic_0(\Sigma)$ via $Pic_0(\Sigma) \ni L \to L \otimes S \in Pic_k(\Sigma)$, once a reference line bundle $S \in Pic_k(\Sigma)$ has been chosen. In the particular case of Chern class $h-1$, an important observation is that a choice of homology basis A_I, B_I determines a particular spin bundle $S[0]$. Thus the space $Pic_{h-1}(\Sigma)$ gets identified correspondingly with $Pic_0(\Sigma)$.

A line bundle L with $c_1(L) = 0$ can be characterized by a flat connection, which can be characterized in turn by the holonomy of its sections around the basis of homology cycles

$$\varphi(z + A_I) = \exp\left(2\pi i \delta_I'\right)\, \varphi(z)\,, \qquad \varphi(z + B_I) = \exp\left(-2\pi i \delta_I''\right)\, \varphi(z)\,. \tag{2.12}$$

Here we have denoted informally by $z+C$ the effect of transporting z along a closed cycle C. The spaces $Pic_0(\Sigma)$, and hence $Pic_{h-1}(\Sigma)$, can be identified in this way with the space of characteristics $\delta = (\delta', \delta'') \in [0,1)^h \times [0,1)^h$. More concretely, define the function $\theta[\delta](z|\tau)$ on $\boldsymbol{C}^h \times \mathcal{H}$ by

$$\theta[\delta](Z|\Omega) = \sum_{n\in\mathbf{Z}^h} \exp\left[\pi i \left(n_I + \delta'_I\right) \Omega_{IJ} \left(n_J + \delta''_J\right) + 2\pi i \left(n_I + \delta'_I\right) \left(Z + \delta''_I\right)\right] . \tag{2.13}$$

It transforms as follows under the above shifts

$$\begin{aligned}\theta[\delta](Z + M + \Omega N|\Omega) \; = \; & \exp\left(-\pi i N_I \Omega_{IJ} N_J - 2\pi i N_I \left(Z_I + \delta''_I\right)\right. \\ & \left. + 2\pi i \delta'_I M_I\right) \theta[\delta](Z|\Omega) .\end{aligned} \tag{2.14}$$

Then sections s of the bundle L with the transformations (2.12) can be obtained by imbedding the surface Σ in $Pic_0(\Sigma)$ by the Abel map

$$\Sigma \ni z \to \int_P^z \omega_I \in \boldsymbol{C}^h / \left(\boldsymbol{Z}^h + \Omega \boldsymbol{Z}^h\right) = Pic_0(\Sigma) \tag{2.15}$$

and essentially restricting the θ-function to the image of Σ

$$s(z) = \frac{\theta[\delta]\left(Z + \int_P^z \omega|\Omega\right)}{\theta[0]\left(Z + \int_P^z |\Omega\right)} . \tag{2.16}$$

Within $Pic_{h-1}(\Sigma)$, the spin bundles can be identified as the bundles with $\delta \in (\frac{1}{2}\boldsymbol{Z})^h \times (\frac{1}{2}\boldsymbol{Z})^h$. They can be divided into even and odd spin bundles, depending on the parity of $4\,\delta' \cdot \delta''$. Generically, the even spin bundles do not admit any non-trivial global holomorphic sections, while the odd spin bundles admit one non-trivial global holomorphic section. There is a close relationship between the determinants of the Laplacians $\bar\partial^\dagger \bar\partial[\delta]$ on a spin bundle $S[\delta]$ and the θ-constant $\theta[\delta](0|\Omega)$. In genus $h=1$, it is

$$\det\left(\bar\partial^\dagger \bar\partial[\delta]\right) \; = \; \left|\frac{\theta[\delta](0|\Omega)}{\eta(\Omega)}\right|^2 , \tag{2.17}$$

$$\det{}'\left(\bar\partial^\dagger \bar\partial\right)[\delta] \; = \; \tau_2^2 |\eta(\Omega)|^4 , \qquad \delta = \left[\tfrac{1}{2}\tfrac{1}{2}\right] , \tag{2.18}$$

where $\eta(\Omega) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$, $q \equiv e^{2\pi i\Omega}$, all determinants are the ζ-regularized product of all the eigenvalues, and $\det'$ is the ζ-regularized product of the strictly positive eigenvalues. In higher genus, similar relations hold between the determinants of the Laplacians on spin bundles, the determinant of the Laplacian on scalars, and θ-constants.

The moduli space of Riemann surfaces is not a compact manifold. In the limit when a homology cycle degenerates to a point, the Riemann surface degenerates to a surface with nodes. What happens near degenerations is crucial for both geometry and physics. In algebraic geometry, the Deligne–Mumford compactification $\overline{\mathcal{M}_h}$ of $\mathcal{M}_h$ is obtained by adding the divisor of Riemann surfaces with nodes, and one tries to determine whether holomorphic sections on $\mathcal{M}_h$ extend to $\overline{\mathcal{M}}_h$, possibly with poles. In differential geometry, one can study the asymptotic behavior of many canonical metrics on $\mathcal{M}_h$ in the region near the divisor of surfaces with nodes.

There are several natural metrics on the two basic bundles on $\mathcal{M}_h$, whose fibers are, respectively, the space of holomorphic $(1,0)$-forms and the space of quadratic differentials. First, observe that the vector bundle $H^0(\Sigma, \Lambda^{1,0})$ on $\mathcal{M}_h$ carries the following metric

$$\|\omega\|^2 = i \int_{\Sigma} \omega \wedge \overline{\omega} \tag{2.19}$$

which is defined only in terms of the complex structure of Σ since $\omega \wedge \overline{\omega}$ is a $(1,1)$ form. The Hodge bundle λ over $\mathcal{M}_h$ inherits a corresponding metric

$$\|\omega_1 \wedge \ldots \wedge \omega_h\|^2 = \det \langle \omega_I | \omega_J \rangle = \det\left(\mathrm{Im}\, \Omega_{IJ}\right), \tag{2.20}$$

where the last equality on the right is a consequence of the Riemann identities. On the space of quadratic differentials, one can consider the L^2 metric, once one has made a choice of metric g_{ij} representing the complex structure. One natural choice is the metric of constant scalar curvature, and the corresponding metric on moduli space is called the Weil–Petersson metric. It is known that the Weil–Petersson metric is Kähler, incomplete, and has negative Ricci and holomorphic sectional curvature [5,6,7]. Other distances arise naturally from the realization of the Teichmüller space as a pseudoconvex domain: they include the Caratheodory and the Kobayashi metrics, and the Kähler–Einstein metric with negative Ricci curvature constructed by Cheng and Yau [8] and Mok and Yau [9]. There has been considerable progress recently in relating these metrics and determining their behavior near the divisor of surfaces with nodes, see [10] for a comprehensive discussion.

3. The bosonic string

It is now easy to see why the moduli of Riemann surfaces should play a major role in string theory. Strings are one-dimensional objects which span in their evolution a surface Σ, called the world-sheet, inside space-time. At the order h of perturbation theory, Σ is a surface of genus h. String amplitudes receive contributions from fluctuations of the world-sheet, which are parametrized in the Polyakov formulation by metrics g_{mk} on Σ. A basic principle of string theory is conformal invariance, that is, the contribution of g_{mk} depends only on its complex structure. Thus, after factoring out the gauge group, string amplitudes should be given by integrals over the moduli space $\mathcal{M}_h$.

The simplest string model is the bosonic string. Here space-time is a Riemannian manifold (X^d, G_{MN}), and the world-sheet evolution is described by a metric g_{mk} on Σ, and d functions $\Sigma : \xi \to (x^\mu(\xi)) \in X^d$, parametrizing the world-sheet inside space-time. The action is

$$I(g_{ij}, x) = \frac{1}{4\pi} \int_\Sigma d^2\xi \sqrt{g} g^{k\ell} \partial_k x^M \partial_\ell x^N G_{MN}(x) \,. \tag{3.1}$$

String amplitudes are given then by functional integrals

$$\begin{aligned}\left\langle \prod_{i=1}^N V_i \right\rangle \equiv & \int_{\mathcal{M}_h} \int_{\Sigma^N} Dg_{ij} \prod_{i=1}^N d\xi_i \sqrt{g(\xi_i)} \int Dx^M e^{-I(g_{ij},x)} \\ & \prod_{i=1}^N V_i \left(g_{km}, x(\xi_i)\right)\end{aligned} \tag{3.2}$$

where the V_i are vertex operators, that is, formally, random variables on the measure space defined by $Dg_{ij} Dx^M$. The principle of conformal invariance implies that these amplitudes are expressible as

$$\left\langle \prod_{i=1}^N V_i \right\rangle = \int_{\mathcal{M}_h} \int_{\Sigma^h} \nu(z_1, \ldots, z_N) \,, \tag{3.3}$$

where ν is a $(1,1)$-form in each z_i, valued in the line bundle $K_{\mathcal{M}_h} \otimes \overline{K_{\mathcal{M}_h}}$ on $\mathcal{M}_h$. There is however an important subtlety: although the action $I(g_{km}, x)$ depends only on the complex structure of g_{mk}, the functional measure $Dg_{mk} Dx^M$ does change under Weyl scalings. This is the famous conformal anomaly [11]. So the principle of conformal invariance turns out to hold only under two important conditions, namely that $d = 26$ and the space-time metric $G_{\mu\nu}$ is Ricci flat, which we assume henceforth.

For simplicity, we shall restrict ourselves to the partition function $Z = \langle 1 \rangle$, for strings evolving in flat Euclidian space-time, $X^d = M^{25,1}$, G_{MN} is the corresponding Euclidian metric (the more realistic case of Minkowski space is easily derived from the Euclidian case by analytic continuation.) In this case, $\nu(z_1, \ldots, z_N)$ reduces to a section ν of $K_{\mathcal{M}_h} \otimes \overline{K_{\mathcal{M}_h}}$ on $\mathcal{M}_h$, and

$$Z = \int_{\mathcal{M}_h} \nu \,. \tag{3.4}$$

String perturbation theory can be viewed as the problem of determining ν for all genus h.

This problem can be completely solved if we choose to represent complex structures by their representative metrics $\hat{g}_{mk}$ of constant scalar curvature. In this case, the string volume form ν was shown in [12] to be

$$Z = \int_{\mathcal{M}_h} Z'(1)^{-13} Z(2)\, \omega_{WP}^{3h-3} \tag{3.5}$$

(up to a multiplicative constant c_h depending only on the genus h, which we ignore in this survey), and $Z(s)$ is the Selberg zeta function, defined by

$$Z(s) = \prod_{\gamma \,\text{primitive}} \prod_{k=1}^{\infty} \left(1 - e^{-(s+k)\ell_\gamma}\right) , \tag{3.6}$$

where γ runs over all primitive closed geodesics on Σ, and $\ell(\gamma)$ is the length of γ with respect to the metric of constant scalar curvature. The asymptotics of $Z'(1)$, $Z(2)$, and ω_{WP} in the degeneration limit have been obtained by Wolpert [13] and Masur [14]. They confirm that the integrand in Z develops a non-integrable singularity near the divisor of surfaces with nodes, which is the geometric consequence of the physical fact that the spectrum of the model contains a tachyon.

An alternative description of ν relies more fundamentally on the fact that $\mathcal{M}_h$ is a complex manifold. It is a classic result of Mumford [15] that $K(\mathcal{M}) \otimes \lambda^{-13}$ is flat, and Belavin and Knizhnik [16] showed that ν is simply the square of the norm of a global holomorphic section s of $K(\mathcal{M}) \otimes \lambda^{-13}$, with respect to the canonical metric on the Hodge bundle λ^{-13},

$$\nu = \|s\|^2_{\lambda^{-13}} \tag{3.7}$$

which is then a positive section of $K(\mathcal{M}) \otimes \overline{K(\mathcal{M})}$, and hence a volume form on $\mathcal{M}_h$. More explicitly, if we fix a canonical homology basis (A_I, B_I), the dual basis ω_I of holomorphic differentials provide a trivialization $\omega_1\wedge\ldots\wedge\omega_h$ of the Hodge bundle λ. If we express s locally as

$$s = (\omega_1 \wedge \ldots \wedge \omega_n)^{-13} \otimes d\mu_B(\Omega)\,, \tag{3.8}$$

with $d\mu_B(\Omega)$ a local section of $K_{\mathcal{M}}$, and make use of the formula (2.20), then we obtain the following more explicit expression for ν

$$\nu = \det{}^{-13}(\mathrm{Im}\Omega_{IJ})\, d\mu_B(\Omega) \wedge \overline{d\mu_B(\Omega)}\,. \tag{3.9}$$

The presence of a tachyon in the spectrum of the bosonic string gets an even simpler interpretation in this complex formalism: from the Riemann–Roch theorem, it follows that the section s is non-vanishing and regular in $\mathcal{M}_h$, but on $\overline{\mathcal{M}_h}$, it develops a pole of order 2 along the divisor of Riemann surfaces with nodes. The power of the complex structure of strings has been particularly emphasized by Friedan and Shenker [17].

In general, $h = 2$ and $h = 3$, the section ν can be written down completely explicitly in terms of θ-constants [18, 19], thanks largely to a good understanding of modular forms in these cases. In a sense, this corresponds to the expression (3.5) of ν in terms of special values of the Selberg zeta function, with the crucial advantage that θ-constants are manifestly holomorphic on $\mathcal{M}_h$. It remains an open problem to this day to determine whether this can be done for higher genera. This could help considerably the study of superstring perturbation theory which we discuss in the next section. We note that expressions of ν in terms of θ-functions with additional points on the surface have been obtained by many authors, using *e.g.* bosonization formulas. But the presence of these additional points limits the applicability of these expressions, as their modular transformation properties are then hard to see.

4. Moduli of Riemann surfaces and super Riemann surfaces

Because of the tachyon and the ensuing divergences, the bosonic string is by itself an ill-suited candidate for a realistic unified string theory. A cure is obtained by adding fermionic degrees of freedom on the world-sheet Σ, and by imposing local supersymmetry. Thus a new geometric structure emerges, that of two-dimensional supergeometries. It is the co-existence of this new structure with the classical complex structures which underlie much of superstring perturbation theory. We discuss now these issues in greater length.

4.1. Two-dimensional supergeometries

Let Σ be a smooth oriented compact surface as before, and fix now a spin structure δ. A two-dimensional supergeometry is a pair $({e_m}^a, \chi_{m\alpha})$, where ${e_m}^a$ is a local frame. We can view ${e_m}^a$ as defining a metric

$g_{mk} = e_m{}^a e_k{}^b \delta_{ab}$ on Σ, and the "gravitino field" $\chi_{m\alpha}$ as a section of $\Lambda^{1,0}(\Sigma) \otimes S$, where S is a spin bundle. The groups of infinitesimal diffeomorphisms, local Weyl, and local U(1) acts on the space of two-dimensional supergeometries in the usual way they act on tensors and spinors. The novel feature of two-dimensional supergeometries is that they carry three more actions, namely local supersymmetry

$$\delta e_m{}^a = \delta\zeta \gamma^a \chi_m \,, \qquad \delta \chi_m{}^\alpha = -2\nabla_m \delta\zeta^\alpha \tag{4.1}$$

generated by a spinor $\delta\zeta^a$, and local super Weyl and super U(1) transformations. A supergeometric structure, or super Riemann surface, is then defined to be an equivalence class of supergeometries, modulo the above actions. The supermoduli space $\int\mathcal{M}_h$ is by definition the space of all supergeometries. Thus

$$s\mathcal{M}_h = \{(e_m{}^a, \chi_{m\alpha})\}/s\mathrm{Diff} \times \mathrm{Diff} \times \mathrm{Weyl} \times s\mathrm{Weyl} \times s\mathrm{U}(1)\,. \tag{4.2}$$

The Riemann–Roch theorem gives the dimension of $s\mathcal{M}_h$,

$$\dim s\mathcal{M}_h = (3h-3|2h-2) \qquad \text{if} \quad h \geq 2\,, \tag{4.3}$$

where $(m|n)$ indicates m bosonic degrees of freedom and n Grassmann degrees of freedom. For $h=0$, $\dim s\mathcal{M}_h = (0|0)$, and for $h=1$, $\dim s\mathcal{M}_h = (1|0)$ or $(1,1)$, depending on whether the spin structure δ is even or odd.

There is a very suggestive interpretation of this structure in terms of superspace. In the superspace formalism, we consider a super surface $s\Sigma$ whole local coordinates $\boldsymbol{z} = (z^M)$, $M = (m,\alpha)$ are given by $z^m = (z,\bar z)$, $z^\alpha = (\theta, \bar\theta)$, with θ an anti-commuting parameter. A supergeometry corresponds to a frame $E^A = dz^M E_M{}^A$, and a U(1) superconnection $\Omega = dz^M \Omega_M$ satisfying the Wess–Zumino torsion constraints

$$T_{ab}{}^c = T_{\alpha\beta}{}^\gamma = 0\,, \qquad T_{\alpha\beta}{}^c = 2(\gamma^c)_{\alpha\beta}\,, \tag{4.4}$$

where the torsion $T_{AB}{}^C$ and curvature R_{AB} are defined by $[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}{}^C \mathcal{D}_C + inR_{AB}$, $\mathcal{D}_A V = E_A{}^M(\partial_M V_B + in\Omega_M V)$ is the covariant derivative on fields V of U(1) weight n, and γ^c are two-dimensional Dirac matrices. The group $s\mathrm{Diff}(\Sigma)$ acts on supergeometries by

$$\delta E_M{}^A = E_M{}^A \left(\mathcal{D}_A(\delta V)^B - \delta V^C T_{CA}{}^B + \delta V^C \Omega_C E_A{}^B\right)\,. \tag{4.5}$$

In Wess–Zumino gauge, defined by $E_\mu{}^\alpha \sim \delta_\mu{}^\alpha + \theta^\nu e^{*\alpha}_{\nu\mu}$, $E_\mu{}^a \sim \theta^\nu e^{**a}_{\nu\mu}$, the component $E_m{}^a$ can be expanded as

$$E_m{}^a = e_m{}^a + \theta\gamma^a\chi_m - \frac{i}{2}\theta\bar\theta e_m{}^a A\,. \tag{4.6}$$

The field A turns out to have no dynamics and can be set to 0. Thus the supergeometry $(E_M{}^A, \Omega_M)$ in the superspace formalism can be identified with the supergeometry $(e_m{}^a, \chi_m{}^\alpha)$ in the component formalism. Furthermore, decomposing the vector superfield δV^M into components δv^m and $\delta\zeta^\alpha$, the transformations (4.5) decompose correspondingly into local diffeomorphisms and local supersymmetry (4.1).

4.2. Superholomorphicity

A supergeometry endows the super Riemann surface $s\Sigma$ with a structure analogous to and as rich as a complex structure on a surface. Recall that the local coordinates on $s\Sigma$ are (z,θ). A super 1/2-form on $\sum \Sigma$ is a form of the type $\hat\omega = \theta\,\omega_+(z) + \omega_0(z)$, where $\omega_+(z)$ and $\omega_0(z)$ are forms on Σ of U(1) weights 1 and 1/2 respectively. In analogy with line integrals of 1-forms on Riemann surfaces, we can define the line integrals of super 1/2-forms by

$$\int_{\mathbf{P}}^{\boldsymbol{z}} \hat\omega = \int_P^z \left(dz\,\omega_+ - \tfrac{1}{2} d\bar z\,\chi_{\bar z}^+ \hat\omega_0\right) + \theta_z \omega_0(z) - \theta_P \hat\omega_0(P)\,. \tag{4.7}$$

A supergeometry defines a notion of superholomorphic forms: a form $\hat\omega(\boldsymbol{z})$ is superholomorphic if it satisfies the equation

$$\partial_z \omega_0 + \tfrac{1}{2}\chi_{\bar z}^+ \omega_+ = 0\,, \qquad \partial_{\bar z}\omega_+ + \tfrac{1}{2}\partial_z(\chi_{\bar z}^+ \omega_0) = 0\,. \tag{4.8}$$

The line integrals $\oint_C \hat\omega$ depend then only on the homology class of C. Generically, the space of odd superholomorphic forms is of dimension h for a surface of genus h, and we can define the super period matrix $\hat\Omega$ by

$$\hat\Omega_{IJ} = \oint_{B_I} \hat\omega_J\,. \tag{4.9}$$

The super period matric $\hat\Omega$ can be viewed as a supersymmetric correction of the period matrix defined by the metric g_{ij}. Define a modified Szegö kernel $\hat S_\delta(z,w)$ by

$$\partial_{\bar z}\hat S_\delta(z,w) + \frac{1}{8\pi}\chi_{\bar z}^+ \int d^2u\,\chi_{\bar u}{}^+ \partial_z\partial_u \ln E(z,u)\hat S_\delta(u,w) = 2\pi\delta(z,w)\,, \tag{4.10}$$

where $E(z,u)$ is the prime form, and $\partial_z\partial_u \ln E(z,u)$ the meromorphic form in z with double pole at u and zero A periods. Then $\hat\Omega$ can be expressed as

$$\hat\Omega_{IJ} = \Omega_{IJ} - \frac{i}{8\pi}\int d^2z d^2w\,\omega_I(z)\chi_{\bar z}^+\,\hat S_\delta(z,w)\,\chi_{\bar w}^+ \omega_J(w)\,. \tag{4.11}$$

Note that, since $\chi_{\bar z}^+$ is a Grassmann variable, the modified Szegö kernel can be expanded into a finite series in $\chi_{\bar z}^+$, starting with the Szegö kernel $S_\delta(z,w)$, which is the Green's function for the $\bar\partial$ operator on spinors,

$$\partial_{\bar z} S_\delta(z,w) = 2\pi\delta(z,w)\,. \tag{4.12}$$

This results into a similar expansion for $\hat\Omega$, starting with Ω.

4.3. Construction of superstring scattering amplitudes

Two-dimensional geometries play the same role in the construction of superstrings as two-dimensional geometries play in the construction of the bosonic string. We restrict to Minkowski space-time, and fix a spin bundle $S[\delta]$. The evolution of the superstring in space-time is governed by the action

$$\begin{aligned} I(g_{mk},\chi_{m\alpha}) \;=\; \frac{1}{4\pi}\int_\Sigma d^2z\,\big(&\partial_z x^\mu \partial_{\bar z} x^\mu - \psi_+^\mu \partial_z \psi_+^\mu - \psi_-^\mu \partial_z \psi_-^\mu \\ &+\chi_{\bar z}^+ \psi_+^\mu \partial_z x^\mu + \chi_z{}^- \psi_- \partial_z x^\mu - \tfrac{1}{2}\chi_{\bar z}^+ \chi_z{}^- \psi_+^\mu \psi_-^\mu\big)\,, \end{aligned} \tag{4.13}$$

where the fields ψ_+^μ are sections of $S[\delta]$, and are the supersymmetric partners of the scalar fields x^μ, just as the gravitino field $\chi_{\bar z}^+$ was the supersymmetric partner of the metric g_{mk}. The action (4.13) is that of two-dimensional supergravity, coupled to the d matter superfields $(x^\mu, \psi_+^\mu, \psi_-^\mu)$. It is invariant under the same symmetries as for $(e_m{}^a, \chi_m{}^a)$, and in particular under Weyl and super Weyl scalings, diffeomorphisms, and local supersymmetry.

We describe now the construction of superstring scattering amplitudes. It starts from the following functional integral, defined for any fixed spin structure δ,

$$\begin{aligned} \boldsymbol{A}[\delta] \;=\; &\int DE_M{}^A D\Omega_M \int \prod_{i=1}^N d^{2|2}\boldsymbol{z}_i E(\boldsymbol{z}_i) \\ &\int DX^\mu e^{-I_m} \prod_{i=1}^N V(\boldsymbol{z}_i, \bar{\boldsymbol{z}}_i; \varepsilon_i, \bar\varepsilon_i, k_i)\,. \end{aligned} \tag{4.14}$$

Here $d^{2|2}\boldsymbol{z}E(\boldsymbol{z})$ is the volume form on $s\Sigma$, and is given by $d^{2|2}\boldsymbol{z}E(\boldsymbol{z}) = d\bar\theta \wedge e^z \wedge d\theta \wedge e^z$, with $e^z = dz - \frac{1}{2}\theta\chi_{\bar z}^+ d\bar z$. The random variable $V(\boldsymbol{z}, \bar{\boldsymbol{z}}; \varepsilon, \bar\varepsilon, k)$ is the vertex for the emission of a particle of momentum k and polarization thensor ε. We shall discuss only the vertices for the emission of a particle in the graviton multiplet. The fundamental assumption of superstring theory, to be proven, is that for each spin structure δ, the amplitude $\boldsymbol{A}[\delta]$ can be expressed as

$$\boldsymbol{A}[\delta] = \int dp_I^\mu \int\limits_{\mathcal{M}_h \times \Sigma^N} \mathcal{H}[\delta]\left(z_i; k_i; \varepsilon_i; p_I^\mu\right) \wedge \overline{\mathcal{H}[\delta]\left(z_i; k_i; \varepsilon_i; p_I^\mu\right)}\,, \quad (4.15)$$

where $\mathcal{H}[\delta]$ is holomorphic in both moduli and insertion points z_i. We refer to a statement of the form (4.15) as a *holomorphic splitting* of the amplitude $\boldsymbol{A}[\delta]$. The superstring amplitude would then be defined by

$$\begin{aligned}\boldsymbol{A}_{II}(k_i; \varepsilon_i) \; = \; & \int dp_I^\mu \int\limits_{\mathcal{M}_h \times \Sigma^N} \sum_{\delta\bar\delta} \varepsilon_{\delta,\bar\delta} \mathcal{H}[\delta]\left(z_i; k_i; \varepsilon_i; p_I^\mu\right) \\ & \wedge \overline{\mathcal{H}[\delta]\left(z_i; k_i; \varepsilon_i; p_I^\mu\right)} \end{aligned} \quad (4.16)$$

with a suitable choice of phases $\varepsilon_{\delta,\bar\delta}$. There should be only two possible inequivalent choices of such phases, differing in the relative signs between the contributions of odd and even spin structures, leading to the Type IIA and the Type IIB superstrings.

The prescription of splitting holomorphically the functional integral $\boldsymbol{A}[\delta]$ is the analogue for Euclidian signature of the prescription of taking ψ_+ to be Majorana–Weyl spinors for Minkowski signature. The prescription of summing over the spin structures $\delta, \bar\delta$ is the Goddard–Kent–Olive, or GKO prescription. Its role is to truncate the spectrum of the superstring and eliminate the tachyon. It is one of the striking features of superstring theory that this prescription produces a space-time supersymmetric theory.

4.4. Holomorphicity and superholomorphicity

The main problem of superstring perturbation theory is to establish the existence of the holomorphic splitting (4.15), and evaluate the holomorphic blocks $\mathcal{H}[\delta]$. We describe here what is known as well as some of the key difficulties.

- The first step is to factor out all the gauge symmetries, and reduce the functional integrals $\boldsymbol{A}[\delta]$ to finite-dimensional integrals. This can be done by the standard Faddeev–Popov gauge-fixing procedures of quantum field theory [4]. The amplitude $\boldsymbol{A}[\delta]$ is reduced to an integral over the quotient space of all supergeometries by all symmetries, which is by definition the supermoduli space $s\mathcal{M}_h$ (in presence of vertex insertions, the integral is over $s\mathcal{M}_h \times s\Sigma^N$). Note that integrals over supermoduli space incorporate integrals over $2h-2$ fermionic degrees of freedom.

- The integrals over supermoduli space obtained by gauge-fixing turn out to have a lot of structure. The chiral splitting theorem of [20] asserts that their integrands can be split into superholomorphic sections,

$$\begin{aligned} \boldsymbol{A}[\delta] \;=\; & \int dp_I \int_{s\mathcal{M}_h \times (s\Sigma)^N} \prod_{i=1}^{N} d^{2|2}\boldsymbol{z}_i \, E(\boldsymbol{z}_i) \\ & \left| \prod_{A=1}^{3h-3|2h-2} dm^A \mathcal{F}[\delta](\boldsymbol{z}_i; k_i, \varepsilon_i; p_I) \right|^2 . \end{aligned} \tag{4.17}$$

  Here m^A are local coordinates for $s\mathcal{M}_h$, and $\mathcal{F}[\delta]$ is superholomorphic in each variable $\boldsymbol{z}_i$ away from coincident points $\boldsymbol{z}_i = \boldsymbol{z}_j$ for $i \neq j$.

- Comparing the preceding expression with the desired expression (4.15), we see that the $2h-2$ Grassmann degrees of freedom of supermoduli space need to be integrated out, and that the main problem now is how to extract holomorphic forms on moduli space from superholomorphic forms on supermoduli space.

The difficulty is that, for a general supergeometry, the notions of holomorphicity and superholomorphicity do not appear directly related. If $\chi_{m\alpha} = 0$, then the superholomorphicity of the 1/2 differential $\hat{\omega} = \theta\omega + \omega_+$ is equivalent to the holomorphicity of ω and of ω_+ with respect to the complex structure defined by g_{mk}. However, the condition $\chi_{m\alpha} = 0$ is not invariant under supersymmetry, and even this relation in this particular case is not well-defined. For general $\chi_{m\alpha}$, there does not even appear to be any candidate for a relation between the two notions.

This is the difficulty that was overcome in [21] in the case of genus $h = 2$. The basic idea is to consider the complex structure defined by the super period matrix $\hat{\Omega}$. The relation between superholomorphicity with respect to $(e_m{}^a, \chi_{m\alpha})$ and holomorphicity with respect to $\hat{\Omega}$ turns out to be encoded in a beautiful hybrid cohomology theory, mixing de Rham and Dolbeault cohomology. The simplest example of this is the following relation between superholomorphic 1/2-forms $\hat{\omega}(z,\theta)$ and forms 1-forms $\omega(z)$ holomorphic with respect to $\hat{\Omega}$,

$$\int d\theta \wedge e^{z}\, \hat{\omega}(z,\theta) = dz\, \omega(z) + d\lambda(z)\,, \tag{4.18}$$

where $\lambda(z)$ is a smooth and globally defined scalar function. Similar relations were established for more complicated objects, such as the correlation functions of scalar superfields. More specifically, a deformation of complex structures to local holomorphic coordinates with respect to $\hat{\Omega}$ and an integration of the Grassmann variables gives

$$\boldsymbol{A}[\delta] = \int dp_I \int_{\mathcal{M}_2 \times \Sigma^N} \left| \mathcal{B}[\delta]\left(z_i; k_i, \varepsilon_i; p_I\right) \right|^2 \tag{4.19}$$

for some forms $\mathcal{B}[\delta]$. Then the most important property of the form $\mathcal{B}[\delta]$ is

$$\sum_\delta \mathcal{B}[\delta]\left(z_i; k_i, \varepsilon_i; p_I^\mu\right) - \sum_{j=1}^N d\bar{z}_j \partial_{\bar{z}_j} \mathcal{S}_j\left(z_i; k_i, \varepsilon; p_I^m u\right) = \mathcal{H}\left(z_i; k_i; p_I^\mu\right) \tag{4.20}$$

with $\mathcal{H}$ a holomorphic $(1,0)$-form in each z_i. Here we have restricted the number of insertions is $N \leq 4$, and the summation is only over even spin structures δ, because only even spin structures contribute to the superstring amplitudes for $N \leq 4$ at this order of perturbation theory. All the relative phases $\varepsilon_{\delta,\delta'}$ in (4.15) can be taken to be 1, and the holomorphic form $\mathcal{H}$ can be equated with the desired form $\sum_\delta \mathcal{H}[\delta]$ there. All the resulting expressions can be evaluated explicitly in terms of θ-constants. We obtain in this manner the following final answer for the superstring measure

$$A = \int_{\mathcal{M}_2} \det(\mathrm{Im}\Omega)^{-5} \sum_{\delta,\delta'} d\mu[\delta](\Omega) \wedge \overline{d\mu[\delta'](\Omega)} \tag{4.21}$$

with the contribution $d\mu_2[\delta]$ of each spin structure δ given by

$$d\mu_2[\delta] = \frac{1}{16\pi^2} \frac{\Xi_6[\delta](\Omega)\theta[\delta](\Omega)^4}{\Psi_{10}(\Omega)} \prod_{I \leq J} d\Omega_{IJ}\,. \tag{4.22}$$

Here $\Psi_{10}(\Omega) = \prod_{\delta\,\mathrm{even}} \theta[\delta](\Omega)^2$, and the key new form $\Xi_6[\delta](\Omega)$ is given by

$$\Xi_6[\delta](\Omega) = \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \theta[\nu_i + \nu_j + \nu_k]^4(\Omega)\,. \tag{4.23}$$

The sum over δ of $d\mu[\delta]$ vanishes identically over moduli space. This can be viewed as the generalization to genus $h = 2$ of the Jacobi identity for θ constants, and a manifestation of space-time supersymmetry. For the 4-point function, we obtain

$$A(k_1, \varepsilon_1; \ldots; k_4, \varepsilon_4) = \frac{K\bar{K}}{2^{12}\pi^4} \int_{\mathcal{M}_2 \times \Sigma^4} \frac{|\prod_{I \leq J} d\Omega_{IJ}|^2}{(\det \mathrm{Im}\Omega)^5} |\mathcal{Y}_S|\,|^2 \exp\left(-\sum_{i<j} k_i \cdot k_j G(z_i, z_j) \right)\,. \tag{4.24}$$

Here k_i are the momenta of the gravitons, ε_i are their polarization tensors, $K, \bar{K}$ are kinematic invariants depending on the k_i and ε_i, $G(z_i, z_j)$ is the conformally invariant Green's function

$$G(z,w) = -\ln|E(z,w)|^2 + 2\pi \left(\mathrm{Im}\Omega^{-1}_{IJ} \left(\mathrm{Im} \int_z^w \omega_I \right) \left(\mathrm{Im} \int_z^w \omega_J \right) \right) \quad (4.25)$$

and $\mathcal{Y}_S$ is the following form in the variables z_i

$$\begin{aligned} 3\mathcal{Y}_S \ = \ & (k_1 - k_2)\cdot(k_3 - k_4)\Delta(z_1, z_2)\Delta(z_3, z_4) \\ & + (k_1 - k_3)\cdot(k_2 - k_4)\Delta(z_1, z_3)\Delta(z_2, z_4) \\ & + (k_1 - k_4)\cdot(k_2 - k_3)\Delta(z_1, z_4)\Delta(z_2, z_3)\,, \end{aligned} \quad (4.26)$$

with the form $\Delta(z, w)$ defined by

$$\Delta(z,w) = \omega_1(z)\omega_2(w) - \omega_1(w)\omega_2(z)\,. \quad (4.27)$$

4.5. Ansatze from factorization constraints

The derivation of scattering amplitudes to all orders of perturbation theory can already be technically very involved for field theories. As we have seen, string theories present formidable additional geometric difficulties. An old idea is to circumvent a derivation from first principles, and construct the amplitudes instead by the constraints of unitarity and factorization. This still proved to be prohibitively difficult in the past. However, with the new insights gained from the explicit formulas for the genus 2 amplitudes, it was proposed in [22] to try this approach anew, by incorporating this time suitable generalizations of the modular properties of the genus 2 amplitudes. Thus an Ansatz for the genus h superstring measure is of the form

$$A = \int_{\mathcal{M}_h} \det(\mathrm{Im}\Omega)^{-5} \sum_{\delta,\delta'} d\mu[\delta](\Omega) \wedge \overline{d\mu[\delta'](\Omega)} \quad (4.28)$$

with $d\mu[\delta](\Omega)$ given by

$$d\mu[\delta](\Omega) = d\mu_B(\Omega)\, \Xi[\delta](\Omega)\,, \quad (4.29)$$

where $d\mu_B(\Omega)$ is the bosonic string volume form, and $\Xi[\delta]$ a generalization of the genus 2 expression in θ-constants, to be determined by suitable modularity and factorization constraints. The first breakthrough in this line of investigation was due to Cacciatori, Dalla Piazza, and van Geemen [23], who found a striking candidate for the superstring measure in genus 3, and showed that it was unique. This led to rapid progress in several directions, including a generalization to all genera by Grushevsky [24] of the modular forms of the types found in genus 2 and 3, and the identification of the genus 4 superstring measure by Grushevsky, Cacciatori, Dalla Piazza, van Geemen, and Salvati Manni and their collaborators [25]. A considerable simplification and synthesis of these developments, including a historical perspective of attempts from the early 1980s to find the superstring measure, was given by Morozov [26]. However, starting from genus 5, new difficulties with such generalizations were uncovered by Matone and Volpato [27]. A lucid discussion can be found in [27], as well as in the paper of Dunin-Barkowski, Morozov, Sleptsov [28]. The surrounding questions about modular forms and θ functions seem to have many ramifications (see *e.g.* [29]), and to be of great interest in their own right.

An alternative approach to superstring theory, which avoids the summation over spin structures and can recover the perturbative results described above to two-loops, is the pure spinor formalism developed by Berkovits and co-authors [30].

5. Moduli space of target manifolds

In the preceding section, we discussed the problem of evaluating scattering amplitudes for superstrings in 10-dimensional Minkowski space-time. The known physical world suggests that space-time should be instead of the form $X = M^{1,3} \times K$, where K is a compact manifold so tiny that it has not been detected at presently available energy levels. The conformal invariance of the quantized theory requires that the dimension of X still be 10, so that the dimension of K is 6. It also requires that the metric $G_{\mu\nu}$ of X be Ricci-flat. But there are further conditions that can be imposed on X that are desirable from the point of view of phenomenology, and which turn out to be also geometrically attractive. We discuss these briefly now.

5.1. Supersymmetry and Calabi–Yau

Although we have described so far the Type IIA and Type IIB superstring theories, the theory whose compactification received the greatest attention is the heterotic string. The heterotic string is a hybrid combination of the left-moving sector of the superstring, with the right-moving sector of the bosonic string in 26 dimensions, with 16 dimensions compactified to a torus $\boldsymbol{R}^{16}/2\pi\Gamma$, where Γ is the root lattice of $E_8 \times E_8$ or of $\mathrm{SO}(32)/\boldsymbol{Z}_2$. The resulting low-energy theory is $N = 1$ supergravity coupled to $N = 1$ super Yang–Mills theory, with $E_8 \times E_8$ or $\mathrm{SO}(32)/\boldsymbol{Z}_2$ gauge group.

An early and very influential Ansatz for compactification was put forth by Candelas, Horowitz, Strominger, and Witten [31]. A key requirement was to preserve $N = 1$ supersymmetry for the effective 4-dimensional theory, which was expected to help address the famous gauge hierarchy problem. Together with taking the product metric in $M^{1,3} \times K$ (in physical terms, setting this dilaton field to be constant), and taking the gauge field to be an SU(3) connection on K, they showed that K should be a Calabi–Yau 3-fold, that is, a Kähler 3-manifold with SU(3) holonomy. A more general Ansatz, still preserving the $N = 1$ supersymmetry of the effective 4-dimensional theory, but relaxing the product metric on $M^{1,3} \times K$ to a warped product metric, was subsequently proposed by Strominger [32]. The Strominger system of equations is for a complex Hermitian 3-manifold (X, ω) with a non-vanishing holomorphic 3-form Ω, curvature R, and a vector bundle E with curvature F, satisfying the following

$$F^{2,0} = F^{0,2} = 0\,, \tag{5.1}$$
$$F \wedge \omega^2 = 0\,, \tag{5.2}$$
$$i\partial\bar{\partial}\omega = \frac{\alpha'}{4}(\mathrm{Tr} R \wedge R - \mathrm{Tr} F \wedge F)\,, \tag{5.3}$$
$$d^*\omega = i(\bar{\partial} - \partial) \ln \|\Omega\|\,. \tag{5.4}$$

Note that the first condition means that E is a holomorphic vector bundle. The second condition means that the metric on E is Hermitian–Einstein, which is the vector bundle generalization of the Ricci-flat condition. Thus ω Kähler, $F = R$, with (X, ω) a Calabi–Yau 3-fold is a special solution, and the Strominger system can be viewed as a generalization to a non-Kähler setting of the Ricci-flat equation solved by Yau [33] and of the Hermitian–Einstein equation solved by Donaldson [34] and Uhlenbeck–Yau [35].

5.2. Chern numbers of moduli of CY

The moduli space of solutions to the Strominger system (5.1) can be viewed as the moduli space of supersymmetric vacua for superstring theory. Within this moduli space is the moduli space of Calabi–Yau 3-folds, and moduli theory plays again a prominent role in theoretical physics.

The moduli space of Calabi–Yau is a rich subject. It is impossible to do it justice in a short article, and we shall limit ourselves to quoting a few results. In parallel with the moduli space of Riemann surfaces, it comes naturally equipped with a Weil–Petersson geometry. More precisely, the analogue of the Hodge bundle λ is given now by the bundle of the holomorphic 3-form Ω, equipped with the natural metric

$$\|\Omega\|^2 = \int_X \Omega \wedge \overline{\Omega}\,. \tag{5.5}$$

The Weil–Petersson metric is the curvature of this bundle

$$\omega_{\mathrm{WP}} = -i\partial\bar{\partial}\,\log \|\Omega\|^2\,. \tag{5.6}$$

The curvature of the Weil–Petersson metric itself on the moduli space of Calabi–Yau has been determined by Strominger. In particular, its curvature is negative. Because the moduli space is open (Calabi–Yau manifolds can degenerate), and because it is not known how to compactify it, even the finiteness and rationality of its characteristic numbers is not easy to see and has been established only recently by Lu and Sun [36] and Douglas and Lu [37]. On the other hand, there has been considerable progress in many directions motivated by the remarkable phenomenon of mirror symmetry, for which we refer to the collection [38] of survey books.

5.3. Solutions of Strominger systems

By contrast, it is only recently that non-Kähler solutions of the Strominger system were found. The first non-trivial solutions were found by Li and Yau [39] as perturbations of Calabi–Yau 3-folds, and the first solutions on manifolds that do not admit a Kähler structure were found, non-perturbatively and rather recently, by Fu and Yau [40]. The moduli theory of Strominger systems is still in its infancy. But Strominger systems are undoubtedly a very natural and deep generalization of Calabi–Yau manifolds, and they will undoubtedly play a major role in complex geometry.

6. Moduli of meromorphic differentials and Seiberg–Witten theory

A rather unexpected emergence of moduli theory occurs in the Seiberg–Witten effective solution of $N = 2$ supersymmetric gauge theories in 4 dimensions.

6.1. $N = 2$ supersymmetric gauge theories

In 4-dimensions, the constraints of $N = 2$ supersymmetry are so powerful that the $N = 2$ supersymmetric Yang–Mills theory is completely determined by the field content. The $N = 2$ pure gauge multiplet consists of $(A_\mu dx^\mu, \phi, \lambda, \psi)$, ϕ a scalar, λ and ψ spinors, all valued in the adjoint representation of the gauge group G. The bosonic part of the action is

$$I = -\frac{1}{2g^2}\mathrm{Tr}\, F \wedge *F - \frac{\theta}{16\pi^2}\mathrm{Tr}\, F \wedge F - \frac{1}{2g^2}\mathrm{Tr}\, D_\mu\phi^\dagger D^\mu\phi + \frac{1}{2g^2}\mathrm{Tr}\,\left[\phi^\dagger, \phi\right]^2 . \tag{6.1}$$

The theory can be coupled to an $N = 2$ hypermultiplet, all fields of which lie in a representation $\mathcal{R}$ of G.

The vacua of the theory are given by $F_{\mu\nu} = 0$, constant fields ϕ, $D_\mu \phi = 0$, with $[\phi, \phi^\dagger] = 0$. Thus they can be expressed as

$$\bar{\phi} = \begin{pmatrix} \bar{a}_1 & \cdot & 0 \\ 0 & \bar{a}_2 & 0 \\ 0 & \cdot & \bar{a}_N \end{pmatrix}, \tag{6.2}$$

where N is the rank of the gauge group, and $\sum_{i=1}^N \bar{a}_i = 0$. While the $N = 2$ supersymmetry is unbroken, the gauge symmetry is spontaneously broken to gauge fields commuting with $\bar{\phi}$, and we obtain a theory of N interacting $N = 2$ supersymmetric U(1) gauge fields A_j. Again by supersymmetry, the effective action for these U(1) fields must be of the form

$$I_{\text{eff}} = \tfrac{1}{4}\text{Im}(\tau_{ij})\, F_i \wedge *F_j + \tfrac{1}{4}\text{Re}(\tau_{ij})\, F_i \wedge F_j + d\bar{\phi}^j \wedge d\phi_{D_j} + \text{fermions}\,. \tag{6.3}$$

Here F_j is the curvature of A_j, ϕ_j is the scalar field in the multiplet of the gauge field A_j, and ϕ_{Dj} is the dual gauge scalar. Furthermore, the effective gauge coupling τ_{ij} and the dual scalar ϕ_{Dj} are determined by a single holomorphic function $\mathcal{F}(\phi, \Lambda)$, called the prepotential

$$\tau_{ij} = \frac{\partial^2 \mathcal{F}}{\partial \phi_i \partial \phi_j}(\phi, \Lambda)\,, \qquad \phi_{Dj} = \frac{\partial \mathcal{F}}{\partial \phi_j}(\phi, \Lambda)\,. \tag{6.4}$$

Here Λ is a scale introduced by renormalization. For scale invariant theories, it is replaced by the microscopically well-defined parameter $\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{g^2}$.

6.2. Effective actions and the Seiberg–Witten Ansatz

The moduli space of classical vacua is the trivial space S of configurations $\bar{\phi}$ of the form (6.2). Unexpectedly, a rich moduli theory of complex geometric structures enters through the effective quantum theory, and the striking Ansatz of Seiberg and Witten [41]: according to this Ansatz, the effective prepotential $\mathcal{F}$ can be obtained from a fibration of Riemann surfaces Γ over S, each equipped with a meromorphic differential $d\lambda$, and

$$\frac{\partial \mathcal{F}}{\partial a_I} = a_{DI}\,, \tag{6.5}$$

where a_J and a_{DJ} are the periods of $d\lambda$ along a basis of canonical homology cycles (A_I, B_I)

$$a_I = \oint_{A_I} d\lambda\,, \qquad a_{DI} = \oint_{B_I} d\lambda\,. \tag{6.6}$$

The correct fibration for a given $N = 2$ supersymmetric gauge theory is determined by the requirement that, in the low-energy regime of Λ small, we must have

$$\mathcal{F}(a, \Lambda) \sim -\frac{1}{8\pi i}\left[\sum_{\alpha\in\mathcal{R}(G)} (a\cdot\alpha)^2 \ln\frac{(a\cdot\alpha)^2}{\Lambda^2} - \sum_{\lambda\in W(R)} (\lambda\cdot a + m)^2 \ln\frac{(\lambda\cdot a+m)^2}{\Lambda^2}\right] \tag{6.7}$$

up to a series in powers of Λ, which represents the contributions of instantons. For example, in the original Seiberg–Witten solution of the SU(2) pure Yang–Mills theory, the fibration Γ is given by the elliptic curves with differential $d\lambda$

$$y^2 = \prod_{j=1}^{2}(x-\bar{a}_j)^2 - \Lambda^2\,, \qquad d\lambda = \frac{x}{y} d\prod_{j=1}^{2}(x-\bar{a}_j)\,. \tag{6.8}$$

The Ansatz of Seiberg–Witten has now been explained from several points of view, including branes by Witten [42], and it has also been derived from first principles by Nekrasov [43].

6.3. Moduli of Riemann surfaces and Abelian integrals

Very early on, it was observed by Gorsky *et al.* [44] that the surfaces Γ and form $d\lambda$ in the Seiberg–Witten solution of the SU(2) theory coincided with the solution of the Whitham hierarchy for the SU(2) Toda lattice. This correspondence between Seiberg–Witten theory and integrable models was exploited in the solution by Donagi and Witten [45] of the SU(N) theory with matter in the adjoint representation. It was developed subsequently more fully by [45,46,48,49], and others (see [50] for a more extensive bibliography). Here we shall concentrate on the viewpoint developed in [48], as it leads as yet to another moduli space of complex structures.

The key underlying idea in the construction of [48] is motivated by the theory of integrable models, where integrability is just the commutativity of two operators, with two corresponding sets of eigenvalues. Thus it is natural to introduce the moduli space of $(\Gamma, E(z), Q(z))$, where $E(z)$ and $Q(z)$ are Abelian integrals of given poles. More precisely, fix a smooth surface Σ of genus h, with a marked point p, and integers $n \neq 0, m$. If $E(z)$ is an Abelian integral with pole of order n at p, then the equation $E(z) = z^{-n} + (\mathrm{Res}_p E)\log z$ defines a holomorphic coordinate near p. Let $\mathcal{M}_h(n, m)$ be the moduli space of triples (Γ, E, Q), where Γ is a complex structure on Σ, and E, Q are Abelian integrals with poles of order n, m at p. As Abelian integrals, $E(z)$, $Q(z)$ are defined only on a surface cut open along homology cycles, but all final expressions will end up independent of such choices. Set

$$d\lambda = EdQ\,. \tag{6.9}$$

Then $\mathcal{M}_h(n,m)$ has dimension $5h+n+m$, and the open set where the divisors $\{z, dE=0\}$ and $\{z; dQ=0\}$ admit the following natural coordinates:

- the $n+m$ pole coefficients in the Laurent expansion of $d\lambda$ near p;
- the $4h$ periods of dE and dQ;
- the A periods $a_I = \oint_{A_I} d\lambda$ of the differential $d\lambda$.

If we fix all the coordinates except for the periods a_I, we obtain a foliation of the moduli space $\mathcal{M}_h(n,m)$, the leaves of which are precisely of the dimension of the genus. It is shown in [48] that the leaves of this foliation coincide precisely with the moduli spaces for the Seiberg–Witten solution of the SU(N) gauge theory with hypermultiplets in the fundamental representation. This is a more systematic generalization of the original observation of Gorsky *et al.* [44], and establishes a direct link between Seiberg–Witten solutions and the Whitham theory, as developed earlier by Flaschka, Forrest, McLaughlin [51] and Krichever [52].

In general, the fibration of Riemann surfaces may degenerate to surfaces with nodes along certain divisors inside S, so the quantum moduli space is now a space with highly non-trivial geometry. In fact, both the geometry and the physics of the theory near these degeneration points is of considerable interest. However, it has not been fully elucidated as yet, even for the simplest gauge theories as above, where the fibration (Γ, E, Q) is relatively simple and known explicitly. This problem of describing the prepotential near the degeneration points can be expected to become even more challenging for the solutions of more sophisticated models. These include the $N=2$ pure Yang–Mills with arbitrary simple gauge group [46, 47], and the $N=2$ Yang–Mills with matter in the adjoint representation, solved in the case of SU(N) with a Hitchin system in [45], and in the case of arbitrary G with a twisted Calogero–Moser system in [49].

6.4. Symplectic forms

Essentially from the beginning, Seiberg and Witten [41] and Donagi and Witten [45] had stressed that the fibration Γ led to a natural symplectic structure on the fibration with the symmetric product $\mathrm{Sym}^h(\Gamma)$ as fiber

$$\omega = \sum_{I=1}^{h} \delta(d\lambda(z_I))\,, \qquad (z_1, \ldots, z_h) \in \mathrm{Sym}^h\Gamma\,, \tag{6.10}$$

and that this symplectic structure sufficed to determine the quantum prepotential $\mathcal{F}$. (Here we have denoted by δ the exterior differential in all variables, including the base variables, to distinguish from the differential d on each curve Γ.) Since a relation had been found between Seiberg–Witten fibrations and fibrations from integrable models as described in the previous section, it is natural to ask whether there exists a relation between the symplectic forms from the Seiberg–Witten fibrations and the Lax pairs from integrable models. The answer turned out to be affirmative, and the following formula was found in [48]

$$\omega = \mathrm{Res}_{k=\infty} \left\langle \psi^\dagger(z,k)\delta L \wedge \delta\psi(z,k) \right\rangle\,, \tag{6.11}$$

where L is the usual operator in the Lax pair, and $\psi(z,k)$ is the Baker–Akhiezer function. A fuller description of these notions can be also found in [48]: roughly speaking, the phase space is the space of all Lax operators L, $\psi(z,k)$ can be viewed as a function on this space, and δ the exterior differential on it, so that ω is a 2-form. As an unexpected by-product, this formulation of symplectic forms led to a new Hamiltonian theory of integrable models [48], based on the Lax pair directly instead of on the R-matrix, as had been developed by Faddeev and Takhtajan [53]. The new Hamiltonian formulation applies to even 2+1 integrable models, a convenient Hamiltonian formulation of which had not been available. We refer to [48] for a detailed discussion.

7. Moduli and canonical metrics

We conclude this survey with a brief discussion of the problem of moduli and canonical metrics. It may happen that a given geometric structure may admit a representative metric with "best" curvature properties. In this case, we refer to such a metric as a "canonical metric", and the moduli space of geometric structures can be identified with a moduli space of canonical metrics. We have encountered canonical metrics all along these lectures. For a complex structure on a surface, a canonical metric is a metric of constant curvature. For a holomorphic vector bundle $E \to (X, \omega)$, where ω is a Kähler form, a canonical metric $H = (H_{\bar\alpha\beta})$ is a metric satisfying the Hermitian–Einstein equation,

$$g^{j\bar k} F_{\bar k j}{}^{\alpha}{}_{\beta} = \mu\, \delta^{\alpha}{}_{\beta}\,, \tag{7.1}$$

where $F_{\bar k j} = -\partial_{\bar k}(H^{-1}\partial_j H)$ is the curvature of $H_{\bar\alpha\beta}$. For a Kähler manifold X with definite first Chern class, a canonical metric $g_{\bar k j}$ is a Kähler–Einstein metric

$$R_{\bar k j} = \mu\, g_{\bar k j}\,, \tag{7.2}$$

where μ is ± 1 or 0, depending on the sign of the first Chern class or whether it vanishes. Note that the Kähler–Einstein condition can be viewed as a more non-linear version of the Hermitian–Einstein condition, when $E = T^{1,0}(X)$, and $g_{\bar k j} = H_{\bar\alpha\beta}$ is also unknown. Clearly, the Kähler–Einstein equation is an Euclidian version of the Einstein equation in general relativity. Both Hermitian–Einstein and Kähler–Einstein equations appeared in the compactification of the heterotic string, as we saw earlier.

The existence of a Hermitian–Einstein metric on a holomorphic vector bundle E was shown by Donaldson [34] and Uhlenbeck–Yau [35] to be equivalent to the stability of E in the sense of Mumford–Takemoto. The existence of a Kähler–Einstein metric on a Kähler manifold was shown, respectively, by Yau [33] in the case of vanishing first Chern class (this is the famous Calabi conjecture, and the foundation of the theory of Calabi–Yau manifolds), and by Yau [33] and Aubin [54] in the case of negative first Chern class.

The case of Kähler–Einstein metrics for positive first Chern class is still open at this time. A classic conjecture of Yau [55] says that the existence of such a metric should be equivalent to the stability of X in the sense of geometric invariant theory. In fact, this conjecture extends to a broader question of which Kähler–Einstein metrics is only one example: let $L \to X$ be a positive holomorphic line bundle over a compact complex manifold X. When does there exist a metric $\omega \in c_1(L)$ with constant scalar curvature? Note that when $L = K_X^{\pm 1}$, the constant scalar curvature condition is actually equivalent to the condition of constant Ricci curvature, so this question is indeed a generalization of the question about the existence of Kähler–Einstein metrics. In this more general context, the conjecture of Yau would assert that the existence of a metric in $c_1(L)$ with constant scalar curvature should be equivalent to the stability of the line bundle $L \to X$ in the sense of geometric invariant theory. Specific notions of stability have been formulated by Tian [56] and Donaldson [57,58]. These notions do not fit entirely in the most traditional framework of geometric invariant theory, nevertheless, it can be hoped that they will lead to a well-behaved, Hausdorff moduli space of stable line bundles over X, just as stable curves led to a well-behaved, Hausdorff moduli of curves.

It is instructive to write down explicitly the equation for a metric in $c_1(L)$ with constant scalar curvature. Fix a reference metric $g^0_{\bar{k}j}$ in $c_1(L)$, and look for $g_{\bar{k}j}$ in $c_1(L)$ under the form $g_{\bar{k}j} = g^0_{\bar{k}j} + \partial_j \partial_{\bar{k}} \varphi$. Then the Ricci curvature $R_{\bar{k}j}$ of the metric $g_{\bar{k}j}$ is given by $R_{\bar{k}j} = -\partial_j \partial_{\bar{k}} \log \det(g_{\bar{q}p})$, and hence the constant scalar curvature equation is

$$- g^{j\bar{k}} \partial_j \partial_{\bar{k}} \log \det \left(g^0_{\bar{q}p} + \partial_p \partial_{\bar{q}} \varphi \right) = A\,. \tag{7.3}$$

This is a fourth-order equation in φ, which combines intriguingly basic features of a fully non-linear equation such as the Monge–Ampère equation, and a quasi-linear equation such as the minimal surface equation. The necessity of several notions of stability for the existence of constant scalar curvature metrics has been established by Tian [56], Donaldson [59], and Stoppa [60] in different situations. The sufficiency of K-stability has been established by Donaldson for two-dimensional toric varieties. For recent progress on the Kähler–Einstein problem for positive first Chern class, see [61]. A survey of various developments around this problem can be found in [62].

The problem of moduli of positive line bundles and constant scalar scalar curvature Kähler metrics is currently a very active research area in differential geometry and partial differential equations. It does not seem to have emerged as yet in theoretical physics, except of course for the well-known cases of Hermitian–Einstein and Kähler–Einstein metrics. However, if past history is any guide, it may reveal itself someday to be closely related to some important problems from physics.

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