QUANTUM GRAVITY AND NON-COMMUTATIVE SPACE-TIMES IN THREE DIMENSIONS: A UNIFIED APPROACH*

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These notes summarise a talk surveying the combinatorial or Hamiltonian quantisation of three dimensional gravity in the Chern–Simons formulation, with an emphasis on the role of quantum groups and on the way the various physical constants $(c, G, \Lambda, \hbar)$ enter as deformation parameters. The classical situation is summarised, where solutions can be characterised in terms of model space-times (which depend on c and Λ) together with global identifications via elements of the corresponding isometry groups. The quantum theory may be viewed as a deformation of this picture, with quantum groups replacing the local isometry groups, and non-commutative space-times replacing the classical model space-times. This point of view is explained, and open issues are sketched.

1. Introduction and motivation

1.1. Historical remarks

Giving a talk on three dimensional (3d) gravity at a meeting in Cracow is like carrying coal to Newcastle: the beginnings of the subject are usually traced back to the paper [1] by Andrzej Staruszkiewicz, alumnus and later professor at the Jagiellonian University in Cracow. Staruszkiewicz's paper, published in 1963, is about classical 3d gravity and its special features. The subject of 3d *quantum* gravity started only five years later with the realisation by Ponzano and Regge [2] that angular momentum theory plays an important role in this context.

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Gravity in 3d is now a large subject in its own right, which I cannot possibly review here. However, in this introductory part of the talk I will at least attempt to identify a few of the main themes and relate them to the approach followed here. Influential papers by Deser, 't Hooft and Jackiw written in the 1980s [3,4,5,6] on classical and quantum scattering of particles demonstrated the possibility of carrying out non-perturbative calculations of quantum scattering processes in 3d gravity. As we shall see, they also contain indications of the relevance of the braid group in describing such processes. These indications are elaborated in the later literature, see for example [7, 8, 9], and turn out to be closely related to the quantum group approach pursued in this paper.

The Chern–Simons formulation of 3d gravity, observed in [10] and elaborated in [11], establishes a connection between 3d gravity and a host of areas in mathematical physics, including topological field theory, knot theory, the theory of Poisson–Lie groups and of quantum groups. Since this paper is based on the Chern–Simons approach, we will see many of these connections.

The early paper by Ponzano and Regge, mentioned above, provides the foundation of the spin foam approach to 3d quantum gravity. This is perhaps the approach to 3d quantum gravity that contains the most directly useful lessons for 4d quantum gravity. I will not discuss this approach in this paper, and shall not attempt to summarise the large literature on it. However, it is worth pointing out that there are close links with Chern–Simons theory (spin foam state sums may be viewed as discretisation of the path integral) and to quantum groups, see [12] for an early paper and [13,14] for examples of recent papers with many references.

The possibility that non-commutative geometry is needed to describe space-time at the quantum level has long been a theme in quantum gravity research [15], see [16] for a recent discussion with some references. It is therefore interesting to ask if one can use the relatively tractable 3d situation to establish the role of non-commutative geometry in quantum gravity in a mathematically convincing way. Early discussions of non-commutative space-time coordinates appear in the paper [17]. Spacetime non-commutativity in 3d quantum gravity is studied, in different approaches, in [18,19,20,21]. Putting these approaches into one coherent picture is one of the objectives of this paper.

Finally, I should mention two further interesting themes of 3d gravity research which I will not be able to touch on here. One is the study of black holes, an introduction to which can be found in the book [22]. The other is the relation to 3d hyperbolic geometry, where [23] may provide a good starting point.

1.2. Topological degrees of freedom and interactions in 3d gravity

The Einstein field equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = -8\pi G T_{ab}$$

(in units where the speed of light is 1) determine the Ricci tensor of a space-time in terms of the energy momentum tensor. In space-time dimensions greater than three, the Ricci tensor contains less information than the Riemann tensor and it is possible to have metrically non-trivial (*i.e.* curved) space-times satisfying the vacuum ($T_{ab} = 0$) field equations. In three space-time dimensions, this is not possible. The Ricci tensor determines the Riemann tensor and, as a result, the only vacuum solutions of the Einstein equations with vanishing cosmological constant are flat [22]. This result simplifies Einstein's theory of gravity in 3d dramatically, but does not render it trivial. There are non-trivial solutions of the Einstein equations in the presence of matter, and, if the topology of the three-dimensional manifold representing the universe is non-trivial, there may be vacuum solutions which, though flat, have non-trivial holonomy. These observations are often summarised in the slogan that in 3d gravity there are no gravitational waves but that the theory has topological degrees of freedom.

The simplest solution of the Einstein equations illustrating the previous paragraph is the space-time surrounding a point-particle. The energy-momentum tensor is a Dirac delta-function with support on the world line of the particle.The metric solving the field equations is flat away from the world line and is singular on the world line. More precisely it is a direct product of a cone (space) and $\mathbb{R}$ (representing time) [22]. The line element, in terms of polar coordinates (r, ϕ), with $r > 0$, and a time coordinate t is simply

$$ds^2 = c^2 dt^2 - dr^2 - r^2 d\phi^2 \,. \tag{1.1}$$

However, the range of ϕ is $[0, 2\pi - \mu)$, where the parameter μ is related to the particle's mass m and to Newton's constant G via

$$\mu = 8\pi G m \,.$$

In three dimensions, the physical dimension of G is that of an inverse mass so that μ is a dimensionless, angular parameter. The effect of a particle on the geometry of space-times is, then, to cut out a wedge of size μ from the space-time surrounding the particle's world line.

It is instructive to consider the effect of the geometry (1.1) on light test particles. Such particles travel on geodesics, which are simply straight lines on the cone after it has been cut open. It is easy to check that geodesics

passing the particle of mass m on one side are deflecting relative to particles which pass it on the other side by the angle μ (in the coordinate system (t, r, ϕ)). This relative deflection is illustrated in Fig. 1 and is independent of the distance of closest approach between the heavy particle of mass m and the test particles (impact parameter). The interaction is topological in the sense that it only depends on whether the test particle passes on the left or the right of the heavy particle, and not on the relative distance. This kind of interaction is familiar from the Aharonov–Bohm interaction between electrons and a magnetic flux, and this analogy can be made precise: both interactions can be related to the braiding of the world lines of the interacting particles [9].
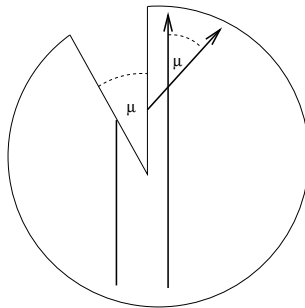

Fig. 1. Geodesics in the space surrounding a conical singularity with deficit angle μ.

1.3. Physical constants entering 3d quantum gravity

The four physical constant entering 3d quantum gravity are the speed of light c, Newton's constant G, Planck's constant $\hbar$ and the cosmological constant Λ. From these, we can form two length constants (remembering that the dimension of G is an inverse mass), namely

$$\text{Planck length } \ell_P = \frac{\hbar G}{c}, \qquad \text{Cosmological length scale } \ell_C = \frac{1}{\sqrt{\Lambda}}. \quad (1.2)$$

In this paper we will deal with both Lorentzian and Euclidean gravity, and we parametrise Euclidean and Lorentzian metrics in a unified fashion by allowing $c^2 < 0$ in the Euclidean situation. As a result, both the length parameters in (1.2) may be imaginary, depending on the sign of c^2 and Λ. From the ratio of the two length parameters we can form a dimensionless quantity. We define the deformation parameter

$$q = e^{-\frac{\hbar G \sqrt{\Lambda}}{c}}, \quad (1.3)$$

which may take values on the real line or the unit circle in the complex plane.

It is useful to clarify the role played by the various constants in 3d gravity in general terms at this stage. The observation of the previous section that, in the absence of matter, solutions of the Einstein equations are locally flat generalises in the presence of a cosmological constant to the statement that vacuum solutions are locally isometric to model space-times, which depend on the parameters c and Λ. For Lorentzian gravity with vanishing cosmological constant, for example, the model space-time is Minkowski space while for Euclidean gravity with positive cosmological constant it is the four-sphere with the round metric. The isometry groups of the model space-times inherit a dependence on c and Λ. In the examples above they are, respectively, the Poincaré group in 3d and the 4d rotation group $SO(4)$. Newton's constant G enters when one studies the dynamics of space-time and plays the role of a parameter in the Poisson structure and that of a coupling constant to matter. Finally, $\hbar$ enters in the quantisation and the dimensionless parameter q in (1.3), combining all four constant, controls the quantum theory when all the constants $1/c, G, \Lambda, \hbar$ are non-zero.

1.4. Motivation and outline of the paper

The goal of this paper is give a unified account of aspects of classical and quantum gravity in 3d, in which the physical parameters of the previous section enter as deformation parameters. Our account of classical gravity is based on the formulation of 3d gravity as a Chern–Simons gauge theory, where the local isometry groups play the role of the gauge groups. As we shall see, the parameters c and Λ enter in this description via the structure constants of the Lie algebra of the gauge group, while the parameter G enters via the inner product (or trace) on the Lie algebra which is used in the Chern–Simons action. We sketch the description of the phase space of 3d gravity as the moduli space of flat connections, and review the description of its Poisson structure in a formulation, due to Fock and Rosly [24], which makes essential use of classical r-matrices.

The description of the Poisson structure in terms of r-matrices is tailor-made for the quantisation via the combinatorial or Hamiltonian scheme pioneered in [25, 26] and [27]. In this scheme, the quantisation is controlled by quantum groups which are deformations of the local isometry groups of the model space-times, with deformation parameters G and $\hbar$ in addition to c and Λ. These quantum groups naturally act on non-commutative spaces, which one may interpret as deformations of the classical model space-times. This framework thus provides a concrete mathematical setting for exploring the proposal that, in quantum gravity, space-time should be mathematically modelled in terms of non-commutative geometry. We end with an evaluation of the successes and limitations of this approach to 3d quantum gravity.

2. Model space-times and isometry groups

The following treatment of the model space-times follows closely that in [28]. We use Roman letters $a, b, c \ldots$ for 3d space-times indices, with range for $\{0, 1, 2\}$ (in both the Euclidean and Lorentzian case). The model space-times arising in 3d gravity can be described in a simple and unified fashion in terms of the metric

$$g_{\mu\nu} = \text{diag}\left(-c^2, 1, 1, \frac{1}{\Lambda}\right) \tag{2.1}$$

in an auxiliary $\mathbb{R}^4$. Here we use Greek indices for the range $\{0, 1, 2, 3\}$. The model space-times can be realised as embedded hypersurfaces via

$$H_{c,\Lambda} = \left\{(t, x, y, w) \in \mathbb{R}^4 | -c^2 t^2 + x^2 + y^2 + \frac{1}{\Lambda}w^2 = \frac{1}{\Lambda}\right\}. \tag{2.2}$$

This two-parameter family includes the three-sphere S^3 ($c^2 < 0$, $\Lambda > 0$), doubles covers of hyperbolic space H^3 ($c^2 < 0$, $\Lambda < 0$), de Sitter space dS^3 ($c^2 > 0$, $\Lambda > 0$) and Anti-de Sitter space AdS^3 ($c^2 > 0$, $\Lambda < 0$). Double covers of Euclidean space E^3 and Minkowski M^3 space arise in the limit $\Lambda \to 0$, which one should take *after* multiplying the defining equation in (2.2) by Λ. In Fig. 2 we show the embedded model space-times (with one spatial dimension suppressed).

In order to be able to take the limit $\Lambda \to 0$ for the associated isometry groups it is best to work with the inverse metric

$$g^{\mu\nu} = \text{diag}\left(-\frac{1}{c^2}, 1, 1, \Lambda\right). \tag{2.3}$$

The Lie algebra generators of the isometry groups of (2.3) can conveniently be defined in terms of the Clifford algebra associated to (2.3) [28]. Thus we define generators γ^μ via

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu}, \tag{2.4}$$

so that the six Lie algebra generators are given by

$$M^{\mu\nu} = \tfrac{1}{4}[\gamma^\mu, \gamma^\nu]. \tag{2.5}$$

They have the commutation relations

$$\left[M^{\kappa\lambda}, M^{\mu\nu}\right] = g^{\kappa\mu}M^{\lambda\nu} + g^{\lambda\nu}M^{\kappa\mu} - g^{\kappa\nu}M^{\lambda\mu} - g^{\lambda\mu}M^{\kappa\nu}. \tag{2.6}$$

The advantage of the Clifford algebra approach is that one can immediately write down two naturally defined invariant bilinear forms. One, denoted
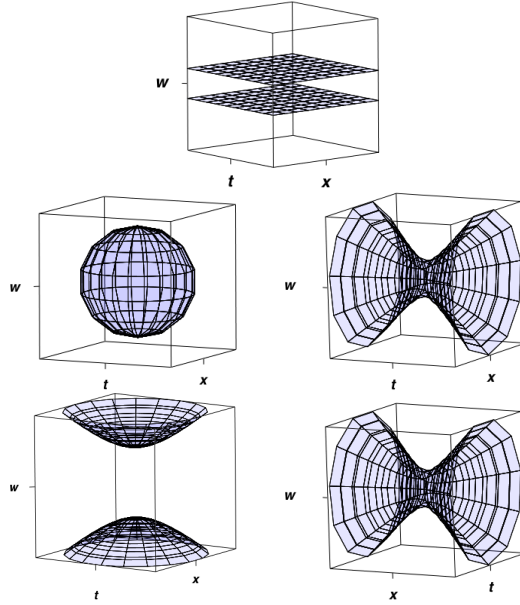
Fig. 2. (Double covers of) Model space-times for 3d gravity, shown as 2d models embedded in a 3d auxiliary space with coordinates (t, x, w) according to (2.2) (the second spatial coordinate y is suppressed). Euclidean and Minkowski space at the top, spherical and de Sitter space in the middle, hyperbolic and Anti-de Sitter space at the bottom.

$\langle \cdot, \cdot \rangle$ is defined by carrying out the Clifford multiplication and projecting onto the invariant, central element $\gamma^5 = \gamma^0 \gamma^1 \gamma^2 \gamma^3$. Multiplying by -4 for later convenience, the resulting inner product is non-zero whenever the indices on the basis vectors are complementary, for example

$$\langle M^{12}, M^{03} \rangle = -1, \qquad \langle M^{12}, M^{01} \rangle = 0, \qquad etc.$$

Another bilinear form $(\cdot, \cdot)$ is obtained by carrying out the Clifford multiplication and projecting onto the identity. Again rescaling by -4 for convenience we have a non-zero answer whenever the indices on the basis vectors match

$$\left(M^{12}, M^{12} \right) = 1, \qquad \left(M^{01}, M^{01} \right) = -\frac{1}{c^2}, \qquad \left(M^{13}, M^{13} \right) = \Lambda, \qquad etc.$$

As we shall see shortly, this is the Killing form on the Lie algebra.

We now express the above generators in more conventional 3d notation. For this purpose we define the three-dimensional totally antisymmetric tensor with downstairs indices via $\epsilon_{012} = 1$. Then we define the rotation generator $\tilde{J}_0$, the boost generators $\tilde{J}_1, \tilde{J}_2$ and translation generators $\tilde{P}_a$ via

$$\tilde{J}_a = \tfrac{1}{2}\epsilon_{abc}M^{bc}, \qquad \tilde{P}_a = g_{ab}M^{b3}, \qquad (2.7)$$

where we used the space-time part of the 4d metric $g_{\mu\nu}$ (2.1) to lower indices, and refer to [28] for a discussion of physical dimensions and interpretation of these generators (which are denoted by the same letters, but without tilde there). The Lie algebra brackets are now

$$\begin{aligned}
\left[\tilde{J}_a, \tilde{J}_b\right] &= \epsilon_{abc}J^c, & \left[\tilde{J}_a, \tilde{P}_b\right] &= \epsilon_{abc}\tilde{P}^c, \\
\left[\tilde{P}_a, \tilde{P}_b\right] &= -c^2\Lambda\epsilon_{abc}\tilde{J}^c,
\end{aligned} \qquad (2.8)$$

with indices raised via the inverse metric g^{ab}. The combination $-c^2\Lambda$ which occurs in the Lie brackets plays an important role in what follows, and we introduce

$$\lambda = -c^2\Lambda. \qquad (2.9)$$

The bilinear form $(\cdot,\cdot)$, already advertised as the Killing form, is

$$\left(\tilde{J}_a, \tilde{J}_b\right) = \kappa_{ab}, \qquad \left(\tilde{P}_a, \tilde{P}_b\right) = \lambda\kappa_{ab}, \qquad (2.10)$$

where

$$\kappa_{ab} = -\frac{1}{c^2}\,g_{ab} = \mathrm{diag}\left(1, -\frac{1}{c^2}, -\frac{1}{c^2}\right). \qquad (2.11)$$

The metric κ_{ab} is the most natural one on the Lie algebra $so(3)$ respectively $so(2,1)$ spanned by $\tilde{J}_0, \tilde{J}_1$ and $\tilde{J}_2$. Note that it differs from the space-time metric g_{ab}, but that it has the right physical dimensions and that imaginary c gives the usual Euclidean metric, as required.

It is one of the coincidences of 3d that space-time and the Lie algebra of rotations and/or boosts are both three-dimensional. Both are equipped with Euclidean respectively Lorentzian metrics, but our derivation shows that, in a physically natural normalisation and construction, the space-time and Lie algebra metrics come out differently. This is potentially confusing in calculations where indices are raised and contracted with these metrics, and most papers on 3d gravity use conventions where the two kinds of metrics coincide. We can achieve this by switching from the physical Lie algebra basis used thus far to a geometrical basis according to

$$\begin{aligned}
\tilde{J}_0 &\to J_0 = -\frac{|c|^2}{c^2}\tilde{J}_0, & \tilde{J}_1 &\to J_1 = |c|\tilde{J}_1, & \tilde{J}_2 &\to J_2 = |c|\tilde{J}_2, \\
\tilde{P}_0 &\to P_0 = -\frac{|c|^2}{c^2}\tilde{P}_0, & \tilde{P}_1 &\to P_1 = |c|\tilde{P}_1, & \tilde{P}_2 &\to P_2 = |c|\tilde{P}_2. \quad (2.12)
\end{aligned}$$

In this geometrical basis, all the generators J_a are dimensionless, and all the translation generators P_a have the dimension of inverse time. One checks that the Killing metric now takes the form

$$(J_a, J_b) = \eta_{ab} := \operatorname{diag}\left(1, -\frac{|c|^2}{c^2}, -\frac{|c|^2}{c^2}\right), \qquad (2.13)$$

which is $\operatorname{diag}(1,1,1)$ in the Euclidean and $\operatorname{diag}(1,-1,-1)$ in the Lorentzian case. Moreover, the Lie brackets take the *same* form as in (2.8),

$$[J_a, J_b] = \epsilon_{abc}J^c, \qquad [J_a, P_b] = \epsilon_{abc}P^c, \qquad [P_a, P_b] = \lambda\epsilon_{abc}J^c, \quad (2.14)$$

but all indices are now raised with the Lie algebra metric η_{ab}. This is convenient and we shall work in this basis for the remainder of this paper. We denote the Lie algebra with these brackets by $\mathfrak{g}_\lambda$. The conventions regarding the metric then agree with [29], but the convention regarding the naming of λ agrees with [11] and differs from [29], where Λ was used for what we call λ now. Conventions regarding the naming of the cosmological constant and the combination (2.9) differ in the literature, and the reader will need to take good care when comparing results from different sources.

The other bilinear form introduced in the Clifford language gives the following non-zero pairings

$$\langle J_a, P_b \rangle = c^2 \eta_{ab}. \qquad (2.15)$$

This pairing is non-degenerate for any value of λ and is crucial for the Chern–Simons formulation of 3d gravity, as we shall see.

In Table I we list Lie groups whose Lie algebras are (2.14). We have used the isomorphisms $SU(2)/\mathbb{Z}_2 = SO(3)$ and $SL(2,\mathbb{R})/\mathbb{Z}_2 = SO(2,1)_0$, the identity component of $SO(2,1)$. The isometry groups are determined by their Lie algebras only up to coverings, and our choice in Table I is one of convenience. In the following, we write G_λ for this family of Lie groups.

TABLE I

Local isometry groups in 3d gravity.

Cos. constant	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$SU(2) \ltimes \mathbb{R}^3$	$SL(2,\mathbb{R}) \ltimes \mathbb{R}^3$
$\Lambda > 0$	$SU(2) \times SU(2)$	$SL(2,\mathbb{C})$
$\Lambda < 0$	$SL(2,\mathbb{C})$	$SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$

3. The Chern–Simons formulation of 3d gravity

In Cartan's approach to Riemannian geometry [30] the fundamental geometrical object is a connection which combines an orthonormal frame field (or vielbein) e_a and the spin connection ω_{ab} on the orthonormal frame bundle into the so-called Cartan connection. Concretely, in the case of 3d geometry, we combine the dreibein with the translation generators P_a of (2.7) and the local connection one-forms $\omega^a = \frac{1}{2}\epsilon^{abc}\omega_{bc}$ with the rotation and/or Lorentz generators J^a into the local one-form

$$A = e_a P^a + \omega_a J^a\,, \tag{3.1}$$

taking values in the Lie algebra $\mathfrak{g}_\lambda$. The curvature

$$F_A = dA + \tfrac{1}{2}[A \wedge A] = R + C + T \tag{3.2}$$

of the Cartan connection combines the Riemann curvature of the spin connection $\omega = \omega_a J^a$,

$$R = d\omega + \tfrac{1}{2}[\omega \wedge \omega]\,,$$

a cosmological term

$$C = \frac{\lambda}{2}\epsilon^{abc} e_a \wedge e_b J_c\,,$$

and the torsion

$$T = (de^c + \epsilon^{abc}\omega_a \wedge e_b)P_c\,.$$

In the Cartan approach to general relativity (in any dimension), the Einstein–Hilbert action is expressed in terms of the vielbein and the connection, which are treated as independent variables. The action is called the Palatini action when interpreted in this way. In this approach, the condition of vanishing torsion (in the absence of spin sources) follows as a variational equation rather than as an a priori condition. It turns out that, in three dimensions, the Einstein–Hilbert (or Palatini) action is simply the Chern–Simons action for the Cartan connection (3.1), with the bilinear form (2.15) used as an inner product [10, 11]. However, beyond the equality of the actions, the relationship between the Chern–Simons formulation and the Einstein formulation of 3d gravity is subtle: non-invertible dreibeins e_a may occur in the Chern–Simons formulation but are ruled out in the Einstein approach, based on metrics. This changes the nature of gauge orbits in the two cases, so that the physical phase spaces are, in general, different. This was pointed out in a $1+1$ dimensional context in [31] and was demonstrated in an explicit example involving four particles in 3d gravity in [32]. Our approach to 3d gravity in the remainder of this paper is based on the Chern–Simons formulation.

We discuss the Chern–Simons action in terms of the general bilinear form

$$(\cdot,\cdot)_{\alpha\beta} = \alpha\langle\cdot,\cdot\rangle + \beta(\cdot,\cdot) \tag{3.3}$$

on the Lie algebra $\mathfrak{g}_\lambda$. This form is non-degenerate iff [33]

$$\alpha^2 - \lambda\beta^2 \neq 0, \tag{3.4}$$

and the associated action

$$
\begin{aligned}
I_{\alpha\beta}(A) &= \int_{M^3} (A \wedge dA)_{\alpha\beta} + \tfrac{1}{3}(A \wedge [A, A])_{\alpha\beta} \\
&= \alpha \int_{M^3} \left(2e^a \wedge R_a + \frac{\lambda}{3}\epsilon_{abc}e^a \wedge e^b \wedge e^c \right) \\
&\quad + \beta \int_{M^3} \left(\omega^a \wedge d\omega_a + \tfrac{1}{3}\epsilon_{abc}\omega^a \wedge \omega^b \wedge \omega^c + \lambda e^a \wedge T_a \right), \quad (3.5)
\end{aligned}
$$

contains the gravitational action (the terms proportional to α), the Chern–Simons action for the spin connection and additional terms involving torsion. This general action was first considered by Mielke and Baekler [34] and recently revisited in [35], where the analogy between the terms proportional to β and the Immirzi term in 4d was stressed. The variational equations which follow from the general action (3.5) are simply the flatness condition for the Cartan connection, *i.e.* the vanishing of (3.2), provided the form (3.3) is non-degenerate. This appears to imply that the family of actions (3.5) leads to equivalent physics provided the condition (3.4) holds. However, as argued in [33], the induced canonical structure of the phase space does depend on the ratio of α and β. Since we are only interested in the Chern–Simons formulation of 3d gravity here, we set

$$\alpha = \frac{1}{16\pi G}, \qquad \beta = 0 \tag{3.6}$$

from now onwards.

The gauge formulation of 3d gravity can easily and naturally be extended to include minimal coupling between the gauge field and point particles. This was first discussed in detail in [36] and is reviewed in our notation in [33], where the dependence of the coupling on the parameters α and β is also discussed. We are not able to discuss the coupling to particles, the Poisson structure and the division by gauge equivalence in the space available here. Instead, we summarise the results in the next section, and motivate them in general, geometric terms.

4. Classical r-matrices and Poisson brackets
on the space of holonomies

Having established that, in the Chern–Simons formulation, classical solutions of the field equations are flat G_λ-connections, we can characterise the phase space of 3d gravity on a manifold M^3 in the Chern–Simons formulation as the space of flat G_λ-connections on M^3, modulo gauge transformations. In order to make this precise and concrete, we consider 3d universes of topology $M^3 = \mathbb{R} \times S$, where S is a two-dimensional manifold representing space. Then one can show [11] that the phase space is the moduli space of flat G_λ-connections on S (*i.e.* the space of flat G_λ-connections moduli gauge transformations), equipped with the Atiyah–Bott symplectic structure [37,38], which is defined in terms of the bilinear form used in the Chern–Simons action. With the choice (3.6) this bilinear form is

$$\frac{1}{16\pi G}\langle\cdot,\cdot\rangle. \tag{4.1}$$

Therefore, in the Chern–Simons formulation, and assuming the factorisation $M^3 = \mathbb{R} \times S$, the task of constructing a theory of quantum gravity amounts to quantising the moduli space of flat G_λ-connections on S, with a symplectic structure induced by (4.1).

Despite the elegance and generality of this result, a precise mathematical description of the moduli space and a rigorous quantisation remains a difficult task. In the case where S is a compact surface of genus $g \geq 2$, the moduli space can be characterised in terms of the moduli space $\mathcal{A}_S$ of flat $SU(2)$ connections in the Euclidean case and in terms of Teichmüller Space $\mathcal{T}_\mathrm{S}$ (a component of the moduli space of flat $SL(2,\mathbb{R})$ connections) in the Lorentzian case. These results, due to [23,39,40] and also discussed in more physical language in [41], are summarised in Table II.

TABLE II

Phase space of 3d gravity for universes of the form $\mathbb{R} \times S$, with S compact and genus ≥ 2 (quoted from [41]).

Cos. constant	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$T^*\mathcal{A}_S$	$T^*\mathcal{T}_\mathrm{S}$
$\Lambda > 0$	$\mathcal{A}_S \times \mathcal{A}_S$	$\mathcal{T}_\mathrm{S} \times \mathcal{T}_\mathrm{S} \sim T^*\mathcal{T}_\mathrm{S}$
$\Lambda < 0$	$\mathcal{T}_\mathrm{S} \times \mathcal{T}_\mathrm{S} \subset T^*\mathcal{T}_\mathrm{S}$	$\mathcal{T}_\mathrm{S} \times \mathcal{T}_\mathrm{S} \sim T^*\mathcal{T}_\mathrm{S}$

For each of the symplectic manifolds in the table, one may in principle attempt a quantisation and subsequent interpretation in terms of 3d quantum gravity. In this paper I summarise a description of the moduli space and

its Poisson structure which is closely based on the parametrisation in terms of G_λ-valued holonomies, and which uses a concrete and unified description of the Poisson structure, which is tailor-made for quantisation. The idea for this description is due to Fock and Rosly [24]. It is the foundation of the combinatorial or Hamiltonian quantisation programme for Chern–Simons theory, described in [25, 26, 27].

Fock and Rosly's description of the phase space starts with the observation that flat connections on a manifold are characterised by their holonomies along non-contractible paths. The moduli space of flat connections on a surface S can thus be parametrised by the set of holonomies along closed paths which generate the fundamental group of S, modulo gauge transformations at the common starting and end point of those paths. So far we have assumed that S is a compact manifold without boundary, but in the Fock and Rosly description it is easy to include punctures decorated with co-adjoint orbits of G_λ. This is desirable in the context of 3d gravity, since a co-adjoint orbit of G_λ physically correspond to the phase space of a point particle, and the 'decoration' of a puncture with a co-adjoint orbit is precisely the effect of minimal coupling between the Cartan connection (3.1) and the point particle's degrees of freedom. Moreover, this minimal coupling correctly reproduces the gravitational coupling between a point particle and the gravitational field, with momentum acting as a source of curvature and spin acting as a source for torsion. For details we refer the reader to the papers [33, 36] and for a relatively brief but pedagogical account to [42].

The effect of the minimal coupling to co-adjoint orbits on the holonomies can be summarised as follows. Using the inner product (4.1), co-adjoint orbits can be written as adjoint orbits. For particles with mass m and spin s, these orbits are of the form

$$O_{ms} = \left\{ g \left(-\mu J_0 - \sigma P_0 \right) g^{-1} | g \in G_\lambda \right\} ,$$

where

$$\mu = 8\pi G m , \qquad \sigma = 8\pi G s .$$

Decorating a puncture on S with such an orbit forces the holonomy around the puncture to lie in the conjugacy class

$$\mathcal{C}_{\mu\sigma} = \left\{ g \left(\exp(-\mu J_0 - \sigma P_0) \right) g^{-1} | g \in G_\lambda \right\} .$$

For a genus g surface S with n punctures and orbit labels μ_i, σ_i, $i = 1 \ldots n$, a set of generators of the fundamental groups is shown in Fig. 3. The moduli space of flat G_λ-connections can be written in terms of the extended phase space

$$\tilde{\mathcal{P}} = G_\lambda^{2g} \times \mathcal{C}_{\mu_n\sigma_n} \times \ldots \mathcal{C}_{\mu_1\sigma_1}, \tag{4.2}$$

by imposing the condition that a suitable composition of the generating loops is contractible (and hence has trivial holonomy), and by dividing by conjugation at the base point

$$\mathcal{P} = \left\{ (A_g, B_g, \ldots, A_1, B_1, M_n, \ldots M_1) \in \tilde{\mathcal{P}} | \right.$$

$$\left. [A_g, B_g^{-1}] \ldots [A_1, B_1^{-1}] M_n \ldots M_1 = 1 \right\}/\text{conjugation} \,. \qquad (4.3)$$
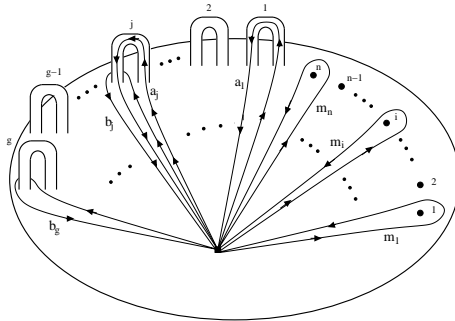

Fig. 3. Generators of the fundamental group of a compact surface with punctures.

The trick introduced by Fock and Rosly is to define a (symplectic) Poisson structure on the extended phase space $\tilde{\mathcal{P}}$ (4.2) in such a way that the G_λ-conjugation action on $\tilde{\mathcal{P}}$ is symplectic and that the symplectic quotient by it gives $\mathcal{P}$ with the Atiyah–Bott symplectic structure. The Poisson structure on $\tilde{\mathcal{P}}$ is defined in terms of a classical r-matrix, *i.e.* an element $r \in \mathfrak{g}_\lambda \otimes \mathfrak{g}_\lambda$ which satisfies the classical Yang–Baxter equation (CYBE)

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0 \,, \qquad (4.4)$$

where we have used standard notation, explained, for example in textbooks like [43] or [44]. The information about the inner product used in the definition of the Atiyah–Bott symplectic structure (or, equivalently, in the Chern–Simons action) is encoded in r via the following compatibility requirement:

Definition *An r-matrix is compatible with a Chern–Simons action if it satisfies the CYBE (4.4) and if its symmetric part is equal to the Casimir associated to the* Ad*-invariant, non-degenerate symmetric bilinear form used in the Chern–Simons action.*

In our case, the relevant Casimir operator for the 'gravitational' bilinear form (4.1) is

$$K = 16\pi G \left(J_a \otimes P^a + P_a \otimes J^a \right) \,. \qquad (4.5)$$

A family of compatible r-matrices is given by [29, 45]

$$r = 32\pi G \left(P_a \otimes J^a + \epsilon_{abc} n^a J^b \otimes J^c \right) , \qquad n_a n^a = -\lambda , \qquad (4.6)$$

where we use the metric (2.13) to lower and contract indices.

Two comments are in order here. The first concerns the dependence of the solution on the real vector $\boldsymbol{n} = (n^0, n^1, n^2)$ which has to satisfy the given constraint but is otherwise arbitrary. Thus, for $\lambda < 0$, the vector $\boldsymbol{n}$ is any vector of length $\sqrt{-\lambda}$ in the Euclidean (hyperbolic) case, but is necessarily time-like in the Lorentzian (de Sitter) case. For $\lambda = 0$, $\boldsymbol{n}$ vanishes in the Euclidean case but may be any light-like vector in the Lorentzian case. For $\lambda > 0$, $\boldsymbol{n}$ is space-like in the Lorentzian (Anti-de Sitter) case, while there is no real solution in the Euclidean case. However, the Euclidean case with $\lambda > 0$ (and hence $\Lambda > 0$) is the only case where the model space (S^3) and the local isometry group $SU(2) \times SU(2)$ are both compact, and the Chern–Simons theory is simply two copies of $SU(2)$ Chern–Simons theory, which is extensively studied in the literature, see [46] for an early paper. I will not say much about this case in the following, although it seems interesting and worthwhile to relate the many results about $SU(2)$ Chern–Simons theory to the framework discussed here, and to interpret them in terms of 3d gravity. Presumably this would involve using a complex vector $\boldsymbol{n}$ and imposing a suitable reality condition after quantisation.

The second comment concerns the non-uniqueness of the solutions (4.6). These solutions all amount to equipping the Lie algebras $\mathfrak{g}_\lambda$ with the structure of a classical double, see [43, 49] for general background and [29] for an explanation in the context of 3d gravity. However, other r-matrices are known, which are also compatible with the bilinear form (4.1) but which do not belong to the family (4.6), see [33] for examples and the forthcoming paper [47] for a systematic discussion. This gives rise to an ambiguity in the implementation of the Fock–Rosly prescription and the subsequent quantisation, but presumably leads to the same quantum theory. This issue has not been conclusively settled, and is also discussed in [47]. One advantage of working with the r-matrices associated to classical doubles is that one may quantise by going to the associated quantum double. This is what we will review in the next section.

The Fock–Rosly Poisson structure on $\tilde{\mathcal{P}}$ is determined in terms of a compatible r-matrix. The formulae for the brackets are explicit but lengthy, and we refer the reader to [24] or [27] for details. Some understanding of it can be gained from the observation, made in [48], that the Poisson brackets can be 'decoupled' after a suitable coordinate change, and that, as a symplectic manifold, $\tilde{\mathcal{P}}$ is isomorphic to a direct sum of g copies of the Heisenberg double of the Poisson–Lie group G_λ (with the Sklyanin Poisson–

Lie structure defined by r) and the manifolds $\mathcal{C}_{\mu_i \sigma_i}$, $i = 1, \ldots n$ viewed as symplectic leaves of the dual Poisson–Lie group G^*_λ

$$\tilde{\mathcal{P}} \simeq \text{Hei}(G_\lambda) \times \ldots \times \text{Hei}(G_\lambda) \times \mathcal{C}_{\mu_n \sigma_n} \times \ldots \mathcal{C}_{\mu_1 \sigma_1} . \qquad (4.7)$$

The general definitions of the Sklyanin, Heisenberg double and dual Poisson structures can be found in the paper [48] and also in the textbook [43] or the lecture notes [49]. We will give some further background in the next section, but here we note that all of these structures for the family of groups G_λ with the r-matrices (4.6) are explicitly given in [29]. For example, in the case of vanishing cosmological constant (and $\boldsymbol{n}$ vanishing), one finds [50,51]

$$\text{Hei}\left(SL(\mathbb{R}) \ltimes \mathbb{R}^3\right) \simeq T^*(SL(2,\mathbb{R}) \times SL(2,\mathbb{R})) .$$

In the Fock–Rosly description of the phase space (4.3) one still needs to impose a constraint in $\tilde{\mathcal{P}}$, and take a quotient. We will not pursue this here since we are mainly interested in the quantum theory. Our approach to quantisation is to quantise $\tilde{\mathcal{P}}$ first, and then to take the quotient at the quantum level.

5. Quantum groups and 3d quantum gravity

5.1. The combinatorial quantisation programme and associated quantum groups

The task of constructing a quantum theory of 3d gravity in the Chern–Simons approach followed here is that of quantising the Poisson algebra of functions on the physical phase space (4.3), and of finding a unitary, irreducible representation (UIR) of the quantised algebra. By 'quantisation' of a Poisson manifold M we mean, generally speaking, a deformation $F_h(M)$ of the algebra of functions on that manifold with a multiplication depending on a parameter h in such a way that the commutator of two elements in $F_h(M)$ to first order in that parameter equals the Poisson bracket of the classical limit of those elements [43]. Details, for example the precise class of functions ($\mathcal{C}^\infty$ or some algebraic subset), depend on the Poisson manifold in question.

In the combinatorial approach, one simplifies this task by first quantising the extended phase space (4.2), and then imposing the reduction to (4.3) at the quantum level by a suitable condition on the Hilbert space carrying the UIR of the quantisation of (4.2). An important advantage of the combinatorial approach is that one really only needs to carry out the quantisation of the building blocks entering the decomposition of the extended phase space (4.7), and that these, in turn, can all be constructed from *one* quantum group H and its representations.

The quantum group H in question is the quantisation of the so-called dual Poisson–Lie group G_λ^* of G_λ (with the Sklyanin Poisson–Lie structure defined by the r-matrix (4.6)). This is explained in general terms in [25, 26] and in the particular case of semi-direct products like the Euclidean or Poincaré groups in [51]. It can be motivated as follows.

The dual Poisson–Lie group G_λ^* is a non-linear analogue of the Kirillov–Kostant–Souriau (KKS) Poisson structure on the dual $\mathfrak{g}_\lambda^*$ of the Lie algebra $\mathfrak{g}_\lambda$ [49,52]. Since the quantisation of the KKS structure on $\mathfrak{g}_\lambda^*$ is the universal enveloping algebra $U(\mathfrak{g}_\lambda)$, it is not surprising that the quantisation of the Poisson algebra of G_λ^* is a deformation of $U(\mathfrak{g}_\lambda)$. Thus we see already at this general level that the quantum groups H are Hopf algebras obtained by deforming the local isometry groups G_λ (or more precisely, of their group algebras). We therefore refer to them as quantum isometry groups in the following. There is further similarity between the canonical Poisson structure on $\mathfrak{g}_\lambda^*$ and G_λ^*: the symplectic leaves of the former are co-adjoint orbits while the symplectic leaves of the latter are conjugacy classes in G_λ [43,49]. Given the non-degenerate bilinear pairing (4.1) on $\mathfrak{g}_\lambda$, co-adjoint orbits may be thought of as adjoint orbits in $\mathfrak{g}_\lambda$, and conjugacy classes in G_λ may be thought of as non-linear deformations of these.

The irreducible representations of a Lie algebra can be obtained by quantising the KKS Poisson algebra and imposing the conditions which define the co-adjoint orbits in terms of suitable Casimir operators. This analogy, and the general comments of the previous paragraph, go some way towards motivating the result that the quantisation of the conjugacy classes $\mathcal{C}_{\mu_i \sigma_i}$ in the decomposition (4.7) gives UIRs $V_{\mu_i \sigma_i}$ of the quantum group H (with possible quantisation conditions on the labels μ_i, σ_i). The quantisation of the classical Heisenberg double of G_λ is the Heisenberg double of the Hopf algebra H [44]. Its unique irreducible representation, in the cases that have been studied, is a quantum group analogue of the regular representation of a group, and we therefore denote it by $\text{Reg}(H)$. We thus arrive at the following Hilbert space for the quantisation of the extended phase space (4.7)

$$\tilde{\mathcal{H}} = \text{Reg}(H)^g \otimes V_{\mu_n \sigma_n} \otimes \ldots V_{\mu_1 \sigma_1} . \tag{5.1}$$

This space is, by construction, a (reducible) representation of the quantum group H. The Hilbert space for quantisation of the physical phase space (4.3) is the invariant part under this H-action [25, 26, 27]

$$\mathcal{H} = \text{Inv}_H \left(\tilde{\mathcal{H}} \right) . \tag{5.2}$$

In order to carry out the combinatorial quantisation programme in practice one needs to construct the quantum group H and to find the representations appearing in (5.1). The construction of the quantum group H is

facilitated by the fact that the r-matrices (4.6) equip $\mathfrak{g}_\lambda$ with the structure of a classical double of either $sl(2, \mathbb{R})$ (in the Lorentzian case) or $su(2)$ (in the Euclidean case) with suitable bialgebra structures, given in [29]. Following the principle that the quantisation of the double is quantum double of the quantisation [53], the family of quantum groups H can thus easily be found. We list them in Table III, which should be seen as a quantised and 'gravitised' version of Table I of the classical isometry groups. We will not give definitions or lists of generators and relations for any of these quantum groups here, but refer to the standard textbooks [43, 44]. However, to gain some physical understanding it is worth noting that half the generators should be interpreted as rotation/boost generators and the other half as momentum generators. Thus, for example in the Lorentzian case of vanishing cosmological constant

$$D(U(su(1,1))) = U(su(1,1)) \ltimes \mathbb{C}(SU(1,1)), \tag{5.3}$$

as an algebra, where $\mathbb{C}(SU(1,1))$ are complex-valued, smooth functions on $SU(1,1)$. The generators J_a of $U(su(1,1)))$ are simply the rotation generator J_0 and the boost generators J_1, J_2 already encountered in (2.14), while elements of $\mathbb{C}(SU(1,1))$ should be thought of as functions or coordinates on the non-linear momentum space $SU(1,1)$, see [42] for details and references, and also below for further remarks. Finally, the parameter q appearing in the table is the one introduced at the beginning (1.3). It combines all four physical parameters entering quantum gravity with a cosmological constant.

TABLE III

Quantum isometry groups in 3d quantum gravity, $q = e^{-\frac{\hbar G \sqrt{\Lambda}}{c}}$.

Cos. constant	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$D(U(su(2)))$	$D(U(su(1,1)))$
$\Lambda > 0$	$D(U_q(su(2)))$, q root of unity	$D(U_q(su(1,1)))$ $q \in \mathbb{R}$
$\Lambda < 0$	$D(U_q(su(2)))$, $q \in \mathbb{R}$	$D(U_q(sl(2,\mathbb{R})))$, $q \in U(1)$

The combinatorial quantisation programme has been carried out to various degrees of completeness in the different cases. For the Euclidean case with vanishing cosmological constant, the importance of the quantum double $D(U(su(2)))$ was first pointed out in [8], and the proof that it plays the role of the quantum isometry group H in the combinatorial approach to Euclidean quantum gravity without cosmological constant was given in [45]. The Lorentzian case was considered in [9] and the general situation of Chern–Simons theory with certain semidirect product gauge groups was considered

in [51]. The situation where the classical gauge group is $SL(2,\mathbb{C})$ (*i.e.* Euclidean with $\Lambda < 0$ or Lorentzian with $\Lambda > 0$) was studied in [54], with the relevant quantum group already constructed in [55]. The Euclidean case with $\Lambda > 0$ is essentially the Turaev–Viro model. Finally, the very interesting Anti-de Sitter case (Lorentzian and $\Lambda < 0$) has, unfortunately, not received much attention in the framework sketched here.

5.2. Non-commutative momentum addition, braiding and non-commutative space-times

Having constructed the quantum groups which control the construction of 3d quantum gravity according to the combinatorial scheme it is natural to ask what one can learn from them about the physics of 3d quantum gravity.

Formally, the role of the quantum isometry groups listed in Table III is strictly auxiliary. The physical Hilbert space (5.2) is, by definition, invariant under the action of those quantum groups. Physical observables which act on this Hilbert space (see [56] for a discussion of classical examples) are not obviously related to the quantum isometry groups. As already mentioned (and discussed further in the Conclusion), the r-matrix used in the Fock–Rosly scheme, and hence the associated quantum group, is not uniquely determined. Both of these observations suggest that the quantum groups in Table III have only an indirect physical significance.

On the other hand, the quantum isometry groups, their representations and even their quantum R-matrices can be directly related to physical properties of particles in 3d quantum gravity. We will illustrate this for the case of vanishing cosmological constant. In that case, the quantum doubles appearing in Table III are quantum doubles of the Lie groups $SU(2)$ in the Euclidean case and $SU(1,1)$ (which is isomorphic to $SL(2,\mathbb{R})$) in the Lorentzian case. These quantum doubles are semi-direct products as algebras as shown in (5.3), and have a representation theory which is very similar to those of the Euclidean and Poincaré groups [8, 57, 58]. The only difference is that the 'mass shell' in momentum space which characterises UIRs of the Euclidean and Poincaré group become conjugacy classes in the non-linear momentum spaces ($SU(2)$ in the Euclidean case and $SU(1,1)$ in the Lorentzian case). Physically, this means that momenta are no longer vectors but group elements of $SU(2)$ or $SU(1,1)$ and that momentum 'addition' is implemented by group multiplication in $SU(2)$ or $SU(1,1)$ instead of vector addition. These non-linear and non-commutative properties of momentum addition for gravitating particle reflect the use of holonomies in characterising particle properties, as used in early papers on 3d gravity [3, 7]. We can even see it in the simplest non-trivial example of 3d space-time, namely the cone shown in Fig. 1. The space-time is fully characterised by the deficit

angle μ, which is the mass of the particle in units of the Planck mass $1/8\pi G$. However, the angular nature of this parameter fits very well into the picture of $SU(1,1)$-valued momenta: we simply think of μ as a rotation, *i.e.* a particular element of $SU(1,1)$.

A closely related property of gravitating particles is their scattering, as analysed in some of the early papers on 3d quantum gravity [5, 6]. It turns out that the S-matrix for the scattering of two massive and spinning particles can also be interpreted in terms of quantum groups and the sort of topological interactions discussed in Sect. 1.2. As shown in [9], the S-matrix is naturally related to the R-matrix of the quantum double $D(U(su(1,1)))$.

Finally, the curved and non-Abelian nature of the momentum manifold suggests that naturally defined positions coordinates (which should generate translations on momentum space) should be non-commutative. One can argue this more formally by demanding that momentum and position algebras should be dual as Hopf algebras, leading to the family of Hopf algebras shown in Table IV. A particular, and much studied example is the 'spin space-time' with generators X_0, X_1, X_2 and commutation relations

$$[X_a, X_b] = \ell_{\mathrm{P}} \epsilon_{abc} X^c \,, \tag{5.4}$$

where $\ell_{\mathrm{P}} = 8\pi\hbar G$ is the Planck length in 3d gravity, and both the Euclidean and Lorentzian interpretation apply. This non-commutativity of positions was already considered in [17] and [18], and appears naturally in the quantum group theoretical framework considered here. It can also be derived in other approaches, namely in a path integral for particles where gravitational field degrees of freedom have been integrated out [20] or in a coset construction [21], which is analogous to the way the classical space-times (2.2) can be obtained as homogeneous spaces of the classical isometry groups G_λ. Finally, the role of the quantum double $D(U(su(2)))$ as a quantum isometry group of the 3d (Euclidean) was noted in [19], where the latter was studied from the point of view of non-commutative differential geometry.

TABLE IV

Momentum/position algebras in 3d quantum gravity, $q = e^{-\frac{\hbar G \sqrt{\Lambda}}{c}}$.

Cos. const.	Euclidean ($c^2 < 0$)	Lorentzian ($c^2 > 0$)
$\Lambda = 0$	$\mathbb{C}(SU(2)) \,/\, U(su(2))$	$\mathbb{C}(SU(1,1))) \,/\, U(su(1,1))$
$\Lambda > 0$	$\mathbb{C}_q(SU(2)) \,/\, U_q(su(2))$, q root of unity	$\mathbb{C}_q(SU(1,1)) \,/\, U_q(su(1,1))$ $q \in \mathbb{R}$
$\Lambda < 0$	$\mathbb{C}_q(SU(2)) \,/\, U_q(su(2))$, $q \in \mathbb{R}$	$\mathbb{C}_q(SL(2,\mathbb{R})) \,/\, U_q(sl(2,\mathbb{R}))$, $q \in U(1)$

It is interesting that physical arguments, path integrals, coset constructions and general quantum group theoretical considerations all lead to the same non-commutative space-times. One way of exploring the physical significance of this non-commutativity is to study representations of the quantum doubles in Table III in position space. The requires Fourier-transforming the usual formulation of the representations in momentum space, in analogy to the way the UIRs of the Poincaré group can be Fourier transformed into the solution space of the familiar wave equations of relativistic physics (Klein–Gordon, Dirac, Maxwell *etc.*). This was carried out for $D(SU(2))$ in [59] and is considered for the Lorentzian case in [60].

6. Outlook and conclusion

We have seen that the combinatorial quantisation of the Chern–Simons formulation of 3d gravity gives a unified picture of the various regimes of 3d gravity, with the physical parameters c, Λ, G and $\hbar$ entering as deformation parameters in distinctive ways. Quantum groups naturally replace the classical isometry groups in this approach to 3d quantum gravity, and non-commutative space-times replace the classical model space-times. In general, the relation between the quantum isometry groups and the physical Hilbert space of 3d quantum gravity is a formal one, but we have seen that aspects of the quantum isometry groups like the non-commutative momentum addition and the braiding via the quantum R-matrix have a direct physical interpretation. It is worth noting that it is possible to take a Galilean limit $c \to \infty$ in the framework discussed here [28, 61], and that the non-commutative quantum space is the Moyal plane in that case, with a time-dependent non-commutativity of the spatial coordinates.

In order to clarify the physical interpretation of quantum isometry groups and the associated non-commutative space-times it may be useful to consider universes with a boundary instead of the spatially compact universes considered in this paper. The treatment of boundaries in the classical theory is discussed in [32, 62, 63] but a general treatment of the quantisation has not been given. Another approach would be to work directly on the physical phase space as in [56, 64], and to attempt the quantisation there.

Other quantum groups than quantum doubles have been discussed in relation to 3d quantum gravity, notably bicrossproducts or κ-Poincaré algebras which were originally introduced in 4d [65, 66, 67]. As shown in [33], the κ-Poincaré algebra with the usual time-like deformation parameter is *not* compatible with 3d gravity in the combinatorial framework. On the other hand, κ-Poincaré algebras with space-like deformation parameters are possible. This and other quantisation ambiguities of 3d quantum gravity are discussed in the forthcoming paper [47].

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