COLOR–FLAVOR TRANSFORMATION AND ITS APPLICATIONS TO LATTICE FIELD THEORY*

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We overview the color–flavor transformation which has various applications to problems as diverse as lattice gauge theory, random network models, and dynamical systems with disorder. We present this transformation in the context of the fermionic sector of lattice QCD and induced lattice gluodynamics. Application to low energy QCD on a lattice leads to a theory where the inverse number of colors appears as expansion parameter. We use a saddle point approximation to estimate the partition function both in the pure mesonic sector and in the case of a single baryon on a mesonic background. The effective chiral Lagrangian of QCD is recovered up to the terms of order $\mathrm{O}(p^4)$. We also consider the color–flavor transformation applied to the pure lattice gluodynamics, in which the gauge theory is induced by a heavy chiral scalar field. The effective, color–flavor transformed theory is expressed in terms of gauge singlet matrix fields carried by the lattice links.

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1. Introduction

The complexity of quantum chromodynamics (QCD) originates from the random character of the gauge field in the low-energy regime, while at high energy (small scales) the theory is asymptotically free. A hierarchy of scales in QCD is provided by $\Lambda_\chi \sim 1$ GeV, the scale of chiral symmetry breaking, and $\Lambda_{\mathrm{QCD}} \sim 0.18$ GeV, the scale at which confinement occurs. Perturbative QCD governs the scales of momenta p up to Λ_χ, where the coupling constant $g(\Lambda_\chi) \sim 1$. It blows up at the scale of about Λ_{QCD}.

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In the past two decades a great deal was learned about the non-perturbative structure of QCD vacuum at scales between Λ_χ and Λ_{QCD}. The guiding idea is to introduce a low-energy effective Lagrangian which encodes the symmetries of the underlying QCD.

To compute these effective theories, one may start from the full QCD action, and integrate out the high-energy degrees of freedom (quarks and gluons) in order to construct a low-energy action in terms of physically significant variables (meson and baryon fields). In this way one was able to recover the QCD chiral Lagrangian [1,2], which had been phenomenologically introduced by Weinberg [3]. Another approach to study non-perturbative QCD is to use the lattice regularization scheme [4].

It has been shown how to perform the separation between effective and non-effective degrees of freedom of the continuum theory, starting from the lattice formulation [5]. Namely, a lattice effective Lagrangian describes the long-distance dynamics of lattice QCD (LQCD). From there, one gets the continuum chiral Lagrangian by expanding the lattice effective theory in powers of the lattice spacing and external momenta [6].

This procedure was actually initiated a long time ago [7, 8]; it relied on a "bosonization" of the the strong-coupling LQCD action, and a large-N_c or large-dimension expansion. More recently, an alternative kind of bosonization was introduced [9], relying on a more sophisticated mathematical formalism, the "color–flavor transformation". This formalism relates two different formulations of a certain class of theories, in other words, this is a transformation of duality. The color–flavor transformation has been applied to the truncated lattice model which describes the quarks coupled to the background $\mathrm{SU}(N_c)$ gauge field [10, 11].

Another approach to analyze non-perturbative QCD, so-called Induced QCD, was initiated in 90s [12, 13]. In the Kazakov–Migdal model [12] the Wilson lattice action is induced by the auxiliary heavy scalar matrix fields which are taken in adjoint representation of the gauge group $\mathrm{SU}(N_c)$. This field can be diagonalized by a gauge transformation so that in the large N_c limit the functional integral over eigenvalues of the scalar field is saturated by a saddle point of the effective action. Thus, such a theory seems to be solvable in the large N_c limit but its continuum limit is questionable since for a fields in adjoint representation there is an extra local Z_N symmetry which leads in the large N_c limit to infinite string tension (so-called local confinement) rather that conventional area law for the Wilson loop [14]. That is also the case of the adjoint fermion model of the induced QCD [13].

Recently we discussed some other approach where the dual formulation of the induced gluodynamics on a lattice is related with the same trick of the "color–flavor transformation" applied to a $\mathrm{U}(N_c)$ pure lattice gauge model.

In such a case the gauge theory is induced by a heavy chiral scalar field [15]. An alternative approach related with auxiliary fermion field was suggested in [16].

In this short review, entirely based on the works [10,15,17,18], we present some applications of the color–flavor transformation in the context of the lattice gauge theory.

The paper is organized as follows: Section 2, we briefly review the Villain form of the Abelian gauge theory on the lattice and related transformation of duality of lattice QED. Then, we describe the color flavor transformation for the groups U(N_c) and SU(N_c). This technique is applied to transform the partition function of the strongly coupling QCD and decompose the partition function into disconnected sectors labeled by the baryonic charge. The integration over the quarks is performed in the pure mesonic sector, it leads us to a theory of a collective field Z. We make use of the long distance approximation in combination with a gradient expansion to recover the corresponding low energy effective action. Next, in Section 3 we study the Euclidean U(N_c) pure gauge model on a d-dimensional hypercubic lattice. Here, the color–flavor transformation is applied to express the dual theory in terms of gauge singlet matrix fields carried by lattice links.

2. Color–flavor transformations and QCD low-energy effective action

2.1. Duality of compact lattice QED

The starting point of our discussion is the Euclidean SU(N_c) lattice gauge theory [19] described by the action

$$S = S_{\text{gauge}} + S_{\text{quarks}}\,, \tag{1}$$

where S_{gauge} is the standard kinetic term for the Yang–Mills action on the lattice

$$S_{\text{gauge}} = -\frac{1}{4g^2}\sum_{\text{plaquettes}} \mathrm{Tr}\left(UUU^\dagger U^\dagger\right) + \text{c.c.} \tag{2}$$

and

$$\begin{aligned} S_{\text{quarks}} \;=\; & \frac{1}{2a}\sum_{\text{sites n}}\sum_{\mu=1}^{d}\Big(\bar{q}^i_a(n)\gamma_\mu U^{\dagger ij}_\mu(n) q^j_a(n+e_\mu) \\ & -\bar{q}^i_a(n+e_\mu)\gamma_\mu U^{ij}_\mu(n) q^j_a(n)\Big) + \bar{q}^i_a(n) M_{ab} q^i_b(n) \end{aligned} \tag{3}$$

is a discrete counterpart of the Dirac operator, which couples the quarks to the gauge field. In other words, in the lattice formulation, the fermions, which are labeled by an integer n, are placed at the nodes of a four-dimensional hypercubical lattice. The gauge fields are associated with the links joining the nearest neighbor sites. Then a parallel transport along a link is given by a unitary matrix

$$U_\mu(n) \equiv U(n, n+e_\mu) = \exp\{iaeA_\mu(n)\} \in \mathrm{SU}(N)\,, \tag{4}$$

where a is a lattice spacing, $e_1 = (1,0,0,0)$, *etc.* If the lattice spacing a is small compared with the characteristic scales of the theory, the continuum limit can be recovered from the expansion of the link variable (4), $U_\mu(n) \approx 1 + iaeA_\mu + O(a^2)$. Indeed, in the lowest order in lattice spacing we have

$$\sum_n \sum_{\mu<\nu} \Re\ \mathrm{Tr}\{1 - \mathit{\Pi}_{\mu\nu}(n)\} \approx a^4 \sum_n \frac{1}{2} \sum_{\mu,\nu} \frac{1}{2N} \mathrm{Tr} F_{\mu\nu}F^{\mu\nu} + O\left(a^6\right)\,,$$

where, for a given plaquette, we trade the sum over directions $\mu < \nu$ from a point n for half of the sum over all directions: $\sum\limits_{\mu<\nu} \to \frac{1}{2}\sum\limits_{\mu,\nu}$. Thus, in the continuum limit a standard gauge action is recovered

$$S_{\text{gauge}} \quad \underset{a\to 0}{\longrightarrow} \frac{1}{e^2} \int d^4x\, \mathrm{Tr}\ F_{\mu\nu}F^{\mu\nu}\,.$$

Let us consider the simplest case of the U(1) theory on the lattice. Such a model is called the *compact QED*, because the corresponding action can be written in terms of angular variables. Indeed, we now have $U_\mu(n) = e^{i\theta_\mu(n)}$, where θ is a phase that can be integrated over the compact domain $[-\pi;\pi]$. Moreover, each plaquette can also be characterized by an angular variable (*cf.* Fig. 1) and the theory is compact, since the range of functional integration is finite.

$$\mathit{\Pi}_{\mu\nu}(n) = e^{i(\theta_\mu(n)+\theta_\nu(n+e_\mu)-\theta_\mu(n+e_\nu)-\theta_\nu(n))} \equiv e^{i\theta_{\mu\nu}(n)}\,.$$

Note that the plaquette variable $\theta_{\mu\nu}(n)$ describes a field flux through the plaquette. Indeed, in the limit of small lattice spacing, we have

$$\theta_{\mu\nu} = ae(\mathit{\Delta}_\nu\theta_\mu - \mathit{\Delta}_\mu\theta_\nu) \approx -a^2eF_{\mu\nu}\,, \tag{5}$$

where the operator of the nearest-neighbor finite differences $\mathit{\Delta}$ replaces the usual derivative

$$\mathit{\Delta}_\nu\theta_\mu(n) \equiv \theta_\mu(n+e_\nu) - \theta_\mu(n) \approx a\partial_\nu\theta_\mu(n)\,. \tag{6}$$

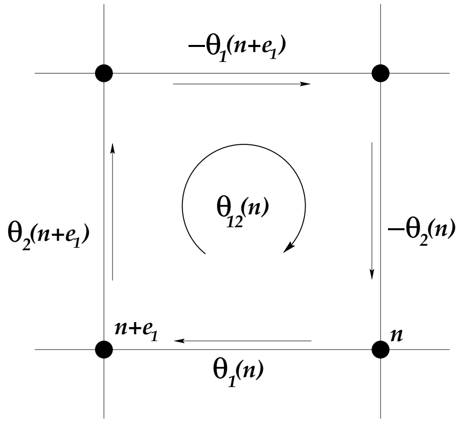

Fig. 1. Plaquette variables of the lattice QED.

We can also define the operator of backward differences

$$\Delta^{\mathrm{d}}_{\nu}\theta_{\mu}(n) \equiv \theta_{\mu}(n) - \theta_{\mu}(n-e_{\nu})\,, \tag{7}$$

with the property $\Delta^{\mathrm{d}} = -\Delta^{\dagger}$.

Thus, the action of compact electrodynamics can be represented in so-called cosine form

$$S_{\mathrm{gauge}} = -\frac{1}{e^2}\sum_{n}\sum_{\mu,\nu}\cos\theta_{\mu\nu}(n)\,. \tag{8}$$

The very simple structure of this expression was used by Villain to apply a remarkable transformation of the partition function (2) [20]. Indeed, in the Abelian theory, the integrand is a periodic function of variable $\theta_{\mu\nu}$, so we may expand it in Fourier series. In the weak coupling limit, this procedure yields the Villian approximation (see *e.g.*, [23, 24])

$$\exp\left\{\frac{1}{e^2}\cos\theta_{\mu\nu}\right\} \underset{e\to 0}{\longrightarrow} \sum_{m_{\mu\nu}=-\infty}^{m_{\mu\nu}=\infty}\exp\left\{-\frac{1}{2e^2}\left|\theta_{\mu\nu}-2\pi m_{\mu\nu}\right|^2\right\}\,. \tag{9}$$

The partition function with the action in the Villain form has the same symmetry properties as for the Wilson action. Moreover, it was proved that the phase structure of both models is similar.

Dual formulation of the compact QED was first investigated by Banks, Myerson and Kogut [23]. They found a transformation that brings the lattice theory to the form describing a monopole gas with magnetic Coulomb interaction. The idea is to make use of the usual electromagnetic duality which can be considered as a transformation that changes the variables of the functional integration. In particular, we may make use of a Gaussian integration over an auxiliary field that can be promoted to the dual variable. We can apply this idea to the partition function of the compact QED in the Villain form with the action (9) [25]

$$\mathcal{Z} = \int \prod_{n,\mu,\nu} \int_{-\pi}^{\pi} \frac{d\theta_\mu(n)}{2\pi} \sum_{n_{\mu\nu}=-\infty}^{n_{\mu\nu}=\infty} \exp\left[-\frac{1}{2e^2}\left|\theta_{\mu\nu}(n) - 2\pi n_{\mu\nu}(n)\right|^2\right] . \tag{10}$$

The starting point here is the Gaussian integration over an auxiliary tensor field $\Theta_{\mu\nu}$, which, up to a normalization factor, allows us to represent the Villain partition function as

$$\begin{aligned}\mathcal{Z} \simeq & \int \prod_{n,\mu<\nu} \int d\Theta_{\mu\nu}(n) \int_{-\pi}^{\pi} d\theta_\mu(n) \sum_{n_{\mu\nu}=-\infty}^{\infty} \\ & \exp\left\{-\frac{e^2}{2}\left[\Theta_{\mu\nu}^2(n) + i\Theta_{\mu\nu}(n)\left[\theta_{\mu\nu}(n) - 2\pi n_{\mu\nu}(n)\right]\right]\right\} .\end{aligned} \tag{11}$$

Note that, unlike $\theta_{\mu\nu}$, the field $\Theta_{\mu\nu}$ is defined on the plaquette, not on the links. Now, the summation over integer numbers in (11) can be performed using the Poisson sum formulae

$$\sum_{l=-\infty}^{\infty} \delta(x-l) = \sum_{m=-\infty}^{\infty} e^{2\pi imx} , \qquad \sum_{l=-\infty}^{\infty} f(l) = \sum_{m=-\infty}^{\infty} \int d\alpha f(\alpha) e^{2\pi im\alpha} . \tag{12}$$

Thus, the field $\Theta_{\mu\nu}$ is forced to take integer values. Furthermore, integration over θ_μ yields the equation $\Delta_\mu^d \Theta_{\mu\nu} = 0$, which is satisfied automatically, if we define a dual vector potential as

$$\widetilde{\Theta}_{\mu\nu}(*n) \equiv \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}\Theta_{\rho\sigma}(n) = \left(\Delta_\mu^d \widetilde{\theta}_\nu(*n) - \Delta_\nu^d \widetilde{\theta}_\mu(*n)\right) , \tag{13}$$

where $\widetilde{\theta}_\mu(*n) \in \mathbb{Z}$. Thus, the dual transformation of the partition function yields [23, 25]

$$\mathcal{Z} \simeq \sum_{\widetilde{\theta}_\mu \in \mathbb{Z}} \prod_{*n,\mu,\nu} \exp\left[-\frac{e^2}{2}\widetilde{\Theta}_{\mu\nu}^2(*n)\right] . \tag{14}$$

Clearly, the weak coupling limit of the model (11) corresponds to the strong coupling limit of the model (14), that is, the duality maps these limits again. Thus, the four-dimensional compact gauge theory is dual to a

$\mathbb{Z}_n$ gauge theory. Similarly, it was shown that there is a duality transformation from the compact U(1) gauge theory into a non-compact Abelian Higgs model [26]. However, in both cases, the dual variables $\widetilde{\theta}_\mu$ are defined not on the links of original lattice Λ^4, but on the cubes of the *dual lattice* $*\Lambda^4$, whose sites are labeled by an integer $*n$. For a hypercubic lattice, it is obtained by shifting the original lattice by half of the lattice spacing in all dimensions. Thus, the lattice duality not only transforms the variables of the functional integral, but also incorporates a transfer to the dual lattice.

2.2. Color–flavor transformation

Let us describe another transformation of duality, namely the "color–flavor transformation" which was suggested some time ago by Zurnbauer [9] as a generalization of the Hubbard–Stratonovich transformation in the context of models describing disordered systems in condensed matter physics. This trick found a number of applications in various models.

We start with a simple example, which illustrates the general form of the transformation [9]. For four fermion fields $\bar{q}_+, q_+$ and $\bar{q}_-, q_-$ one can easily check the relation

$$\frac{1}{2\pi}\int\limits_0^{2\pi} d\alpha\ \exp\left(\bar{q}_+ e^{i\alpha} q_+ + \bar{q}_- e^{-i\alpha} q_-\right) = 1 + \bar{q}_+ q_+ \bar{q}_- q_-$$
$$= 2\int\limits_{\mathbb{C}} \frac{d\mu(z,\bar{z})}{1+z\bar{z}}\ \exp(\bar{q}_+ z \bar{q}_- - q_+ \bar{z} q_-)\,. \tag{15}$$

Evidently, the integration variable α has a meaning of the U(1) group parameter, so the l.h.s. of this expression is actually and integral over the U(1) group. The integration measure $d\mu(z,\bar{z}) = \frac{\text{const.}}{(1+z\bar{z})^2}\, dz d\bar{z}$ on the r.h.s. is the natural measure on the space S^2 expressed in stereographical coordinates $z, \bar{z}$.

In 1996 Zirnbauer suggested very interesting generalization of thus identity [9]. Let q_a^i and $\bar{q}_a^i$ be two independent sets of Grassmann variables, referred to as *quark fields*, and consider the color group integral

$$\mathcal{Z}(q,\bar{q}) = \int\limits_{\mathrm{U}(N_c)} dU \exp\left(\bar{q}^i_{+a} U^{ij} q^j_{+a} + \bar{q}^i_{-b} \bar{U}^{ij} q^j_{-b}\right)\,. \tag{16}$$

The Haar measure dU of $\mathrm{U}(N_c)$ is understood to be normalized by $\int_{\mathrm{U}(N_c)} dU = 1$.

The color–flavor transformation will replace the integral (16) by an integral over the flavor group $\mathrm{U}(2N_f)$. A key step in doing the transformation is to interpret $\mathcal{Z}(q,\bar{q})$ as the matrix element of an operator $\mathcal{P}$ that projects on the colorless sector (or flavor sector) of Fock space. This sector is the subspace of all states $|\text{flavor}\rangle$ which are invariant under the color group: $T_{\mathrm{U}}|\text{flavor}\rangle = |\text{flavor}\rangle$ for all $\mathrm{U}\in \mathrm{U}(N_c)$.

The first step towards the color–flavor transformation is to express the projector $\mathcal{P}$ as

$$\mathcal{P} = \int\limits_{\mathrm{U}(N_c)} dU\, T_{\mathrm{U}}\,. \tag{17}$$

Let us now introduce the fermion coherent states

$$\left\langle\bar{\Psi}\right| = \langle 0|\exp\left(\bar{q}^{i}_{-a} f^{i}_{-a} + \bar{q}^{i}_{+a} f^{i}_{+a}\right)\,, \qquad |\Psi\rangle = \exp\left(\bar{f}^{i}_{-a} q^{i}_{-a} + \bar{f}^{i}_{+a} q^{i}_{+a}\right)|0\rangle\,. \tag{18}$$

It is straightforward to show that

$$\left\langle\bar{\Psi}|T_{\mathrm{U}}|\Psi\right\rangle = \exp\left(\bar{q}^{i}_{+a} U^{ij} q^{j}_{+a} + \bar{q}^{i}_{-b}\bar{U}^{ij} q^{j}_{-b}\right)$$

for $\mathrm{U} \in \mathrm{U}(N_c)$. This yields the simple formula

$$\mathcal{Z}(q,\bar{q}) = \left\langle\bar{\Psi}|\mathcal{P}|\Psi\right\rangle\,. \tag{19}$$

To express $\mathcal{Z}(\psi,\bar{\psi})$ as an integral over the flavor group, we will derive an alternative representation of the projector $\mathcal{P}$, as an integral over coherent states of the flavor sector.

The subspace of states in Fock space which are invariant under $\mathrm{U}(N_c)$ was described in [9]. It consists of the vacuum and of *mesonic* excitations on top of it. The prototype of such an excitation is the “one-meson” state

$$|m_{ab}\rangle = \sum_i E^{ii}_{+a,-b}|0\rangle = \sum_i \bar{f}^{i}_{+a}\bar{f}^{i}_{-b}|0\rangle\,.$$

By the multiple action of the $\mathfrak{gl}(2N_f)$ generators $E^{ii}_{+a,-b}$, one can build states containing up to N_cN_f mesons, with different flavors. These states are automatically $\mathrm{U}(N_c)$-invariant; conversely, all $\mathrm{U}(N_c)$-invariant states are linear combinations of such multi-meson states. The group $\mathrm{U}(2N_f)$ acts irreducibly on this invariant subspace.

The set of $\mathrm{SU}(N_c)$-invariant states is larger. To obtain it, one relaxes the constraint $\hat{Q}|\psi\rangle = 0$. Thus, there exist colorless sectors of Fock space on which the central generator $\hat{Q}$ takes a non-zero value. These sectors contain the *baryons*, which are totally antisymmetric combinations of N_c

quarks. A matrix $g \in \mathrm{GL}(N_c)$ acts on this state simply by multiplication with $\mathrm{Det}(g)$ (resp. $\mathrm{Det}^{N_f-1}(g)$. Therefore, the state is invariant under the color group $\mathrm{SU}(N_c)$.

The values of the baryon charge range from $-N_f$ to N_f, according to Pauli's exclusion principle. As with $Q = \pm 1$, acting on $|B_Q\rangle$ with the algebra $\mathfrak{gl}(2N_f)$ builds the full Q-baryon part of the flavor sector, so the group $\mathrm{U}(2N_f)$ acts irreducibly on this part. This can be proved by using the dual-pair property of the subalgebras $\mathfrak{gl}(2N_f)$ and $\mathfrak{gl}(N_c)$.

To summarize, the flavor sector of Fock space decomposes into $2N_f + 1$ subsectors, characterized by their baryon charges Q. Each sector carries an irreducible unitary representation of the flavor group $\mathrm{U}(2N_f)$.

Having decomposed the flavor sector as described above, we can now express the projector $\mathcal{P}$ in a different way by making use of the generalized coherent states [27]. On each subsector with a fixed baryon charge Q, we consider the *generalized coherent states* built by the action of $G \equiv \mathrm{U}(2N_f)$ on the reference state $|B_Q\rangle$, *i.e.* the states

$$\forall g \in G,\ \forall Q = -N_f, \ldots, N_f : \qquad |g_Q\rangle \stackrel{\mathrm{def}}{=} T_g |B_Q\rangle\,, \qquad \langle g_Q | \stackrel{\mathrm{def}}{=} \langle B_Q | T_g^\dagger\,. \tag{20}$$

The crucial property of coherent states we will now use, is that they supply a resolution of unity. Because of the irreducibility of the $\mathrm{U}(2N_f)$ action on each Q-subsector, the operator

$$\mathcal{P}_Q \stackrel{\mathrm{def}}{=} \alpha_Q \int\limits_G dg\, |g_Q\rangle\langle g_Q| \tag{21}$$

coincides with the orthogonal projector on that subsector, the only provision being that the normalization constant α_Q be chosen appropriately.

For the matrix element (19) of the projector $\mathcal{P}$ on the full flavor sector $\mathcal{P} = \bigoplus\limits_{Q=-N_f}^{N_f} \mathcal{P}_Q$, we now have a new representation

$$\mathcal{Z}(q, \bar{q}) = \sum_{Q=-N_f}^{N_f} \alpha_Q \int\limits_G dg\, \left\langle \bar{\Psi} | g_Q \right\rangle \left\langle g_Q | \Psi \right\rangle\,. \tag{22}$$

The last step of the calculations is to compute the overlaps $\langle \bar{\Psi} | g_Q \rangle$ and $\langle g_Q | \Psi \rangle$. In the case of the $\mathrm{U}(N_c)$ group we simple have $Q = 0$ and this yields following generalization of the expression (15) [9]

$$\begin{aligned} &\int\limits_{\mathrm{U}(N_c)} dU\, \exp\left(\bar{q}^i_{+a} U^{ij} q^j_{+a} + \bar{q}^i_{-b} \bar{U}^{ij} q^j_{-b} \right) \\ &= \int\limits_{\mathbb{C}^{N_f \times N_f}} \frac{D\mu\left(Z, Z^\dagger\right)}{\mathrm{Det}\left(1 + ZZ^\dagger\right)^{N_c}}\ \exp\left(\bar{q}^i_{+a} Z_{ab} \bar{q}^i_{-b} - q^i_{+a} \bar{Z}_{ab} q^i_{-b} \right)\,. \end{aligned} \tag{23}$$

Evidently, for $N_f = 1$ the general identity (23) reduces to the special case considered before. Note that the identity similar to (23) can be obtained not only for the fermionic but also for the bosonic fields [11]. There are also various modifications of the color–flavor transformations for several other groups [9, 21].

Let us consider the "color–flavor" transformation for the special unitary group $\mathrm{SU}(N_c)$. Then, in the simplest case $N_c = 1$ and $N_f = 1$, the integral on the l.h.s. of (15) reduces to a single point (evaluation at unity), and the following statement is immediate

$$\begin{aligned}&\exp(\bar q_+ q_+ + \bar q_- q_-) = 1 + \bar q_+ q_+ + \bar q_- q_- + \bar q_+ q_+ \bar q_- q_- \\ &= 2\int\limits_{\mathbb{C}} \frac{d\mu(z,\bar z)}{1+z\bar z}\exp(\bar q_+ z \bar q_- - q_+ \bar z q_-)\left[1 + \tfrac{1}{2}\bar q_+(1{+}z\bar z)q_+ + \tfrac{1}{2}\bar q_-(1{+}z\bar z)q_-\right].\end{aligned} \tag{24}$$

Again, *cf.* [10], this is a special case of a more general identity

$$\begin{aligned}&\int\limits_{\mathrm{SU}(N_c)} dU\ \exp\left(\bar q^i_{+a} U^{ij} q^j_{+a} + \bar q^i_{-b}\bar U^{ij} q^j_{-b}\right)\\ &= \int\limits_{\mathbb{C}^{N_f\times N_f}} \frac{dZdZ^\dagger}{\mathrm{Det}\,(1{+}ZZ^\dagger)^{N_c}} \exp\left(\bar q^i_{+a} Z_{ab}\bar q^i_{-b} - q^i_{+a}\bar Z_{ab} q^i_{-b}\right) \sum_{Q=-N_f}^{N_f} \chi_Q(Z, Z^\dagger, q, \bar q)\,,\end{aligned} \tag{25}$$

where the integration measure is $D\mu(Z,Z^\dagger)\frac{dZdZ^\dagger}{\mathrm{Det}(1+ZZ^\dagger)^{N_c}}$. The physical interpretation of the equation (25) is as follows: It is a sum over contributions, which come from mesonic excitations over the vacuum ($Q = 0$) or mesonic excitations in the presence of $|Q|$ baryons ($Q > 0$) or $|Q|$ antibaryons ($Q < 0$). The explicit expressions for χ_Q in the vacuum sector and in the one-baryon sector are [10]

$$\chi_0 = \mathrm{const.}\,, \qquad \chi_1 = \mathrm{const.} \sum_{\sigma\in \mathrm{S}_{N_c}} \mathrm{sgn}\,\sigma \prod_{i=1}^{N_c} \bar q^i_{+a}\left(1 + ZZ^\dagger\right)_{ab} \bar q^{\sigma(i)}_{+b}\,. \tag{26}$$

The sum runs over all permutations S_{N_c} of the numbers $1,\dots,N_c$.

2.3. Application to lattice gauge theory

Let us now consider Euclidean $\mathrm{SU}(N_c)$ gauge theory in d dimensions placed on a hypercubic lattice with lattice constant a. We restrict our considerations to an even number d of space-time dimensions.

The fermions on the two neighboring sites are coupled through the gauge field on the connecting link in the gauge invariant way

$$S_{n,\mu}\left(U_\mu(n)\right) = \frac{1}{2a}\left(\bar{q}_a^i(n)\gamma_\mu U_\mu^{\dagger ij}(n)q_a^j(n+e_\mu) - \bar{q}_a^i(n+e_\mu)\gamma_\mu U_\mu^{ij}(n)q_a^j(n)\right) + \bar{q}_a^i(n)M_{ab}q_b^i(n)\,. \tag{27}$$

Here, for the sake of simplicity, we do not worry about the fermion doubling problem and employ naive fermions. The mass matrix $M = \mathrm{diag}(m_1, ..., m_{N_f})$ contains the quark masses and it is diagonal in flavor space. The fermionic fields are labeled also by the Lorentz indices $\nu = 1, \ldots, N_s$ with $N_s = 2^{d/2}$, they are coupled through the γ-matrices. The γ-matrices are the generators of a Clifford algebra over the Euclidean space-time and satisfy the anticommutation relations $\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = 2\delta_{\mu\nu}$.

Let us apply the color–flavor transformation to the gauge field $U_\mu(n)$ separately on each link. Note that the color–flavor transformation of the partition function then turns out to be a color-chiral transformation: it rearranges the fields by decoupling of the left and right components.

In doing so, we get a new action, which is expressed in terms of the flavor field $Z_\mu(n)$. The fermions are now coupled through their *flavor* indices, whereas in the original action the coupling had been mediated by the color degrees of freedom. Moreover, the coupling has become ultralocal: the fermions at a site n couple only to one another, via $Z(n+\frac{\hat{\mu}}{2})$, and so do the fermions at site $n+\hat{\mu}$, via $Z^\dagger(n+\frac{\hat{\mu}}{2})$. Correlations between neighbors are solely due to the relation between Z and $Z^\dagger$ by Hermitian conjugation (*cf.* Fig. 2).

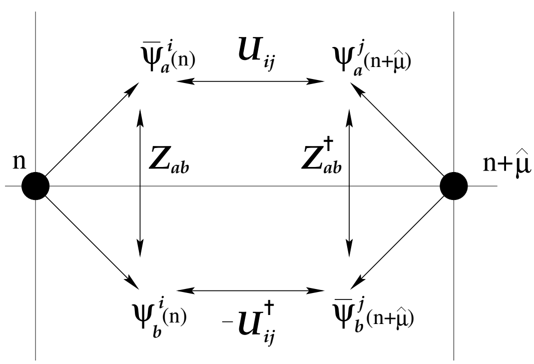

Fig. 2. Coupling of the fermion fields before and after the color–flavor transformation.

The result is a sum over contributions from sectors with different baryonic charge Q and can be written as

$$\mathcal{Z}(q,\bar{q}) = \prod_{n,\mu} \int_{\mathbb{C}^{N_f\times N_f}} D\mu_{N_c}\left(Z_\mu(n), Z_\mu^\dagger(n)\right) \sum_{Q=-N_f}^{N_f} \chi_{Q,\mu}(n) \exp\left(-S_{n,\mu}(Z_{n,\mu})\right) . \tag{28}$$

On each link, we perform the color–flavor transformation, thereby introducing a complex "flavor matrix field" $Z(n+\frac{\hat{\mu}}{2})$, $Z^\dagger(n+\frac{\hat{\mu}}{2})$. The outcome of the transformation reads

$$\begin{aligned} S_{n,\mu}(Z_\mu(n)) &= \frac{1}{2a}\left(\bar{q}_a^i(n+e_\mu)\gamma_\mu Z_{\mu,ab}(n) q_b^i(n+e_\mu) + \bar{q}_a^i(n)\gamma_\mu Z_{\mu,ab}^\dagger(n) q_b^i(n)\right) \\ &\quad + \bar{q}_a^i(n) M_{ab} q_b^i(n) . \end{aligned} \tag{29}$$

Explicit expressions for the prefactors in the case of the vacuum sector ($Q = 0$) and the one-baryon sector ($Q = 1$) are

$$\chi_{0,\mu}(n) = \text{const.}, \tag{30}$$

$$\chi_{1,\mu}(n) = \text{const.} \sum_{\sigma\in S_{N_c}} \operatorname{sgn}\sigma \prod_{i=1}^{N_c} \bar{q}_a^i(n+e_\mu)\left(1 + Z_\mu Z_\mu^\dagger\right)_{ab}(n)\bar{q}_b^{\sigma(i)}(n) . \tag{31}$$

The partition function (28) is actually a sum over all configurations of baryonic fluxes. For most of these configurations, the Grassmann integral vanishes identically [10]. It corresponds to the conservation of the baryonic current: the (algebraic) number of baryons "arriving" at the site n (from the links $n - \hat{\mu}/2$) must equal the number of baryons "leaving" the site (via the links $n + \hat{\mu}/2$). Indeed, the general structure of the partition function (28) corresponds to the hadronic correlation function written in terms of colorless N_c-quark currents [28].

2.4. Integration over the quarks and saddle point approximation

Performing the Gaussian integral over the quarks in the pure mesonic sector ($Q = 0$) and sending the result back to the exponent, we obtain the action

$$\begin{aligned} S_{Q=0}(Z) &= N_c\Bigg(-\sum_n \operatorname{Tr}\ln\left(\sum_{\mu=1}^{d}\gamma_\mu\left(Z_\mu(n-e_\mu) + Z_\mu^\dagger(n)\right) - aM\right) \\ &\quad + \sum_n \sum_{\mu=1}^{d} \operatorname{Tr}\ln\left(1 + Z_\mu(n) Z_\mu^\dagger(n)\right)\Bigg) . \end{aligned} \tag{32}$$

Variation with respect to the independent variables Z and $Z^\dagger$ yields the saddle point equations

$$\begin{aligned}\gamma_\mu\left(\frac{1}{Z_\mu(n)}+Z^\dagger_\mu(n)\right) &= \sum_{\nu=1}^{d}\gamma_\nu\left(Z_\nu(n-e_\nu)+Z^\dagger_\nu(n)\right)-2aM\,,\\ \gamma_\mu\left(\frac{1}{Z^\dagger_\mu(n-e_\mu)}+Z_\mu(n-e_\mu)\right) &= \sum_{\nu=1}^{d}\gamma_\nu\left(Z_\nu(n-e_\nu)+Z^\dagger_\nu(n)\right)-2aM\,.\end{aligned}\tag{33}$$

The solution of the saddle point equations is

$$Z^0_\mu(n)=Z^{0\dagger}_\mu(n)=z\gamma_\mu I \quad \text{with} \quad z=\frac{1}{2d-1}\left(aM\pm\sqrt{2d-1+(aM)^2}\right)\,,\tag{34}$$

where I is the unit matrix in flavor space. Recall the following definitions of γ_5 and the projectors on the chiral components of the spinors

$$\gamma_5=i^{d(d-1)/2}\gamma_1\times\cdots\times\gamma_d\,,\tag{35}$$

$$\gamma_{\rm L}=\tfrac{1}{2}(1+\gamma_5)\,,\qquad \gamma_{\rm R}=\tfrac{1}{2}(1-\gamma_5)\,.\tag{36}$$

In the chiral limit ($M=0$) there is a saddle point manifold

$$Z^0_\mu(n)=Z^{0\dagger}_\mu(n)=z\gamma_\mu g^{\gamma_5}:=z\gamma_\mu\left(\gamma_{\rm L}\otimes g+\gamma_{\rm R}\otimes g^{-1}\right)\,,\tag{37}$$

which is parameterized by unitary matrices $g\in{\rm U}(N_f)$. The lattice gauge theory in the chiral limit ($M=0$) is invariant under the chiral transformations ($(g_{\rm L},g_{\rm R})\in{\rm U}(N_f)_{\rm L}\times{\rm U}(N_f)_{\rm R}$) of the quarks fields

$$\begin{aligned}q(n) &\to (\gamma_{\rm L}\otimes g_{\rm L}+\gamma_{\rm R}\otimes g_{\rm R})\,q(n)\,,\\ \bar q(n) &\to \left(\gamma_{\rm L}\otimes g_{\rm R}^{-1}+\gamma_{\rm R}\otimes g_{\rm L}^{-1}\right)\bar q(n)\,.\end{aligned}\tag{38}$$

The chiral symmetry of the action is preserved by the color–flavor transformation, it leads to invariance of the action (32) and the saddle point manifold under the transformations

$$\begin{aligned}Z_\mu(n) &\to \left(\gamma_{\rm L}\otimes g_{\rm L}^{-1}+\gamma_{\rm R}\otimes g_{\rm R}^{-1}\right)Z_\mu(n)\,(\gamma_{\rm L}\otimes g_{\rm L}+\gamma_{\rm R}\otimes g_{\rm R})\,,\\ Z^\dagger_\mu(n) &\to \left(\gamma_{\rm L}\otimes g_{\rm L}^{-1}+\gamma_{\rm R}\otimes g_{\rm R}^{-1}\right)Z^\dagger_\mu(n)\,(\gamma_{\rm L}\otimes g_{\rm L}+\gamma_{\rm R}\otimes g_{\rm R})\,.\end{aligned}\tag{39}$$

The saddle point (34) is invariant under (39) only if $g_{\rm L}=g_{\rm R}$, *i.e.* it breaks the the chiral ${\rm U}(N_f)_{\rm L}\times{\rm U}(N_f)_{\rm R}$ symmetry to the subgroup ${\rm U}(N_f)$. The above symmetry considerations explain how chiral symmetry breaking takes place in our approach to the gauge theory. On the other hand, the chiral symmetry breaking gives rise to Goldstone bosons, which can be identified with the coset space ${\rm U}(N_f)_{\rm L}\times{\rm U}(N_f)_{\rm R}/{\rm U}(N_f)\cong{\rm U}(N_f)$ and parameterize the saddle point manifold (37).

2.5. Gradient expansion

In this section, our aim is to derive an effective theory, which describes the long range behavior of Goldstone modes $g \in \mathrm{U}(N_f)$. We use the technique of a long distance approximation in combination with a gradient expansion around the saddle point manifold. The lattice action (32) is then replaced by a simple continuum action

$$S_{Q=0}(Z) \longrightarrow S(g) = S_{\mathrm{fl}}(g) + S_{\mathrm{M}}(g)\,, \tag{40}$$

where $S_{\mathrm{fl}}(g)$ is the action of the fluctuations of the Goldstone modes, and $S_{\mathrm{M}}(g)$ is the contribution due to the finite quark masses. More explicitly, $Z_\mu(n)$ is put into correspondence to a continuous field $g(x) \in \mathrm{U}(N_f)$ in the following way

$$Z_\mu(n) = Z_\mu^\dagger(n) = z\gamma_\mu\, g\left(n + \frac{a}{2}e_\mu\right)^{\gamma_5}\,. \tag{41}$$

Inserting into (32) the Taylor expansion

$$g\left(n + \frac{a}{2}e_\mu\right) = g(n) + \frac{a}{2}\partial_\mu g(n) + \frac{1}{2!}\left(\frac{a}{2}\right)^2 \partial_\mu^2 g(n) + \ldots \tag{42}$$

we obtain for the hypercubic lattice

$$S_{\mathrm{fl}}(g) = \frac{N_s}{8d}a^{2-d}\int d^dx\ \mathrm{Tr}\left(\partial g \partial g^{-1}\right)\,, \tag{43}$$

$$S_{\mathrm{M}}(g) = N_s\frac{\sqrt{2d-1}}{2d}a^{-d}\int d^dx\ \mathrm{Tr}\left(aM\left(g+g^{-1}\right)\right)\,. \tag{44}$$

We have carried out the expansion up to fourth order derivatives.

In $d = 4$ space-time dimensions both parts of the actions (43) and (44) diverge when the lattice constant approaches zero. Recall that we are considering the strong coupling limit only, *i.e.* we neglect the kinetic term (2), which is suppressed as $1/g^2$. Therefore, our theory is restricted to the low energy sector and we have to keep the lattice constant at a finite value. There are three ways to estimate the lattice constant a through comparisons with experimental data: Using the relation

$$\langle \bar{q}_f q_f \rangle = -\frac{1}{V}\frac{\partial}{\partial m_f}\bigg|_0 S_{\mathrm{saddle}} = -N_s N_c \frac{d-1}{d\sqrt{2d-1}}a^{1-d} \tag{45}$$

we get from the experimental value for the chiral condensate $a = (166\ \mathrm{MeV})^{-1}$. By looking at the coefficient in front of the fluctuation action,

$$\frac{F_\pi^2}{4} = N_s N_c \frac{1}{8d}\,a^{2-d}\,, \tag{46}$$

we get from the experimental value for the pion decay constant F_π the estimate $a = (76\ \mathrm{MeV})^{-1}$. By looking at the coefficient in front of the fluctuation action,

$$\frac{1}{4}F_\pi^2 m_\pi^2 = N_s N_c \frac{\sqrt{2d-1}}{2d}\, a^{1-d} m_f\,, \tag{47}$$

we get from the experimental values for the pion decay constant, the mass of the pion m_π and the mass of the light quarks m_f the value $a = (76\ \mathrm{MeV})^{-1}$. We conclude that the lattice constant has to be chosen as $a \approx (100\ \mathrm{MeV})^{-1}$ ≈ 2 Fermi to get a realistic description.

3. Color–flavor transformation of induced QCD

Let us consider another possible application of the same trick of the *color–flavor transformation* to the gauge sector of the theory. It allows us to construct a new dual formulation of the induced gluodynamics on a lattice [15, 16].

Here, for the sake of simplicity, we restrict our consideration to the Euclidean $\mathrm{U}(N_c)$ pure gauge model placed on a d-dimensional hypercubic lattice. The lattice sites are labeled by integer vectors $n = (n_1, \ldots, n_d)$, the gauge matrix variables $\mathrm{U}(n + \frac{\hat{\mu}}{2}) \in \mathrm{U}(N_c)$ are placed on the lattice links $(n + \hat{\mu}/2)$, with $\hat{\mu} = \hat{1}, \ldots, \hat{d}$ the basis vectors. The plaquettes are either labeled by an independent index p, or by triplets of the form $(n, \pm\mu, \pm\nu)$. The plaquette field $U_{\mathrm{P}}(n, \mu, \nu)$ is defined (for $\mu < \nu$) as the ordered product along the boundary of the plaquette

$$U_{\mathrm{P}}(n,\mu,\nu) = U\Big(n + \tfrac{\hat{\mu}}{2}\Big)\, U\big(n + \hat{\mu} + \tfrac{\hat{\nu}}{2}\big)\, U^{-1}\Big(n + \hat{\nu} + \tfrac{\hat{\mu}}{2}\Big)\, U^{-1}\big(n + \tfrac{\hat{\nu}}{2}\big)\ .$$

In our model [29,30], the gauge coupling is induced by a massive auxiliary field [31,32], which is in our case a massive complex bosonic field $\phi(n)$ placed on the lattice sites. This field first has "flavor" components $\phi^{(\pm\mu,\pm\nu)}(n)$ associated to each of the $2d(d-1)$ plaquettes $\{(n, \pm\mu, \pm\nu); 1 \le \mu < \nu \le d\}$ adjacent to the site n. Each of them decomposes into two "chiral components" $\phi_{\mathrm{R}}^{(\mu,\nu)}(n)$ and $\phi_{\mathrm{L}}^{(\mu,\nu)}(n)$, hopping in opposite directions. There might be $n_b \ge 1$ "generations" of these fields, such that the auxiliary "flavor" space has dimension $N_b = n_b \times 2d(d-1)$ for each chiral component. Finally, the auxiliary field ϕ also transforms as a vector through the gauge group $\mathrm{U}(N_c)$, so it contains both "color" indices $i = 1, \ldots, N_c$ and "flavor" indices $a = 1, \ldots, N_b$. All field components have the same (large) mass m_b.

This model is a bit different from the model usually considered in induced QCD [31, 32], where the flavor degrees of freedom of the auxiliary fields are not associated with plaquettes. As we will show below, this structure will induce a Wilson-type action in a cleaner way than in the previous models.

We first group the bosonic fields surrounding a given plaquette $p = (n, \mu, \nu)$ into the *plaquette quadruplets*

$$\phi_{\mathrm{L,R}}(p) = \left(\phi^{(\mu,\nu)}_{\mathrm{L,R}}(n),\ \phi^{(-\mu,\nu)}_{\mathrm{L,R}}(n+\hat{\mu}),\ \phi^{(-\mu,-\nu)}_{\mathrm{L,R}}(n+\hat{\mu}+\hat{\nu}),\ \phi^{(\mu,-\nu)}_{\mathrm{L,R}}(n+\hat{\nu})\right). \tag{48}$$

The plaquette action of the "left", resp. "right", bosonic field may then be written compactly as $S_{\mathrm{L}}(p) = \phi^{\dagger}_{\mathrm{L}}(p)M(p)\phi_{\mathrm{L}}(p)$; $S_{\mathrm{R}}(p) = \phi^{\dagger}_{\mathrm{R}}(p)M^{\dagger}(p)\phi_{\mathrm{R}}(p)$, with the $4N_c \times 4N_c$ matrices

$$M(p) \stackrel{\mathrm{def}}{=} \begin{pmatrix} m_b & 0 & 0 & -U^{-1}(n+\frac{\hat{\nu}}{2}) \\ -U(n+\frac{\hat{\mu}}{2}) & m_b & 0 & 0 \\ 0 & -U(n+\hat{\mu}+\frac{\hat{\nu}}{2}) & m_b & 0 \\ 0 & 0 & -U^{-1}(n+\hat{\nu}+\frac{\hat{\mu}}{2}) & m_b \end{pmatrix}. \tag{49}$$

From there, integrating over the auxiliary fields yields the pure gauge action

$$\begin{aligned} S_{\mathrm{plaq}}(p) &= n_b \mathrm{Tr}\left(\ln\left(1 - \beta_b U_{\mathrm{P}}(p)\right) + \ln\left(1 - \beta_b U^{\dagger}_{\mathrm{P}}(p)\right)\right) \\ &\approx -n_b \beta_b \mathrm{Tr}\left(U_{\mathrm{P}}(p) + U^{\dagger}_{\mathrm{P}}(p)\right), \end{aligned}$$

where we set $\beta_b = m_b^{-4}$ and expanded to first order in β_b. We, therefore, recover Wilson's pure gauge action.

We will now treat the inducing action $S_{\mathrm{L}} + S_{\mathrm{R}}$ in an alternative manner, namely by regrouping together the terms containing the gauge field on a given link $(n + \hat{\mu}/2)$. From now on, we restrict ourselves to the case of a single *generation* ($n_b = 1$). Each link is surrounded by $2d - 2$ plaquettes; it is convenient to gather the associated flavor components into *site-link multiplets*[1]

$$\begin{aligned} \Psi(n;\mu) &= \big(\phi^{(\mu,\mu+1)}_{\mathrm{R}}(n),\ \phi^{(\mu,-\mu-1)}_{\mathrm{L}}(n), \ldots, \phi^{(\mu,d)}_{\mathrm{R}}(n), \\ &\qquad \phi^{(\mu,-d)}_{\mathrm{L}}(n), \ldots, \phi^{(\mu,-\mu+1)}_{\mathrm{R}}(n),\ \phi^{(\mu,\mu-1)}_{\mathrm{L}}(n)\big) \end{aligned}$$

and their chiral conjugates $\Phi(n;\mu)$. In terms of these multiplets, the interacting part of the action $S_{\mathrm{L}} + S_{\mathrm{R}}$ on the link $(n + \hat{\mu}/2)$ may be written in the compact form

$$\begin{aligned} -S_U\left(n + \tfrac{\hat{\mu}}{2}\right) &= \bar{\Phi}^i_a(n+\hat{\mu};-\mu)U^{ij}\left(n + \tfrac{\hat{\mu}}{2}\right)\Phi^j_a(n;\mu) \\ &\quad + \bar{\Psi}^i_a(n;\mu)U^{\dagger ij}\left(n + \tfrac{\hat{\mu}}{2}\right)\Psi^j_a(n+\hat{\mu};-\mu). \end{aligned}$$

[1] For the details see [15].

We can now perform on this action the bosonic $U(N_c)$ color–flavor transformation, which replaces the integration over $U(n+\frac{\hat{\mu}}{2})$ by an integral over complex matrices $Z(n+\frac{\hat{\mu}}{2})$ of dimension $2d-2$ satisfying $1-Z^\dagger Z>0$. This yields a dual local action, in which the auxiliary fields at the site n are coupled as follows[2]

$$-S_Z[n] = \sum_{\mu=1}^{d}\Big[\bar{\Psi}_b^i(n;\mu)\bar{Z}_{ab}\left(n+\tfrac{\hat{\mu}}{2}\right)\Phi_a^i(n;\mu) + \bar{\Phi}_a^i(n;-\mu)Z_{ab}\left(n-\tfrac{\hat{\mu}}{2}\right)\Psi_b^i(n;-\mu)\Big] \,. \tag{50}$$

The bosonic fields are now coupled *ultralocally* via the Z matrices through their flavor indices, while the color indices decouple. As a result, the partition function of this dual action factorizes into N_c identical Gaussian integrals over the auxiliary fields. Performing this integral yields an effective action in the field Z, which reads to second order in $1/m_b$

$$-S[Z] = \sum_n \left(m_b^{-2} \sum_{1\le\mu<\nu\le d} zz_{\mu\nu}(n) + \sum_{\alpha=1}^{d} \mathrm{Tr}\ln\left(1-Z\left(n+\tfrac{\hat{\alpha}}{2}\right)Z^\dagger\left(n+\tfrac{\hat{\alpha}}{2}\right)\right)\right),$$

where the terms $zz_{\mu\nu}(n)=Z^\dagger_{\nu,\nu}(n+\frac{\hat{\mu}}{2})Z^\dagger_{\mu,\mu}(n+\frac{\hat{\nu}}{2})+Z^\dagger_{-\nu,-\nu}(n+\frac{\hat{\mu}}{2})Z_{\mu,\mu}(n-\frac{\hat{\nu}}{2})+Z_{\nu,\nu}(n-\frac{\hat{\mu}}{2})Z^\dagger_{-\mu,-\mu}(n+\frac{\hat{\nu}}{2})+Z_{-\nu,-\nu}(n-\frac{\hat{\mu}}{2})Z_{-\mu,-\mu}(n-\frac{\hat{\nu}}{2})$ only involve diagonal elements of the Z-fields.

The factor N_c in front of the action $S[Z]$ allows to study the large-N_c limit of the theory. The corresponding saddle point equations admit the trivial configuration $Z\equiv 0$ as solution (this is the unique solution if $m_b \gg 1$). Expansion of the action $S[Z]$ to quadratic order in Z, $Z^\dagger$ yields

$$-S[Z]_{\mathrm{quad}} = \sum_n \left(m_b^{-2} \sum_{1\le\mu<\nu\le d} zz_{\mu\nu}(n) - \sum_{\alpha=1}^{d}\sum_{a,b}\left|Z_{ab}\left(n+\tfrac{\hat{\alpha}}{2}\right)\right|^2\right).$$

This quadratic action contains no propagating mode. Yet, the propagation can be induced by including higher-order terms in $ZZ^\dagger$ when expanding the logarithm: one obtains a quartic contribution $\mathrm{Tr}(ZZ^\dagger)^2$, which couples different diagonal elements through non-diagonal ones.

[2] Schlittgen and Wettig independently applied the $SU(N)$ color–flavor transformation to a similar, yet different QCD-inducing model [16].

3.1. d = 2 effective action

For the sake of illustration, let us consider in more detail the simplest possible situation, which corresponds to the model placed on the 2-dimensional square lattice spanned by two orthogonal unit vectors $\hat{1}$ and $\hat{2}$.

Consider the four links having the lattice site n in common. The space of auxiliary fields at n is of dimension 8 and the site-link multiplets (here, doublets) read

$$\begin{aligned} \Psi(n;1) &= \begin{pmatrix} \phi_{\mathrm{R}}^{(1,2)}(n) \\ \phi_{\mathrm{L}}^{(1,-2)}(n) \end{pmatrix}; \qquad \Psi(n;-1) = \begin{pmatrix} \phi_{\mathrm{R}}^{(-1,2)}(n) \\ \phi_{\mathrm{L}}^{(-1,-2)}(n) \end{pmatrix}, \\ \Psi(n;2) &= \begin{pmatrix} \phi_{\mathrm{R}}^{(2,-1)}(n) \\ \phi_{\mathrm{L}}^{(2,1)}(n) \end{pmatrix}; \qquad \Psi(n;-2) = \begin{pmatrix} \phi_{\mathrm{R}}^{(-2,-1)}(n) \\ \phi_{\mathrm{L}}^{(-2,1)}(n) \end{pmatrix}. \end{aligned} \tag{51}$$

The 4 chirally conjugated doublets $\Phi(n;\pm\hat{\alpha})$ are obtained by exchanging $\mathrm{L} \leftrightarrow \mathrm{R}$. These doublets are coupled through the 2×2 matrices $Z^\dagger(n+\frac{\hat{1}}{2})$, $Z^\dagger(n+\frac{\hat{2}}{2})$, $Z(n-\frac{\hat{1}}{2})$ and $Z(n-\frac{\hat{2}}{2})$ carried by the four links adjacent to the site n. To give an example, the matrix $Z^\dagger(n+\frac{\hat{1}}{2})$ has the following index structure

$$Z^\dagger\left(n+\tfrac{\hat{1}}{2}\right) = \begin{pmatrix} Z^\dagger_{2,2}\left(n+\frac{\hat{1}}{2}\right) & Z^\dagger_{2,-2}\left(n+\frac{\hat{1}}{2}\right) \\ Z^\dagger_{-2,2}\left(n+\frac{\hat{1}}{2}\right) & Z^\dagger_{-2,-2}\left(n+\frac{\hat{1}}{2}\right) \end{pmatrix}. \tag{52}$$

Together with the link carrying the matrix, the lower pair of indices represent the plaquettes associated with the field components coupled by the matrix element: the diagonal element $Z^\dagger_{2,2}(n+\frac{\hat{1}}{2})$ couples different fields associated with the same plaquette $(n,1,2)$, while the non-diagonal element ${Z^\dagger}_{-2,2}(n+\frac{\hat{1}}{2})$ couples fields associated with the two plaquettes $(n,1,2)$ and $(n,1,-2)$.

We want to write an effective action uniquely in terms of the Z fields, by integrating over the bosonic fields. For this aim, we need to describe the coupling between each pair or flavors in the action.

As we already mentioned, the site-link multiplets (51) are not independent of each other, so we now group the auxiliary fields at the lattice site n into chirally conjugated *site quadruplets*

$$\Phi(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_{\mathrm{R}}^{(1,2)}(n) \\ \phi_{\mathrm{L}}^{(1,-2)}(n) \\ \phi_{\mathrm{L}}^{(-1,2)}(n) \\ \phi_{\mathrm{R}}^{(-1,-2)}(n) \end{pmatrix}; \qquad \Psi(n) \stackrel{\text{def}}{=} \begin{pmatrix} \phi_{\mathrm{L}}^{(1,2)}(n) \\ \phi_{\mathrm{R}}^{(1,-2)}(n) \\ \phi_{\mathrm{R}}^{(-1,2)}(n) \\ \phi_{\mathrm{L}}^{(-1,-2)}(n) \end{pmatrix}. \tag{53}$$

The union of these two quadruplets contain each bosonic component once. The color–flavor transformed action can be written in terms of these quadruplets via two complex 4×4 matrices in the flavor space, $V(n)$ and $W(n)$, which contain the components of the Z-fields

$$-S_Z[n] = \Phi^\dagger(n)V(n)\Psi(n) + \Psi^\dagger(n)W(n)\Phi(n)\,. \tag{54}$$

The matrices $V(n)$ and $W(n)$ can be compactly written

$$V(n) \stackrel{\text{def}}{=} \begin{pmatrix} Z^\dagger(n+\frac{\hat{1}}{2}) & 0 \\ 0 & Z(n-\frac{\hat{1}}{2}) \end{pmatrix}, \; W(n) \stackrel{\text{def}}{=} \tau_{(1,4)} \begin{pmatrix} Z(n-\frac{\hat{2}}{2}) & 0 \\ 0 & Z^\dagger(n+\frac{\hat{2}}{2}) \end{pmatrix} \tau_{(1,4)}\,, \tag{55}$$

where the permutation matrix $\tau_{(1,4)}$ interchanges the first and fourth indices. The integral over auxiliary fields at the site n (including one color component) reads

$$\begin{aligned} \mathcal{Z}[n] \;&=\; \int d\Psi^\dagger(n)d\Psi(n)d\Phi^\dagger(n)d\Phi(n) \\ &\quad\times \exp\left[-m_b\left(\Psi^\dagger\Psi + \Phi^\dagger\Phi\right) + \Phi^\dagger V\Psi + \Psi^\dagger W\Phi\right] \\ &\quad\propto \mathrm{Det}\begin{pmatrix} m_b & -V \\ -W & m_b \end{pmatrix}^{-1} = \mathrm{Det}\left(m_b^2 - VM\right)^{-1} \\ &\quad= \exp\left[-\,\mathrm{Tr}\ln\left(1 - m_b^{-2}VW\right)\right] \approx \exp\left[m_b^{-2}\mathrm{Tr}\big(VW\big)\right]. \end{aligned} \tag{56}$$

In the last line we expanded the logarithm to first order in $1/m_b$. The trace of the product VW can be easily computed

$$\begin{aligned} \mathrm{Tr}\big(V(n)W(n)\big) \;&=\; Z^\dagger_{2,2}\left(n+\tfrac{\hat{1}}{2}\right) Z^\dagger_{1,1}\left(n+\tfrac{\hat{2}}{2}\right) + Z^\dagger_{-2,-2}\left(n+\tfrac{\hat{1}}{2}\right) Z_{1,1}\left(n-\tfrac{\hat{2}}{2}\right) \\ &\quad + Z_{2,2}\left(n-\tfrac{\hat{1}}{2}\right) Z^\dagger_{-1,-1}\left(n+\tfrac{\hat{2}}{2}\right) \\ &\quad + Z_{-2,-2}\left(n-\tfrac{\hat{1}}{2}\right) Z_{-1,-1}\left(n-\tfrac{\hat{2}}{2}\right). \end{aligned} \tag{57}$$

Notice that only the diagonal elements of the Z matrices appear in this leading-order term, which represent couplings between auxiliary fields carried by the same plaquette. In each term of the sum (57), the two matrix elements are carried by different links, but they correspond to fields related to the same plaquette, precisely the plaquette which shares these two links. One can represent the correlations embodied in (57) by *dual links* joining the middles of the two coupled links (see Fig. 3).

To summarize, to leading order in $1/m_b$ the full partition function is given by

$$\mathcal{Z} = \int \left\{ \prod_n \prod_{\alpha=1,2} d\mu\left(Z, Z^\dagger\left(n + \tfrac{\hat{\alpha}}{2}\right)\right) \right\} \exp\left(-N_c\, S[Z]\right)\,, \tag{58}$$

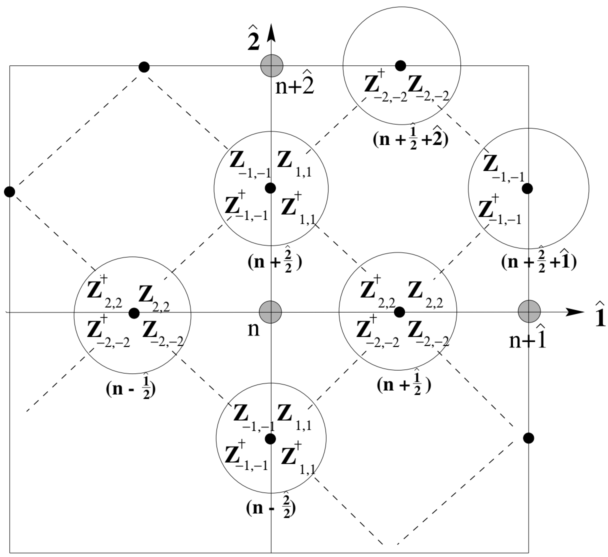

Fig. 3. Schematic representation of the quadratic action (57). Each large circle contains diagonal matrix elements situated at some link. The correlations between matrix elements of different links are depicted by broken lines.

with the effective action depending on the "flavor" matrices Z

$$-S[Z]=\sum_n \left[m_b^{-2}\mathrm{Tr}\,(V(n)W(n))+\sum_{\alpha=1,2}\mathrm{Tr}\ln\left(1-Z\left(n+\tfrac{\hat{\alpha}}{2}\right)Z^\dagger\left(n+\tfrac{\hat{\alpha}}{2}\right)\right)\right]. \tag{59}$$

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