THE GERBE THEORY OF THE BOSONIC σ-MODEL: THE MULTI-PHASE CFT, DUALITIES, AND THE GAUGE PRINCIPLE*

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The theory of gerbes provides us with powerful cohomological and geometric tools that have been successfully employed in the construction and classification of consistent two-dimensional non-linear bosonic σ-models, in both the classical and the quantum régime. The theory does, in particular, naturally accommodate the concept of a duality map between two such models and affords a rigorous formulation of the gauge principle. In the present note, I review recent progress in understanding the geometry of the σ-model from the gerbe-theoretic perspective.

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1. Introduction

The use of geometric and cohomological methods in the study of quantum-mechanical systems coupled to external gauge fields in topologically nontrivial circumstances has a long and successful history. It can be traced back to Dirac's original attempt, reported in [1], at formulating a rigorous description of the dynamics of an electron placed in the field of a magnetic monopole without a global potential. The analysis has yielded a relative quantisation of the electric and magnetic charges carried by the particles. It also has — in the long term — led to the introduction of the concise language of fibre bundles to modern mathematical physics and, ultimately, served the development and a better understanding of gauge theory proper. The basic idea of the approach, which boils down to consistently lifting the charged particle's dynamics from the original space-time target M of its propagation

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to a larger space $\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} M$, surjectively submersed over M and supporting a global primitive of (the pullback along $\pi_{\mathcal{E}}$ of) the closed background-field strength, has proven particularly natural from the point of view of Dirac's quantisation programme of [2] insomuch as it provides us with an explicit definition of a pre-quantum bundle $\pi^*_{\mathsf{T}^*M}\mathcal{E} \xrightarrow{\pi_{\mathcal{E}}} \mathsf{T}^*M$ over the phase space $\mathsf{T}^*M \xrightarrow{\pi_{\mathsf{T}^*M}} M$ of the physical system under investigation. The bundle gives rise to a definition of the Hilbert space of the system as the space of its suitably polarised sections,

$$\mathcal{H} = \Gamma_{\text{pol.}}\left(\pi^*_{\mathsf{T}^*M}\mathcal{E}\right) ,$$

with symmetries realised as bundle automorphisms, *cf.*, *e.g.*, [3].

With the advent, in the 1980s, of a panoply of models of two-dimensional Conformal Field Theory (CFT) with the metric-manifold structure on the fibre M of the covariant configuration bundle over the two-dimensional space-time, termed non-linear σ-models and emerging from a description of critical phenomena at points of a second-order phase transition in models of statistical physics, as effective field theories of quantum spin-chain excitations, and — finally — as Lagrangian models of the critical Polyakov string, the methods previously applied in the study of the charged point-like particle could be transplanted into the new domain of loop dynamics, whereby a novel geometric object was called for. The relevant object, with local data[1] captured by the (real) Deligne hypercohomology in degree 3, was identified in the pioneering papers: [4] by Alvarez, and [5] by Gawędzki, and subsequently further formalised by Brylinski in [6]. A geometric realisation of (a representative of) the relevant class in the second Deligne hypercohomology group was obtained by Murray and Stevenson in Refs. [7,8]. Thus, the fibre bundle associated with the charged point-like particle was supplanted by the (Abelian) bundle gerbe (with curving and Hermitian connection) $\mathcal{G}$ over the target M of the dynamical loop, in which all the features of the original construction central to its rôle in defining a quantisation of the classical model consistent with its symmetries were preserved in consequence of the existence of the transgression map of [5]. The latter canonically associates to $\mathcal{G}$ a line bundle $\mathcal{L}_{\mathcal{G}} \to \mathsf{L}M$ (with connection $\nabla_{\mathcal{L}_{\mathcal{G}}}$) over the free-loop space $\mathsf{L}M$ of M. Its curvature 2-form $\text{curv}(\nabla_{\mathcal{L}_{\mathcal{G}}}) \in \Omega^2(\mathsf{L}M)$ reproduces, upon pullback to the phase space $\mathsf{T}^*\mathsf{L}M \xrightarrow{\pi_{\mathsf{T}^*\mathsf{L}M}} \mathsf{L}M$ of the σ-model, the cohomologically nontrivial (in general) component of the symplectic form $\Omega_\sigma \in \Omega^2(\mathsf{T}^*\mathsf{L}M)$ of the two-dimensional field theory,

$$H^2(\mathsf{T}^*\mathsf{L}M) \ni \left[\Omega_\sigma - \pi^*_{\mathsf{T}^*\mathsf{L}M}\text{curv}(\nabla_{\mathcal{L}_{\mathcal{G}}})\right] = 0 .$$

[1] That is data associated with a choice of an open cover of the target manifold M.

The ensuing pre-quantum bundle of the σ-model, explicitly realised (at least on the formal level) in terms of Feynman's path integrals in a manner originally devised in [9] and adapted to the two-dimensional setting in [5], could next be used in a systematic discussion of quantum lifts of its symmetries.

The unceasing theoretic interest in symmetries of two-dimensional CFT and dualities between various such theories, going beyond the standard applications in the derivation of correlation functions with the help of Ward–Takahashi identities and a methodical reconstruction of the Hilbert space of these theories in the framework of the representation theory of the (current-)symmetry algebras, is of twofold origin. On the one hand, it stems from the long-cherished hope that an in-depth survey of dualities that set outwardly distinct CFTs in correspondence can help to tame the immense moduli space of consistent CFTs (of, say, the σ-model type), an issue of prime significance, *e.g.*, in string theory[2]. On the other hand, it rests upon the old observation made by Martinec in [10] and subsequently elaborated upon by Moore and Seiberg in Refs. [11,12,13] and by a good many of followers, that the gauging of internal (*i.e.* rigid) symmetries of a given CFT, whether discrete or continuous, is a basic, and — under certain circumstances — even exhaustive procedure for building new models from a given class (such as, *e.g.*, rational CFTs) out of a given parent model. In the latter context, it deserves to be emphasised that gerbe theory puts at our disposal the potent tools of equivariant cohomology which, in conjunction with the principle of categorial descent laid out in [14], bring clear-cut answers to the questions of the existence and uniqueness of the derivative σ-models that can be obtained in the gauging procedure. In particular, they provide a neat cohomological characterisation of gauge anomalies, *cf.* Refs. [15, 16] for a general discussion and also Refs. [17,18,19,20] for a detailed treatment of the special case of a discrete gauge group (augmented by the space-time parity group $\mathbb{Z}_2$, of particular relevance in the context of a supersymmetric extension of the theory). These answers fall in perfect agreement with results of alternative approaches to the CFT model-building, such as, *e.g.*, the classification of modular invariants and the TFT-aided realisation of Segal's categorial quantisation programme, whenever the said approaches manage to deliver concrete results susceptible to comparison. But over and above this, they seem to open avenues for straightforward generalisations that promise to elucidate the recent proposals of so-called non-geometric σ-model backgrounds, advanced in [21].

The notion of a conformal duality map constitutes a natural abstraction of the better-studied concept of an internal symmetry of a σ-model with a *fixed* target space M, induced from an isometry of the latter (preserving the gerbe over it in a suitable sense), to the case in which two dis-

[2] The so-called "duality net" of consistent superstring theories lends strong support to such expectations.

tinct σ-models, with target spaces of possibly inequivalent topology, metric and gerbe structure, are mapped to one another by a symplectomorphism between the respective phase spaces that identifies the spectra of the two Hamiltonians and admits a lift to a polarisation-intertwining isomorphism between the two pre-quantum bundles. Given the geometric interpretation of the σ-model target space, this indicates the possibility of a conceptual departure from the Riemannian paradigm of globally smooth geometry, as per a variant of the gauging procedure tailored to background-changing dualities. This is one of the topics currently under investigation. It is hoped to give a rigorous meaning to the aforementioned proposal of loop dynamics in non-geometric backgrounds.

Within the strict categorial quantisation programme, originally put forward by Segal in [22] and later developed by Fröhlich *et al.* in a series of papers that followed [23], a correspondence has been established between (at least some of) the dualities and a class of operators on the Hilbert space of the CFT on either side of a one-dimensional domain wall within the two-dimensional space-time. From the field-theory perspective, the domain wall marks the locus of a distinguished discontinuity of the σ-model field and is accordingly understood to separate distinct phases of a *single* CFT. Domain walls of this sort, termed defect lines and akin to lines of frustration in spin lattice models (such as, *e.g.*, the two-dimensional Ising model in which they describe macroscopic arrays of flipped neighbouring spins), are bound to appear in a Lagrangian formulation of the classical dynamics of the σ-model. A prototypical example of such a situation is encountered when passing from a σ-model with a target space M having a discrete group G as a subgroup of the (gerbe-preserving) isometry group to its G-orbifold, that is a σ-model with the coset target space M/G. The orbifold can be defined in terms of all those patch-wise continuous field configurations of the parent model whose discontinuities, localised at a number of intersecting defect lines $\ell_i,\ i \in \{1, 2, \ldots\}$ embedded arbitrarily densely in the two-dimensional space-time, are given each by the action of the corresponding element $z_i \in \mathrm{G}$ and hence are smoothened upon passing to M/G. The above example motivates the study of multi-phase CFTs in the Lagrangian picture in which they are supported on space-times decorated by arbitrary oriented defect graphs, such as the one shown in Fig. 1.

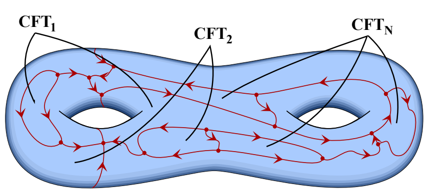


Fig. 1. The two-dimensional space-time of a multi-phase σ-model. The various phases of the CFT are denoted as $\mathrm{CFT}_1, \mathrm{CFT}_2, \ldots, \mathrm{CFT}_\mathrm{N}$.

The very construction of a classical multi-phase CFT in terms of an action functional brings to the fore, as a prerequisite, the full-fledged 2-categorial structure built, along the lines of [14], around the geometric object $\mathcal{G}$ necessitated by the analysis of the σ-model on an un-decorated (closed) space-time. Consistently with the earlier findings, the construction, derived[3] in [28], assigns to the patches of the space-time decorated with a defect graph Γ the 0-cells of the 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$ of Abelian bundle gerbes with curving and Hermitian connection over the extended target space $\mathscr{F} := M \sqcup Q \sqcup T$ of the σ-model. The latter contains, alongside the (now possibly multi-component) target manifold $M = M_1 \sqcup M_2 \sqcup \cdots$ for the patches (*i.e.* for the distinct phases), also the smooth targets Q and T into which — respectively — the edges and the vertices of Γ are mapped by the σ-model field $X : \Sigma \to \mathscr{F}$. To the edges and vertices of Γ, it associates — respectively — specific 1-cells (termed bi-branes) and 2-cells (termed inter-bi-branes) of the said 2-category.

The discussion of the σ-model in the presence of intersecting defect lines very clearly attests the naturality of the 2-categorial (and geometric) language of the theory of gerbes in the context of loop dynamics. This conclusion is further strengthened by the observation, formalised in [29], that the complete 2-categorial content of the classical σ-model on a space-time with defect lines canonically determines a geometric quantisation scheme via transgression, in perfect analogy with the case of an undecorated space-time with or without boundary examined in Refs. [17] and [5], respectively, of which [29] is, in this respect, a logical completion. It is worth pointing out that the thus established framework of canonical analysis of the multi-phase σ-model can be consistently extended to space-times of arbitrary topology[4] to yield a phase- and Hilbert-space picture of the purely geometric splitting-joining interaction of any number of loops. This fact can be viewed as an explicit manifestation, first indicated in [5] (*cf.* also [30]), of Segal's categorial quantisation scheme in terms of generalised transport and holonomy operators defining (the topological term in) the action functional of the σ-model on an interaction-vertex space-time with an embedded defect graph, of the type depicted in Fig. 2. Moreover, it supplies us with methods of direct investigation of the fusion algebra of consistent dualities, an intrinsic feature of defects that distinguishes them from conformal boundaries. The last remark finds explicit corroboration in the recent findings of [28]

[3] The derivation was largely inspired by the earlier constructions for space-times with boundaries, starting with [24], later taken up in [25], and culminating in Refs. [17,26], and especially by the recent CFT-based proposal of [27] for space-times with non-intersecting defects.

[4] In its most elementary form, it gives an account of the canonical structure of the model of a single loop sweeping a world-sheet of the topology of a cylinder in the field space M.

where the Moore–Seiberg fusing matrices of the simple-current sector of the Wess–Zumino–Witten (WZW) σ-model with a simple connected Lie-group target were extracted from the calculus of distinguished 2-isomorphisms of the gerbe 2-category for the σ-model with the so-called central-jump defects, and those of Refs. [31, 32] providing strong evidence of the existence[5] of a purely geometric realisation of the Verlinde fusion ring of chiral sectors of the WZW model in terms of a class of inter-bi-branes associated with junctions of the so-called maximally symmetric defects of [27].

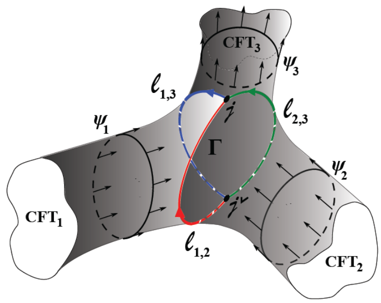


Fig. 2. The basic splitting-joining interaction between states ψ_1, ψ_2 and ψ_3 from the respective phases CFT_i, $i \in \{1,2,3\}$ across the defect graph Γ composed of defect lines $\ell_{1,2}, \ell_{2,3}$ and $\ell_{1,3}$ intersecting at defect junctions $\jmath$ and $\jmath^\vee$.

The gerbe-induced geometric quantisation of the multi-phase σ-model gives us access to the standard field-theoretic methods of study of consistent dualities of the quantised theory and thus paves the way to their cohomological classification and systematic construction. Finally, having put background-preserving symmetries and *bona fide* dualities on common footing, it offers us invaluable insights into the still poorly understood field of non-geometry (or, more adequately, *string* or *loop* geometry) from the vantage point of the by now much explored theory of geometric (internal) symmetries of the σ-model, of an intrinsically groupoidal nature, and the well-developed methodology of their gauging[6]. Hence, it gives promise of continually expanding our hitherto knowledge on the intricate geometry of the field space of the dynamical loop.

In the remainder of the present note, I try to give substance — by way of a more detailed argument or a specific reference to the extensive literature on the subject — to at least some of the observations and claims made in the foregoing paragraphs. In so doing, I hope to provide the reader with a

[5] It was actually proven in the case of the WZW model with the target group SU(2).

[6] For a recent work on the relation between internal σ-model symmetries and Courant-type algebroid structures on the generalised tangent bundle over the extended field space, and for a discussion of the groupoidal interpretation of the gaugeability conditions catalogued in Refs. [15, 16], *cf.* [33].

fairly general, if also unavoidably incomplete, overview of the recent progress in our understanding of the geometry and physics of the two-dimensional σ-model from the gerbe-theoretic perspective.

2. A 2-category for the decorated world-sheet

Two-dimensional field theories with a non-anomalous (local) conformal symmetry have played an important rôle in the development of a variety of fields of modern mathematical physics, serving as a testing ground for a theoretic modelling of complex higher-dimensional phenomena, such as, *e.g.*, the confinement of charge, and bringing into the game their own share of novel effects, such as, *e.g.*, the fermion–boson equivalence. They have been instrumental in setting up an effective field-theoretic description of two-dimensional lattice models of statistical mechanics in terms of Landau–Ginzburg Lagrangians, in the study of the Haldane–Affleck effective dynamics of excitations of quantum spin chains in the continuum limit (resurfacing nowadays as an important aspect of the AdS/CFT correspondence), and in the Lagrangian formulation of classical string theory at criticality.

A prominent place amidst two-dimensional CFTs is occupied by the so-called non-linear σ-model. In its simplest version, it is a Lagrangian theory of C^1-maps $X : \Sigma \to M$ from a two-dimensional manifold Σ, termed the world-sheet, with a Minkowskian or Euclidean[7] metric γ into the smooth fibre M, termed the target space, of the covariant configuration bundle $\Sigma \times M \to \Sigma$. The target space is equipped with a metric g and a 2-form (Kalb–Ramond) field B, which, from the point of view of the two-dimensional field theory, determine the highly non-linear (in general) dynamics of the embedding field X as per the least-action principle applied to the σ-model action functional

$$S_\sigma[X;\gamma] = -\frac{1}{2}\int_\Sigma \mathrm{g}_X\,(\mathsf{d}X \overset{\wedge}{,} \star_\gamma \mathsf{d}X) + \int_\Sigma X^*\mathrm{B}\,. \tag{2.1}$$

Above, the metric g (evaluated at a point $X(p) \in M$ for $p \in \Sigma$) is understood to act on the second factor in $\mathsf{d}X = \partial_a X^\mu\,\mathsf{d}\sigma^a \otimes \partial_\mu$ (the latter being written in local coordinates[8] $\{\sigma^a\}^{a\in\{1,2\}}$ on Σ and $\{X^\mu\}^{\mu\in\overline{1,\dim M}}$ on M), and $\star_\gamma$ is the Hodge operator on $\Omega^\bullet(\Sigma)$. Thus, the 2-form field enters the Lagrangian description through a purely topological Wess–Zumino term, in complete analogy with the manner in which the 1-form potential of the electromagnetic field couples to a charged particle's world-line.

[7] While on world-sheets with elementary topology, such as the cylinder and the torus, the choice of the signature of γ is the standard matter of a field-theoretic convention, obstructions against the existence of a global Lorentzian structure would favour the Euclidean signature on world-sheets of a more involved topology.

[8] Here, and in what follows, we use the notation $\overline{1,n} = \{\ k \in \mathbb{N}\ \mid\ 1 \le k \le n\ \}$.

The metric g and the 2-form field B are mutually constrained by the requirement of cancellation of the Weyl anomaly in the quantised σ-model, *cf.* Refs. [34,35], so that — in particular — we are not at liberty to choose, say, the Kalb–Ramond field arbitrarily once the metric on M has been specified. It is important to note that neither the anomaly constraints, nor the field equations of the σ-model depend on B itself but instead on its exterior derivative, or field strength $\mathsf{d}\mathrm{B} =: \mathrm{H}$. Taking into account a loop variant of the Aharonov–Bohm effect, we infer that quantum-mechanical amplitudes are ultimately bound to depend on the cohomology class of the gauge field B. It may well happen that the anomaly constraints force us to consider a field strength 3-form H that represents a nontrivial class in $H^3(M)$ and hence does not admit a global primitive. This is the case, *e.g.*, if a compact simple 1-connected Lie group G with the Cartan–Killing metric is taken as the target space, as in the seminal [36], whereupon H is found to coincide with an $\mathbb{R}$-multiple of the canonical Cartan 3-form on G whose de Rham class generates $H^3(G) \cong \mathbb{Z}$. It is therefore of essence to understand the topological term in Eq. (2.1) in more intrinsic, geometric terms, devoid of an explicit dependence on the choice of a local primitive B of the field strength H. This motivated a detailed examination, undertaken by Alvarez in [4] and subsequently placed in the appropriate (more) formal context in Refs. [5,6], of the Wess–Zumino term in a topologically nontrivial setting. The result was a local formula for the Wess–Zumino term given by (the logarithm of) a particular Cheeger–Simons differential character evaluated on the *closed* world-sheet Σ along the embedding map X. While written for a specific triangulation of the world-sheet and expressed in terms of components of a Čech–Deligne cochain, composed of local sections of the Deligne complex

$$\mathcal{D}(2)^\bullet_M \ :\ \underline{\mathrm{U}(1)}_M \xrightarrow{\frac{1}{\mathsf{i}}\mathsf{d}\log} \underline{\Omega}^1(M) \xrightarrow{\mathsf{d}} \underline{\Omega}^2(M)$$

of sheaves of locally smooth $\mathrm{U}(1)$-valued maps and locally smooth (real) 1- and 2-forms on M, associated with a specific open cover of M and defining a class in the Deligne hypercohomology group $\mathbb{H}^2\left(M, \mathcal{D}(2)^\bullet_M\right)$ that determines a local trivialisation of H, the formula was demonstrated to be independent of the arbitrary choices entering its definition. By the standard argument, the de Rham class of the field strength was then constrained as

$$[\mathrm{H}] \in H^3(M, 2\pi\mathbb{Z}) \subset H^3(M, \mathbb{R})\,.$$

A similar strategy was subsequently employed in [24] in a derivation of appropriate local boundary corrections to the Wess–Zumino term induced on a world-sheet with $\partial\Sigma \neq \varnothing$. This, in turn, produced an additional constraint[9]

[9] Here, $B^3(Q)$ is the group of exact 3-forms on Q.

$$\iota_Q^* \mathrm{H} \in B^3(Q)$$

for the restriction of H to a submanifold embedded, as per $\iota_Q : Q \to M$, in the target space and defining the codomain of $X|_{\partial\Sigma}$.

The geometric object $\mathcal{G} = (\mathsf{Y}M, \mathrm{B}, L, \mu_L)$ realising a class in $\mathbb{H}^2(M, \mathcal{D}(2)^\bullet_M)$ associated with the given closed 3-form H with periods in $2\pi\mathbb{Z}$ was constructed by Murray in [7] and dubbed the Abelian bundle gerbe with curving and Hermitian connection. It is neatly represented by the diagram

$$\begin{array}{ccc} (L, \nabla_L, \mu_L) & & \\ \Big\downarrow{\scriptstyle \pi_L} & & \\ \mathsf{Y}^{[2]}M & \overset{\mathrm{pr}_1}{\underset{\mathrm{pr}_2}{\rightrightarrows}} & (\mathsf{Y}M, \mathrm{B}) \\ & & \Big\downarrow{\scriptstyle \pi_{\mathsf{Y}M}} \\ & & (M, \mathrm{H}) \end{array}$$

and consists of a surjective submersion $\pi_{\mathsf{Y}M} : \mathsf{Y}M \to M$ and a globally defined 2-form $\mathrm{B} \in \Omega^2(\mathsf{Y}M)$ such that

$$\pi_{\mathsf{Y}M}^* \mathrm{H} = \mathsf{d}\mathrm{B}\,,$$

together with a line bundle $\mathbb{C} \hookrightarrow L \xrightarrow{\pi_L} \mathsf{Y}^{[2]}M$ over the M-fibred square

$$\mathsf{Y}^{[2]}M = \{\ (y_1, y_2) \in \mathsf{Y}M \times \mathsf{Y}M \quad | \quad \pi_{\mathsf{Y}M}(y_1) = \pi_{\mathsf{Y}M}(y_2)\ \} \equiv \mathsf{Y}M \times_M \mathsf{Y}M\,,$$

$$\mathrm{pr}_i \ :\ \mathsf{Y}^{[2]}M \to \mathsf{Y}M \ :\ (y_1, y_2) \mapsto y_i\,, \qquad i \in \{1, 2\}\,,$$

with connection ∇_L of curvature

$$\mathrm{curv}(\nabla_L) = \mathrm{pr}_2^* \mathrm{B} - \mathrm{pr}_1^* \mathrm{B}\,,$$

and of a distinguished isomorphism of line bundles with connection

$$\mu_L \ :\ \mathrm{pr}_{1,2}^* L \otimes \mathrm{pr}_{2,3}^* L \xrightarrow{\cong} \mathrm{pr}_{1,3}^* L$$

over the triple M-fibred product $\mathsf{Y}^{[3]}M \equiv \mathsf{Y}M \times_M \mathsf{Y}M \times_M \mathsf{Y}M$, the latter being equipped with the canonical projection maps

$$\mathrm{pr}_{i,j} \ :\ \mathsf{Y}^{[3]}M \to \mathsf{Y}^{[2]}M \ :\ (y_1, y_2, y_3) \mapsto (y_i, y_j)\,,$$
$$(i, j) \in \{(1, 2), (2, 3), (1, 3)\}\,.$$

The isomorphism is required to obey the associativity constraint

$$\mathrm{pr}_{1,2,4}^*\mu \circ \left(\mathrm{id}_{\mathrm{pr}_{1,2}^* L} \otimes \mathrm{pr}_{2,3,4}^*\mu\right) = \mathrm{pr}_{1,3,4}^*\mu \circ \left(\mathrm{pr}_{1,2,3}^*\mu \otimes \mathrm{id}_{\mathrm{pr}_{3,4}^* L}\right)$$

over the quadruple M-fibred product $\mathsf{Y}^{[4]}M \equiv \mathsf{Y}M \times_M \mathsf{Y}M \times_M \mathsf{Y}M \times_M \mathsf{Y}M$, endowed with the obvious canonical projections $\mathrm{pr}_{i,j,k} : \mathsf{Y}^{[4]}M \to \mathsf{Y}^{[3]}M$ and $\mathrm{pr}_{i,j} : \mathsf{Y}^{[4]}M \to \mathsf{Y}^{[2]}M$. An explicit link to the local description in terms of the Deligne hypercohomology is readily established with the help of local sections of $\mathsf{Y}M \to M$ and $L \to \mathsf{Y}^{[2]}M$ associated with an open cover of M.

Just as line bundles with connection over a given base form a category together with bundle maps between them, gerbes with curving and connection over a given base M give rise to a tensor 2-category in the sense of, *e.g.*, [37], as first observed by Stevenson in [14] and subsequently elaborated in [38]. The 2-category shall be denoted as $\mathfrak{BGrb}^\nabla(M)$ in what follows. Its 1-cells are morphisms

$$\Phi_{1,2} \ : \ \mathcal{G}_1 \to \mathcal{G}_2$$

between gerbes $\mathcal{G}_i = (\mathsf{Y}_i M, \mathrm{B}_i, L_i, \mu_{L_i})$ (*i.e.* between 0-cells), defined as triples $\Phi_{1,2} = (\mathsf{YY}_{1,2}M, E_{1,2}, \alpha_{1,2})$ consisting each of a surjective submersion $\mathsf{YY}_{1,2}M \to \mathsf{Y}_{1,2}M \equiv \mathsf{Y}_1 M \times_M \mathsf{Y}_2 M$, of a rank-$N$ vector bundle $E_{1,2} \to \mathsf{YY}_{1,2}M$ with connection $\nabla_{E_{1,2}}$ of scalar curvature

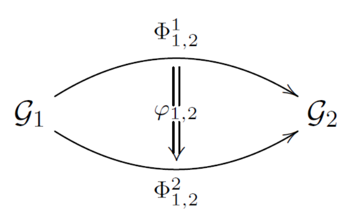
$$\frac{1}{N}\,\mathrm{tr}\left(\mathrm{curv}(\nabla_{E_{1,2}})\right) = \mathrm{pr}_2^*\mathrm{B}_2 - \mathrm{pr}_1^*\mathrm{B}_1$$

(written in terms of the canonical projections $\mathrm{pr}_i : \mathsf{Y}_{1,2}M \to \mathsf{Y}_i M$), and of an isomorphism of vector bundles with connection

$$\alpha_{1,2} \ : \ \mathrm{pr}_{1,3}^* L_1 \otimes \mathrm{pr}_{3,4}^* E_{1,2} \xrightarrow{\cong} \mathrm{pr}_{1,2}^* E_{1,2} \otimes \mathrm{pr}_{2,4}^* L_2$$

over the M-fibred square $\mathsf{Y}^{[2]}\mathsf{Y}_{1,2}M \equiv \mathsf{YY}_{1,2}M \times_M \mathsf{YY}_{1,2}M$, the latter being equipped with the obvious canonical projections $\mathrm{pr}_{1,2}, \mathrm{pr}_{3,4} : \mathsf{Y}^{[2]}\mathsf{Y}_{1,2}M \to \mathsf{YY}_{1,2}M$ and $\mathrm{pr}_{i,i+2} : \mathsf{Y}^{[2]}\mathsf{Y}_{1,2}M \to \mathsf{Y}_i^{[2]}M$. The isomorphism is subject to an additional condition that expresses its compatibility with the μ_{L_i}, *cf.* [38]. The distinguished (invertible) 1-cells associated with rank-1 bundles are called 1-isomorphisms. Finally, 2-cells of $\mathfrak{BGrb}^\nabla(M)$ are secondary morphisms

$\varphi_{1,2} \doteq$



between primary morphisms $\Phi^A_{1,2} = (\mathsf{Y}^A\mathsf{Y}_{1,2}M, E^A_{1,2}, \alpha^A_{1,2})$, $A \in \{1,2\}$, *i.e.* pairs $(\mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M, \beta_{1,2})$ composed each of a surjective submersion $\pi_{\mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M} : \mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M \to \mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M \equiv \mathsf{Y}^1\mathsf{Y}_{1,2}M \times_{\mathsf{Y}_{1,2}M} \mathsf{Y}^2\mathsf{Y}_{1,2}M$ and an isomorphism of vector bundles with connection

$$\beta_{1,2} : p_1^* E^1_{1,2} \xrightarrow{\cong} p_2^* E^2_{1,2} ,$$

expressed in terms of the maps $p_A = \mathrm{pr}_A \circ \pi_{\mathsf{Y}\mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M}$, with the canonical projections $\mathrm{pr}_A : \mathsf{Y}^{1,2}\mathsf{Y}_{1,2}M \to \mathsf{Y}^A\mathsf{Y}_{1,2}M$, and subject to an extra condition of compatibility with the $\alpha^A_{1,2}$, *cf.* *ib.* It is to be noted that the above construction is compatible with the underlying structure of a category of (finite-dimensional) smooth manifolds of which the base M of $\mathfrak{BGrb}^\nabla(M)$ is an object: The 2-category admits a natural notion of transport (or pullback) along smooth maps.

The hypercohomological content of the gerbe 2-category, accessible directly via local sections of the various surjective submersions involved, affords an explicit definition of the σ-model on a world-sheet Σ with an embedded defect graph Γ, first derived in [28], along the lines of Alvarez's original reasoning. The derivation presupposes that patches of Σ into which Γ splits the world-sheet are mapped by a patch-wise C^1-map X into a metric target space (M, g) (possibly a disjoint union of spaces) with a gerbe $\mathcal{G}$ over it, that edges ℓ of Γ are similarly mapped into another manifold Q equipped with a pair of smooth maps $\iota_\alpha : Q \to M$, $\alpha \in \{1,2\}$, and that every intersection vertex $\jmath_n$ of valence n of Γ is sent into a smooth manifold T_n coming with a collection of smooth maps $\pi_n^{k,k+1} : T_n \to Q$, $k \in \overline{1,n}$ (with $\pi_n^{n,n+1} \equiv \pi_n^{1,n}$). The latter satisfy some straightforward conditions of compatibility with the ι_α which follow from their world-sheet interpretation: While the maps ι_α reproduce the two one-sided limits of the patch-embedding map attained at a point $p \in \ell$ from the single value of the edge-embedding map at p, the $\pi_n^{k,k+1}$ relate, in an analogous fashion, the one-sided limiting values of the edge-embedding maps for the n defect lines converging at $\jmath_n$ to the value of the vertex-embedding map assumed at that junction of the defect lines. The defining data of a consistent σ-model on $\Sigma \supset \Gamma$, to be pulled back to the patches of the world-sheet and to the edges and vertices of the defect graph, are then supplied by the 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$ for the composite target $\mathscr{F} = M \sqcup Q \sqcup T \equiv M \sqcup Q \sqcup \bigsqcup_{n \in \mathbb{N}_{\geq 3}} T_n$ of the σ-model field X. More specifically, we arrive at the notion of a string background, *cf.* [29], that is a triple $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ composed of the following geometric structures:

- the target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ consisting of a metric target space (M, g) and a gerbe $\mathcal{G}$ of curvature $\mathrm{H} \equiv \mathrm{curv}(\mathcal{G})$;
- the $\mathcal{G}$-bi-brane $\mathcal{B} = \big(Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\}\big)$ consisting of a $\mathcal{G}$-bi-brane world-volume Q, with a $\mathcal{G}$-bi-brane curvature 2-form ω, and a pair of

smooth maps $\iota_\alpha : Q \to M$, $\alpha \in \{1,2\}$, and of a gerbe 1-isomorphism

$$\Phi \ : \ \iota_1^*\mathcal{G} \xrightarrow{\cong} \iota_2^*\mathcal{G} \otimes I_\omega\,,$$

written in terms of a so-called trivial gerbe I_ω with a global curving ω, obeying the identity

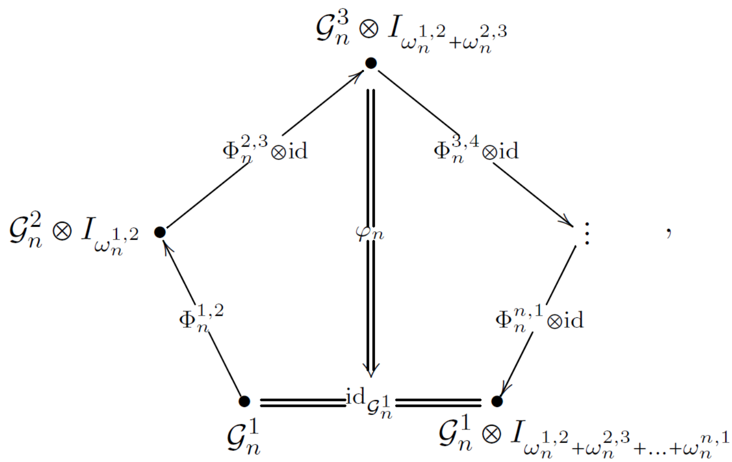
$$\Delta_Q \mathrm{H} = -\mathrm{d}\omega\,, \qquad \Delta_Q := \iota_2^* - \iota_1^*\,;$$

- the $(\mathcal{G},\mathcal{B})$-inter-bi-brane $\mathcal{J} = \big(T_n, (\pi_n^{k,k+1} \mid k \in \overline{1,n}), \varphi_n \mid n \in \mathbb{N}_{\geq 3}\big)$, consisting of the composite $(\mathcal{G},\mathcal{B})$-inter-bi-brane world-volume $T = \bigsqcup_{n \in \mathbb{N}_{\geq 3}} T_n$, with a collection of smooth maps $\pi_n^{k,k+1} : T_n \to Q$ subject to the constraints[10]

$$\iota_2 \circ \pi_n^{k-1,k} = \iota_1 \circ \pi_n^{k,k+1}\,, \qquad k \in \overline{1,n}\,,$$

and of distinguished gerbe 2-isomorphisms

$\varphi_n \quad \doteq$



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written in terms of 1-isomorphisms $\Phi_n^{k,k+1} = \pi_n^{k,k+1\,*}\Phi$ between gerbes $\mathcal{G}_n^k = (\iota_1 \circ \pi_n^{k,k+1})^*\mathcal{G}$, and of the trivial gerbes with global curvings $\omega_n^{k,k+1} = \pi_n^{k,k+1\,*}\omega$. The latter satisfy the Defect-Junction Identity (DJI)

$$\Delta_{T_n}\omega = 0\,, \qquad \Delta_{T_n} := \sum_{k=1}^{n} \pi_n^{k,k+1\,*}\,.$$

[10] In fact, one should also take into account the various possible relative orientations (in-coming *vs.* out-going) of the defect lines converging at a given defect junction. We drop the relevant elements of the formalism for the sake of transparency, assuming all defect lines to be in-coming.

Within this complex but otherwise completely natural geometric and categorial framework, the exponentiated[11] topological term in the action functional of the σ-model acquires a simple interpretation: It is the complex number of unital modulus assigned to $\Sigma \supset \Gamma$ by a generalised Cheeger–Simons differential character on graph-decorated closed two-dimensional manifolds. The number, determined by the background $\mathfrak{B}$, was dubbed the generalised surface holonomy for a network-field configuration $(X \,|\, \Gamma)$ in [28] and denoted as $\mathrm{Hol}_{\mathfrak{B}}(X \,|\, \Gamma)$, *cf.* [29]. In the simplest setting, which is that of $\Gamma = \varnothing$, it is just the standard surface holonomy $\mathrm{Hol}_{\mathcal{G}}(X) \in \mathrm{U}(1)$ given by the image of the class $[X^*\mathcal{G}] \in H^2(\Sigma, \mathrm{U}(1))$ under the isomorphism $H^2(\Sigma, \mathrm{U}(1)) \cong \mathrm{U}(1)$. The appearance of the (sheaf-)cohomology group $H^2(\Sigma, \mathrm{U}(1))$ in this context is a direct consequence of the following crucial corollaries of the relation between the Deligne hypercohomology and sheaf cohomology, *cf.*, *e.g.*, Refs. [6, 17, 39].

Proposition 1 *The set of 1-isomorphism classes of gerbes with a given curvature over a manifold M is a torsor under a natural action of the sheaf-cohomology group $H^2(M, \mathrm{U}(1))$.*

Proposition 2 *The set of 2-isomorphism classes of 1-isomorphisms between two given gerbes over a manifold Q is a torsor under a natural action of the sheaf-cohomology group $H^1(Q, \mathrm{U}(1))$.*

Proposition 3 *The set of inequivalent 2-isomorphisms between two given 1-isomorphisms of gerbes over a manifold T with $|\pi_0(T)|$ connected components is a torsor under a natural action of the sheaf-cohomology group $H^0(T, \mathrm{U}(1)) \cong \mathrm{U}(1)^{|\pi_0(T)|}$.*

Incidentally, the above propositions lead us to a natural cohomological classification of the various constitutive elements of the two-dimensional field theory of interest, to wit,

- inequivalent species of the theory with a fixed target structure $(M, \mathrm{g}, \mathrm{H})$ are enumerated by classes in $H^2(M, \mathrm{U}(1))$;
- inequivalent species of a defect line with a fixed (target and) $\mathcal{G}$-bi-brane structure $(Q, \iota_\alpha, \omega)$ are labelled by classes in $H^1(Q, \mathrm{U}(1))$;
- inequivalent species of a defect junction with a fixed (target, bi-brane and) $(\mathcal{G}, \mathcal{B})$-inter-bi-brane structure $\big(T_n, (\pi_n^{k,k+1} \mid k \in \overline{1,n})\big)$ are in a one-to-one correspondence with elements of $\mathrm{U}(1)^{|\pi_0(T)|}$.

Their precise physical interpretation and direct application in a systematic study of σ-model dualities shall be discussed in the remainder of this note.

[11] Recall that the classical action functional enters the definition of the Feynman amplitude through the expression $\mathrm{e}^{\mathrm{i}\, S_\sigma[X;\gamma]}$.

3. The canonical picture: dualities and state fusion

A field-theoretic model of physical phenomena ought to be understood more adequately as an equivalence class within a suitable category (that is, depending on the régime of the analysis, either the category of symplectic manifolds with a distinguished family of Hamiltonian functionals or the category of Hilbert spaces with a distinguished family of self-adjoint operators) than a particular representative of the said class, defined by, say, a specific action functional. This natural constatation has driven much of the hitherto effort aimed at understanding the so-called moduli space of consistent (conformal) non-linear σ-models, or — more generally — of consistent two-dimensional CFTs, that is the space of all such theories. Amongst the great successes of this line of research, one should count the discovery of the 'duality net' of (super)string theories in the 1990s. In what follows, we stick to the fairly standard nomenclature and call an isomorphism of the suitable (classical or quantum) category of field theories a duality. In so doing, we single out those isomorphisms which do not change a given target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ and merely effect translations within the corresponding state space along flows of Hamiltonian vector fields induced by gerbe-preserving isometries of the target space — these dualities shall be called symmetries of a given σ-model. They are at the focus of the next section of this note.

The smooth manifold structure on (the fibre of) the covariant configuration bundle of the σ-model serves to guide our intuition in the search for a field-theoretic realisation of a duality. The latter is a particular correspondence between states of the theory, and states are classically represented by the Cauchy data, localised on equitemporal slices of a cylindrical world-sheet, of field configurations extremising the σ-model action functional, or — in short — by loops smoothly embedded in the target space and carrying a normal momentum 1-form field. This is just the standard model $\mathsf{T}^*\mathsf{L}M$ of the loop phase space[12]. Consequently, a duality can be represented by a closed contour in a cylindrical world-sheet, separating the space-time supports of the two dual σ-models which are to be regarded as *phases* of a single CFT. In this picture, illustrated in Fig. 3, the two states in correspondence appear as the two one-sided boundary conditions imposed at the identification/discontinuity contour upon the two smooth classical field configurations of the dual models extending away from the contour. As the two field configurations come from two equivalent formulations of a single theory, it is

[12] Having admitted arbitrary embedded defect graphs, we should — in principle — consider both smooth and piece-wise smooth loops in M, the latter spanning the so-called twisted sector of the theory in the presence of time-like defect lines. However, as we are mainly interested in understanding the canonical interpretation of defects, we focus on the former, relegating the discussion of the latter to the final part of the present section in which we present results on the canonical description of the splitting-joining loop interaction.

to be expected that they can be glued consistently across the identification contour and thus define a field configuration of a σ-model with a circular domain wall, or defect line. If this heuristic reasoning is to work, the 'duality data' to be placed at the discontinuity contour should be identified with the data of a suitable bi-brane in whose world-volume, customarily termed the correspondence space in this context, the contour is embedded.

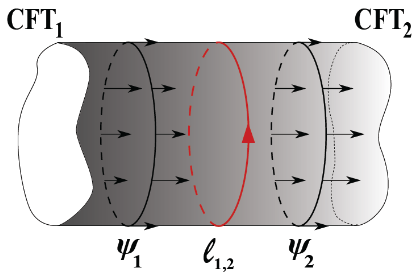

Fig. 3. The correspondence between states mediated by the defect line: The state ψ_1 from the phase CFT_1 is transferred to the state ψ_2 from the phase CFT_2 across the defect line $\ell_{1,2}$.

The above intuition appears to be corroborated by numerous results of studies of defects carried out in the framework of the full-fledged quantum CFT (both algebraic and categorial), and in particular those reported in Refs. [28, 40, 41, 42, 43, 44]. They bear ample evidence of a prominent rôle played by conformal domain walls in establishing correspondences between phases of CFT, in encoding order-disorder dualities among various CFTs, and in mapping into one another their RG flows as well as UV and IR fixed points thereof. Furthermore, they associate with the domain walls the so-called spectrum-generating symmetries of the CFT, relating — via fusion with boundary states — the categories of consistent boundary bi-branes (or D-branes) of a dual pair of CFTs.

The point of departure in the discussion of the anticipated relation between σ-model dualities and defect lines is the derivation of the canonical formalism for the σ-model. This goal is most straightforwardly accomplished within the framework of covariant classical field theory (or first-order formalism) of Refs. [45, 46, 47, 48, 49, 50], giving immediate access to the symplectic structure on the phase space of the field theory of interest (and so also to the Poisson algebra of Hamiltonian functionals). In the case of the (mono-phase) σ-model determined by the choice of the target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$, with its standard phase space

$$\mathsf{P}_{\sigma,\varnothing} = \mathsf{T}^*\mathsf{L}M \xrightarrow{\pi_{\mathsf{T}^*\mathsf{L}M}} \mathsf{L}M \equiv C^\infty\left(\mathbb{S}^1, M\right)\,,$$

it yields the symplectic form

$$\Omega_{\sigma,\emptyset} = \mathsf{d}\theta_{\mathsf{T}^*\mathsf{L}M} + \pi_{\mathsf{T}^*\mathsf{L}M}^* \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{curv}(\mathcal{G})\,,$$

written in terms of the canonical 1-form $\theta_{\mathsf{T}^*\mathsf{L}M} \in \Omega^1(\mathsf{T}^*\mathsf{L}M)$ and of the standard evaluation map $\mathrm{ev}\ :\ \mathsf{L}M \times \mathbb{S}^1 \to M$, *cf.* [29].

The significance of the last result rests upon the observation, originally due to Dirac, that the symplectic structure on the classical state space of the theory can be employed directly to canonically quantise the theory by taking as its Hilbert space the space of suitably polarised sections of a line bundle over the phase space endowed with a Hermitian connection of curvature equal to the symplectic 2-form, *cf.* [2] and, *e.g.*, [3]. This so-called geometric quantisation of the classical theory was originally put to work in the context of the quantum mechanics of a charged point-like particle propagating in the electromagnetic field. However, it has since found highly nontrivial applications such as, *e.g.*, the celebrated Kähler quantisation of the three-dimensional Chern–Simons theory worked out by Witten *et al.* in [51]. In the setting of interest, Dirac's quantisation programme can be carried out, at least formally, due to the following remarkable feature of the gerbe (co)defining the classical σ-model, as first noted and subsequently exploited by Gawędzki.

Theorem 4 *[5] A gerbe $\mathcal{G}$ over M canonically induces a line bundle $\pi_{\mathcal{L}_\mathcal{G}}\ :\ \mathcal{L}_\mathcal{G} \to \mathsf{L}M$ over the free-loop space $\mathsf{L}M$, termed the transgression bundle, with connection $\nabla_{\mathcal{L}_\mathcal{G}}$ of curvature*

$$\mathrm{curv}(\nabla_{\mathcal{L}_\mathcal{G}}) = \int_{\mathbb{S}^1} \mathrm{ev}^*\mathrm{curv}(\mathcal{G})\,.$$

The assignment $\mathcal{G} \mapsto \mathcal{L}_\mathcal{G}$ gives rise to a cohomology map

$$\mathbb{H}^2\left(M, \mathcal{D}(2)^\bullet_M\right) \to \mathbb{H}^1\left(\mathsf{L}M, \mathcal{D}(1)^\bullet_{\mathsf{L}M}\right)\,,$$

termed the transgression map, between cohomology groups of which the latter consists of isomorphism classes of line bundles with connection over $\mathsf{L}M$.

Thus, the gerbe defines a pre-quantum bundle of the (mono-phase) σ-model[13]*,*

$$\pi_{\mathcal{L}_{\sigma,\emptyset}}\ :\ \mathcal{L}_{\sigma,\emptyset} := \pi_{\mathsf{T}^*\mathsf{L}M}^*\mathcal{L}_\mathcal{G} \otimes (\mathsf{T}^*\mathsf{L}M \times \mathbb{C}) \to \mathsf{T}^*\mathsf{L}M\,,$$

where the trivial factor $\mathsf{T}^*\mathsf{L}M \times \mathbb{C}$ is endowed with the global connection 1-form $\theta_{\mathsf{T}^*\mathsf{L}M}$.

[13] In Dirac's geometric quantisation, classical symmetries of the theory are required to lift to polarisation-preserving automorphisms of the pre-quantum bundle. It well deserves to be remarked that this postulate was explicitly realised in the setting of the WZW model by Gawędzki *et al.* in [52].

In the presence of the pre-quantum bundle, we may formalise the concept of a duality and, in this manner, set up the context for a canonical interpretation of a defect and of the attendant bi-brane $\mathcal{B}$.

Definition 5 *A pre-quantum duality of the σ-model with a (composite) target* $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ *is a graph* $\mathfrak{I}_\sigma \subset \mathsf{P}_{\sigma,\emptyset} \times \mathsf{P}_{\sigma,\emptyset} \equiv \mathsf{P}_{\sigma,\emptyset}^{\times 2}$ *isotropic with respect to the 'difference' symplectic form*

$$\Omega^-_{\sigma,\emptyset} := \mathrm{pr}_1^* \Omega_{\sigma,\emptyset} - \mathrm{pr}_2^* \Omega_{\sigma,\emptyset}\,,$$

having the property that the difference

$$\mathcal{H}^-_{\sigma,\emptyset} := \mathrm{pr}_1^* \mathcal{H}_{\sigma,\emptyset} - \mathrm{pr}_2^* \mathcal{H}_{\sigma,\emptyset}$$

of pullbacks of the Hamiltonian density $\mathcal{H}_{\sigma,\emptyset}$ *of the σ-model along the canonical projections* $\mathrm{pr}_i : \mathsf{P}_{\sigma,\emptyset}^{\times 2} \to \mathsf{P}_{\sigma,\emptyset}$ *vanishes identically on* $\mathfrak{I}_\sigma$*, and such that the symplectomorphism of* $\mathsf{P}_{\sigma,\emptyset}$ *thus defined by* $\mathfrak{I}_\sigma$ *lifts to a bundle isomorphism*

$$\mathfrak{D}_\sigma \;:\; \mathrm{pr}_1^* \mathcal{L}_{\sigma,\emptyset}|_{\mathfrak{I}_\sigma} \xrightarrow{\;\cong\;} \mathrm{pr}_2^* \mathcal{L}_{\sigma,\emptyset}|_{\mathfrak{I}_\sigma}\,.$$

✓

At this stage, it remains to identify the phase-space representation of the state identification across the world-sheet defect, a step that clearly calls for a dynamical input. Its source is the principle of least action for the σ-model in the presence of the circular defect under investigation. It yields, alongside the standard field equations for the two phases of the σ-model separated by the defect line, also a gluing condition for the momentum 1-form fields $\mathsf{p}_{|\alpha},\ \alpha \in \{1,2\}$ on either side of the defect line. The condition, dubbed the Defect-Gluing Condition (DGC) in [28], takes the form

$$\mathsf{p}_{|1} \circ \iota_{1*} - \mathsf{p}_{|2} \circ \iota_{2*} - X_* \widehat{t} \lrcorner\, \omega_X = 0\,,$$

expressed in terms of the vector field $\widehat{t}$ tangent to $\mathbb{S}^1$ (modelling the defect line) and the tangent maps $\iota_{\alpha *} : \Gamma(\mathsf{T}Q) \to \Gamma(\mathsf{T}M)$ and $X_* : \Gamma(\mathsf{T}\mathbb{S}^1) \to \Gamma(\mathsf{T}Q)$, the latter being associated to the defect loop (embedding) $X : \mathbb{S}^1 \to Q$. The DGC, in conjunction with the obvious relation

$$X_{|\alpha} = \iota_\alpha \circ X\,, \qquad X \in \mathsf{L}Q$$

between the two loops $X_{|\alpha} \in \mathsf{L}M$ put in correspondence by the circular defect (as being engendered by the single loop in Q introduced previously), defines a subset within $\mathsf{P}_{\sigma,\emptyset}^{\times 2}$ whose properties can be further examined by standard methods of symplectic geometry. The upshot is the sought-after correspondence

Theorem 6 *[29] A $\mathcal{G}$-bi-brane $\mathcal{B} = \big(Q, \iota_\alpha, \omega, \Phi \mid \alpha \in \{1,2\}\big)$ defines a pre-quantum duality of the σ-model with target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$ iff both induced loop maps $\widetilde{\iota}_\alpha : \mathsf{L}Q \to \mathsf{L}M : X \mapsto \iota_\alpha \circ X$ are surjective submersions satisfying a number of straightforward conditions, including a condition for their tangent maps which expresses the preservation, at the defect, of the component of the energy-momentum tensor of the σ-model generating those conformal transformations on the world-sheet which deform the defect. Defects of this kind are termed topological.*

The identification of the topological defects as the ones which correspond to dualities falls in perfect agreement with results of the formerly cited CFT analyses. Also the distinction of surjective submersions from among the possible bi-brane maps has found its interpretation in the context of the principle of categorial descent, to be encountered in the next section, *cf.* [53].

Given the above general result on the correspondence between world-sheet defects and σ-model dualities, it is only natural to reverse the question and look for dualities that canonically give rise to a bi-brane structure. This issue was addressed in [29], following the earlier discussion in Refs. [54, 55], with regard to a class of dualities engendered by symplectomorphisms with generating functionals *linear* in the canonical variables $(X_{\alpha *}\widehat{t}, \mathsf{p}_\alpha)$ — such symplectomorphisms are manifestly distinguished in view of the structure of $\mathcal{H}^-_{\sigma,\varnothing}$, quadratic in these variables. They cover a wide range of important dualities, including isometric symmetries of the σ-model as well as the so-called T-duality of Refs. [56, 57], currently under investigation in the framework of gerbe theory, *cf.*, *e.g.*, Refs. [43, 58]. The upshot of the canonical analysis performed in [29], in which the notion of the gerbe-induced pre-quantum bundle of the σ-model played a key rôle, was the identification of two important classes of dualities, assigned the respective types T and N, that induce a bi-brane structure in the Cartesian square $M \times M$ of the target space of the σ-model. We have

Statement 7 *For every pre-quantum duality of the linear type T of the σ-model with target $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$, there exists a topological defect and a $\mathcal{G}$-bi-brane with a world-volume $Q \subset M \times M$ endowed with a symplectic 2-form induced by its curvature ω. The latter satisfies, together with the pullback of the metric g along the bi-brane maps $\iota_\alpha \equiv \mathrm{pr}_\alpha : Q \to M$ (given by the canonical projections restricted to $Q \subset M \times M$), duality-background constraints akin to the Buscher rules of Refs. [56, 57].*

Similarly, for every pre-quantum duality of the linear type N of the said σ-model, there exists a topological defect and a $\mathcal{G}$-bi-brane of a vanishing curvature, with a world-volume $Q = (\mathrm{id}_M \times F)(M) \subset M \times M$ determined by a smooth isometry $F : M \to M$ of the metric manifold (M, g). The isometry preserves $\mathcal{G}$ in the sense of a 1-isomorphism $\mathcal{G} \xrightarrow{\cong} F^*\mathcal{G}$.

The discussion of the concept of a duality in the framework of the gerbe theory of the σ-model demonstrates, once again, the adequacy and the power of the geometric and cohomological methods which, while themselves safely based in the smooth category (from which the relevant structures $\mathcal{G}$ and $\mathcal{B}$ come), transcend the perimeter of the classical description.

The scope of their application is by no means limited to the study of dualities. Another aspect of the loop mechanics encompassed by the gerbe-theoretic framework advocated herein is the canonical picture of the purely geometric loop interaction via splitting and joining, sometimes called loop fusion. An elementary two-to-one process of this sort is shown in Fig. 2 and the corresponding world-sheet geometry goes under the name of a trinion, or of a 'pair of pants'. This depiction of the physical process immediately suggests where to look for its canonical signature. In analogy with the previously discussed case of a state correspondence realised by a defect line, the trinion interaction is expected to define an isotropic submanifold within the triple Cartesian product of the phase space $\mathsf{P}_{\sigma,\varnothing}$, composed of triples of loops with momentum, subject to gluing conditions that capture the details of the loop fusion (such as, *e.g.*, the (dis)continuity of the embedding fields and their tangent maps). Equipped with the transgression map, we anticipate, moreover, the existence of an isomorphism between the tensor product of pre-quantum bundles associated with the in-coming states and the one for the out-going state. The idea laid out above is, in fact, a (simplified) restatement of some basic elements of Segal's categorial quantisation scheme for two-dimensional CFT, advanced in [22] (*cf.* also Refs. [30, 59] for a lucid account), in which Hilbert spaces would be functorially assigned to connected components of the boundary of the world-sheet representing the in-coming and out-going loops under fusion, and linear operators acting between tensor products of the in-coming Hilbert spaces and those of the out-going Hilbert spaces would be associated to the two-dimensional world-sheet cobordism. That the surface transport operators (obtained from the previously discussed differential characters by pulling the target-space gerbe back to a proper cobordism, with $\partial\Sigma$ given by a disjoint union of circles) realise this scheme in the defect-free setting jointly with the transgression map has been known since Gawędzki's seminal paper [5] and its counterpart [17] for the case of an embedded boundary defect (or, equivalently, an open world-sheet). Seen from this perspective, the findings of the recent survey [29] form a logical completion of the former analyses, extending them to arbitrary (decorated) world-sheets. Thus, for the trans-defect fusion of untwisted loops, representing states from various phases of the σ-model CFT that coalesce at a defect embedded in the interaction geometry, we find

Theorem 8 *[29] Let $\mathfrak{J}(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})$ be the subset in $\mathsf{P}_{\sigma,\varnothing}^{\times 3}$ composed of triples of loops with momentum in the target $\mathcal{M}=(M,\mathrm{g},\mathcal{G})$, subject to relations (of the DGC and gluing type) determined by the data of the $\mathcal{G}$-bi-brane $\mathcal{B}$ and those of the $(\mathcal{G},\mathcal{B})$-inter-bi-brane $\mathcal{J}$ and imposed in a (self-evident) manner that imitates the imposition of pair-wise DGCs along the three defect lines $\ell_{1,2},\ell_{2,3}$ and $\ell_{1,3}$ in Fig. 2, and of the additional gluing constraint[14] at the two defect junctions $\jmath$ and $\jmath^\vee$ in the same figure. The subset is isotropic with respect to the symplectic form*

$$\Omega_{\sigma,\varnothing}^{+-}:=\mathrm{pr}_1^*\Omega_{\sigma,\varnothing}+\mathrm{pr}_2^*\Omega_{\sigma,\varnothing}-\mathrm{pr}_3^*\Omega_{\sigma,\varnothing}$$

on $\mathsf{P}_{\sigma,\varnothing}^{\times 3}$, defined in terms of the canonical projections $\mathrm{pr}_i\ :\ \mathsf{P}_{\sigma,\varnothing}^{\times 3}\to\mathsf{P}_{\sigma,\varnothing}$, $i\in\{1,2,3\}$. Furthermore, the background $\mathfrak{B}=(\mathcal{M},\mathcal{B},\mathcal{J})$ canonically induces a bundle isomorphism

$$\mathfrak{J}_{\sigma,(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}\ :\ (\mathrm{pr}_1^*\mathcal{L}_{\sigma,\varnothing}\otimes\mathrm{pr}_2^*\mathcal{L}_{\sigma,\varnothing})\,|_{\mathfrak{J}(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}\xrightarrow{\cong}\mathrm{pr}_3^*\mathcal{L}_{\sigma,\varnothing}|_{\mathfrak{J}(\circledast\mathcal{B}:\mathcal{J}:\mathcal{B})}\,.$$

The universal character of the previous findings is confirmed by the inspection of the so-called N-twisted sector $\mathsf{P}_{\sigma,\mathcal{B};N}$ of the σ-model on a world-sheet with time-like defect lines, of prime relevance to the study of orbifold target spaces, as in Refs. [60, 61]. States from $\mathsf{P}_{\sigma,\mathcal{B};N}$ are piece-wise smooth loops with momentum in the target $\mathcal{M}=(M,\mathrm{g},\mathcal{G})$. The finite number N of discontinuities of the loop embedding map and its momentum field model discrete jumps of the Cauchy data of classical field configurations on a cylindrical world-sheet with N time-like defect lines, the data being localised at equitemporal slices transversally intersecting the defect lines[15]. Such Cauchy data can be compactly encoded by an $(N+2)$-tuple $(X,\mathsf{p},q_k\ |\ k\in\overline{1,N})$ in which $X\ :\ \mathbb{S}^1\to M$ is a piece-wise smooth embedding map, and p is a piece-wise smooth normal 1-form field on its image, both with N jumps determined (via the maps ι_α and a suitable DGC) in terms of the N points $q_k\in Q$ in the world-volume of the $\mathcal{G}$-bi-brane associated with the defect that sets the N-fold twist of the sector. As in the untwisted setting, the first-order formalism gives us a symplectic 2-form on the space of N-twisted states, to wit,

$$\Omega_{\sigma,\mathcal{B};N}[(X,\mathsf{p},q_k)]=\int_0^{2\pi}\mathsf{d}\varphi\ (\mathsf{dp}_\mu\wedge\mathsf{d}X^\mu)+\int_0^{2\pi}\mathsf{d}\varphi\ (X_*\partial_\varphi\lrcorner\mathrm{H}_X)+\sum_{k=1}^N\omega(q_k)\,,$$

cf. [29], and we readily establish

[14] The gluing constraint ensures that the endpoints of all the half-loops related by the piece-wise DGCs and joining at a given defect junction descend from a single point in the world-volume of $\mathcal{J}$.

[15] Hence, the discontinuities are determined by a variant of the DGC.

Theorem 9 *[29] The pair* $(\mathcal{G},\mathcal{B})$ *canonically induces a line bundle* $\mathcal{L}_{(\mathcal{G},\mathcal{B});N} \to \mathsf{L}_{\mathcal{B};N}M$ *over the* N*-twisted loop space* $\mathsf{L}_{\mathcal{B};N}M$ *with (local) coordinates* $(X, q_k \mid k \in \overline{1,N})$ *as above. The bundle is equipped with a connection* $\nabla_{\mathcal{L}_{(\mathcal{G},\mathcal{B});N}}$ *of curvature*

$$\mathrm{curv}(\nabla_{\mathcal{L}_{(\mathcal{G},\mathcal{B});N}})[(X,q_k)] = \int_0^{2\pi} \mathsf{d}\varphi\ (X_*\partial_\varphi \lrcorner \mathrm{H}_X) + \sum_{k=1}^{N} \omega(q_k)\,.$$

Under the assignment $(\mathcal{G},\mathcal{B}) \mapsto \mathcal{L}_{(\mathcal{G},\mathcal{B});N}$*, (cohomological) equivalence classes of pairs* $(\mathcal{G},\mathcal{B})$ *are mapped into isomorphism classes of bundles with connection.*

The gerbe together with the bi-brane define a pre-quantum bundle of the N*-twisted sector of the σ-model,*

$$\mathcal{L}_{\sigma,\mathcal{B};N} := \pi^*_{\mathsf{P}_{\sigma,\mathcal{B};N}} \mathcal{L}_{(\mathcal{G},\mathcal{B});N} \otimes (\mathsf{P}_{\sigma,\mathcal{B};N} \times \mathbb{C}) \to \mathsf{P}_{\sigma,\mathcal{B};N}\,,$$

where the trivial factor is endowed with the global connection 1-form given by the first integral factor in the definition of $\Omega_{\sigma,\mathcal{B};N}$*, and where* $\pi_{\mathsf{P}_{\sigma,\mathcal{B};N}} : \mathsf{P}_{\sigma,\mathcal{B};N} \to \mathsf{L}_{\mathcal{B};N}M$ *denotes the canonical projection.*

The last result enables us to extend our canonical analysis of the splitting-joining interaction to the twisted sector of the σ-model and thus confirms the general applicability of the proposed mode of description of the loop dynamics. This leads to an important aspect of the theory of defects that sets it apart from, *e.g.*, the theory of conformal boundary conditions[16] and boundary CFTs, namely, the fusion of defects. Restricting our attention to the 1-twisted sector for the sake of simplicity and concreteness, we find

Theorem 10 *[29] Let* $\mathfrak{I}(\circledast\mathcal{B}_{\mathrm{triv}} : \mathcal{J} : \mathcal{B}_{\mathrm{triv}})$ *be the subset in* $\mathsf{P}^{\times 3}_{\sigma,\mathcal{B};1}$ *composed of triples of 1-twisted loops with momentum in the target* $\mathcal{M} = (M, \mathrm{g}, \mathcal{G})$*, subject to relations (of the DGC and gluing type) determined by the data of the trivial* $\mathcal{G}$*-bi-brane* $\mathcal{B}_{\mathrm{triv}} = (M, \mathrm{id}_M, \mathrm{id}_M, 0, \mathrm{id}_{\mathcal{G}})$ *(associated with the distinguished identity (trivial) defect) and those of the* $(\mathcal{G},\mathcal{B})$*-inter-bi-brane* $\mathcal{J}$ *and imposed in a (self-evident) manner that imitates the imposition of (trivial) pair-wise DGCs along the three (invisible) defect lines* $\ell_{1,2}, \ell_{2,3}$ *and* $\ell_{1,3}$ *in Fig. 4, and of the additional gluing constraint[17] at the defect junctions* $\jmath$ *in the same figure. The subset is isotropic with respect to the symplectic form*

$$\Omega^{+-}_{\sigma,\mathcal{B};1} := \mathrm{pr}_1^*\Omega_{\sigma,\mathcal{B};1} + \mathrm{pr}_2^*\Omega_{\sigma,\mathcal{B};1} - \mathrm{pr}_3^*\Omega_{\sigma,\mathcal{B};1}$$

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[16] As observed in [62], non-intersecting circular defects can be understood as world-sheet boundaries in the so-called folded σ-model.

[17] The gluing constraint ensures that the jump coordinates $q_i \in Q$ of the three 1-twisted loops under fusion descend from a single point in the world-volume of $\mathcal{J}$, to which the defect junction $\jmath$ is mapped.

on $\mathsf{P}^{\times 3}_{\sigma,\mathcal{B};1}$, *defined in terms of the canonical projections* $\mathrm{pr}_i : \mathsf{P}^{\times 3}_{\sigma,\mathcal{B};1} \to \mathsf{P}_{\sigma,\mathcal{B};1}$, $i \in \{1,2,3\}$. *Furthermore, the background* $\mathfrak{B} = (\mathcal{M},\mathcal{B},\mathcal{J})$ *canonically induces a bundle isomorphism*

$$\begin{aligned}\mathfrak{J}^{\mathcal{B}}_{\sigma,(\circledast\mathcal{B}_{\mathrm{triv}}:\mathcal{J}:\mathcal{B}_{\mathrm{triv}})} &: \left(\mathrm{pr}_1^*\mathcal{L}_{\sigma,\mathcal{B};1} \otimes \mathrm{pr}_2^*\mathcal{L}_{\sigma,\mathcal{B};1}\right)|_{\mathfrak{I}(\circledast\mathcal{B}_{\mathrm{triv}}:\mathcal{J}:\mathcal{B}_{\mathrm{triv}})} \\ &\xrightarrow{\cong} \mathrm{pr}_3^*\mathcal{L}_{\sigma,\mathcal{B}}|_{\mathfrak{I}(\circledast\mathcal{B}_{\mathrm{triv}}:\mathcal{J}:\mathcal{B}_{\mathrm{triv}})} .\end{aligned} \tag{3.1}$$

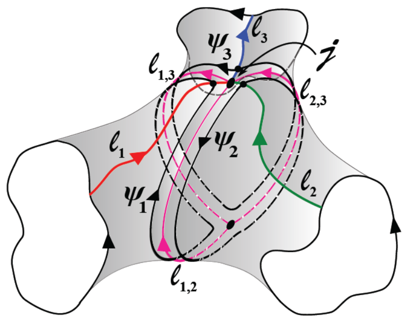

Fig. 4. The basic splitting–joining interaction between 1-twisted states ψ_1, ψ_2 and ψ_3 (drawn in black, the momentum field implicit) across the defect graph composed of the identity-defect lines $\ell_{1,2}, \ell_{2,3}$ and $\ell_{1,3}$ carrying the data of $\mathcal{B}_{\mathrm{triv}}$. The defect lines ℓ_i, $i \in \{1,2,3\}$, defining the $\mathcal{B}$-twist of the respective states ψ_i, join at the defect junction $\jmath$ embedded in the world-volume of $\mathcal{J}$.

The trinion vertex of the 1-twisted sector of the σ-model becomes interesting in its own right as it captures the structure of the fusion ring on the set of (equivalence classes of) conformal defects, which goes beyond their mere set-theoretic enumeration offered by the investigation of defect graphs composed of non-intersecting circular defect lines. Thus, the hands-on reconstruction of the vertex, based on the explicit local presentation (in terms of local data of the background $(\mathcal{M},\mathcal{B},\mathcal{J})$) of the relevant pre-quantum bundle given in [29], is expected to provide us with novel insights into the algebra of σ-model dualities. In the next section of this note, we substantiate this claim by reviewing some recent advances in the gerbe-theoretic study of the distinguished target-preserving dualities of the σ-model, *i.e.* its symmetries, and their gauging.

4. Groupoidal internal symmetries and gauging

An in-depth understanding of the symmetry content of the physical model has its obvious merits: It introduces the potent tools of representation theory and the theory of invariants into the arsenal of field-theoretic methods, thus bringing natural structure into the state space of the model and organising the derivation of its correlation functions. This is particularly eminent in two-dimensional CFT, with its infinite-dimensional loop-algebra extensions of the fundamental Virasoro algebra of the (local) conformal symmetry, obtained from a current realisation of internal (rigid) symmetries. But the rôle of the symmetry principle does not end there. Upon augmentation to the form of the gauge principle, it opens avenues for a systematic construction of new models enjoying a local version of the former rigid symmetry. All these generic aspects of field-theoretic symmetries find their manifestation in the gerbe theory of the σ-model. The theory also brings to the fore the particularities of the latter that stem from the existence of the smooth geometric structure on its covariant configuration bundle, such as, *e.g.*, the generalised-geometric (in the infinitesimal picture) and groupoidal (in the integrated, global picture) nature of its internal symmetries. The lessons drawn from a thorough study of the σ-model in the context of the gauge principle can subsequently be used towards demystification of the intricate non-geometric backgrounds of loop dynamics.

It is customary in field theory to first look — in the spirit of Noether's original argument — for conditions of invariance of the action functional under infinitesimal rigid field transformations, and only later address the issue of their integrability to global symmetries of the model. Applying this strategy to the two-dimensional σ-model on a world-sheet with a generic embedded defect graph, and hence also a generic background $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$, we readily establish that infinitesimal (internal) symmetries of the action functional correspond to certain smooth sections of the disjoint union of generalised tangent bundles

$$\mathsf{E}\mathscr{F} := (\mathsf{T}M \oplus \mathsf{T}^*M) \sqcup (\mathsf{T}Q \oplus (\mathbb{R} \times Q)) \sqcup \mathsf{T}T \equiv \mathsf{E}^{(1,1)}M \sqcup \mathsf{E}^{(1,0)}Q \sqcup \mathsf{T}T$$

over the fibre $\mathscr{F} = M \sqcup Q \sqcup T$ of the covariant configuration bundle of the σ-model, *cf.* Refs. [16, 33]. The appearance of the standard generalised tangent bundle $\mathsf{E}^{(1,1)}M$ and of the attendant generalised geometry *à la* Hitchin and Gualtieri of Refs. [63, 64] in the setting of a defect-free σ-model was first noted and discussed at length in the important paper [65] by Alekseev and Strobl[18]. In the presence of the full-blown 2-categorial content of $\mathfrak{B}$ over the world-sheet, the relevant sections of $\mathsf{E}\mathscr{F}$ are triples $\mathfrak{K} = ({}^M\mathscr{K} \oplus \kappa, {}^Q\mathscr{K} \oplus k, {}^{T_n}\mathscr{K})$ composed each of a vector field $\mathscr{K} \in \Gamma(\mathsf{T}\mathscr{F})$ with restrictions ${}^{\mathscr{M}}\mathscr{K}$ to $\mathscr{M} \in \{M, Q, T_n\}$, of which the first is Killing[19] with respect to g,

$$\mathscr{L}_{{}^M\mathscr{K}}\,\mathrm{g} = 0\,,$$

[18] It is perhaps worth noting that the emergence of a generalised tangent bundle in the description of a system coupled to an external gauge field is by no means peculiar to the two-dimensional setting. Indeed, infinitesimal symmetries of the action functional of a charged point-like particle in an external electromagnetic field are described by smooth sections of the generalised tangent bundle $\mathsf{E}^{(1,0)}M$.

[19] Here, and in what follows, $\mathscr{L}$ denotes the Lie derivative.

and subject to the alignment constraints

$$ {}^M\mathcal{K}\big|_{\iota_\alpha(Q)} = \iota_{\alpha *}\big({}^Q\mathcal{K}\big)\,, \qquad {}^Q\mathcal{K}\big|_{\pi_n^{k,k+1}(T_n)} = \pi_{n*}^{k,k+1}\big({}^{T_n}\mathcal{K}\big)\,, $$

of a 1-form field $\kappa \in \Omega^1(M)$ satisfying the consistency condition

$$ \mathsf{d}\kappa + {}^M\mathcal{K} \lrcorner \mathrm{H} = 0\,, $$

and of a smooth function $k \in C^\infty(Q, \mathbb{R})$ that obeys the relations

$$ \mathsf{d}k + {}^Q\mathcal{K} \lrcorner \omega + \Delta_Q \kappa = 0\,, \qquad \Delta_{T_n} k = 0\,, $$

written in terms of the pullback operators $\Delta_Q = \iota_2^* - \iota_1^*$ and $\Delta_{T_n} = \sum_{k=1}^n \pi_n^{k,k+1*}$. The more immediate physical interpretation of the geometric objects $\mathfrak{K}$ is revealed by considering their canonical lifts to the phase space[20] P_σ of the σ-model and to the pre-quantum bundle $\mathcal{L}_\sigma \to \mathsf{P}_\sigma$. We thus find that — as shown in [33] — to each such section of $\mathsf{E}\mathcal{F}$ there corresponds a Hamiltonian functional resp. a linear (first-order differential) operator on the space of sections of the pre-quantum bundle, generating, in the standard manner, an infinitesimal symmetry transformation on the (classical resp. quantum) state space of the model, *i.e.* we have assignments

$$ \begin{aligned} \Gamma_\sigma(\mathsf{E}\mathcal{F}) \to C^\infty(\mathsf{P}_\sigma, \mathbb{R}) \quad &: \quad \mathfrak{K} \mapsto h_{\mathfrak{K}}\,, \\ \Gamma_\sigma(\mathsf{E}\mathcal{F}) \to \mathrm{End}\,(\Gamma(\mathcal{L}_\sigma)) \quad &: \quad \mathfrak{K} \mapsto \widehat{\mathcal{O}}_{h_{\mathfrak{K}}}\,, \end{aligned} \tag{4.1} $$

related to one another by the canonical quantisation map, *cf.* [3].

The set $\Gamma_\sigma(\mathsf{E}\mathcal{F})$ of all sections of $\mathsf{E}\mathcal{F}$ of the type described above — call them σ-symmetric — comes with an internal bilinear antisymmetric operation, or a bracket, in analogy with the Lie bracket $[{}^M\mathcal{K}_1, {}^M\mathcal{K}_2]$ of Killing vector fields ${}^M\mathcal{K}_i,\ i \in \{1,2\}$ that describe infinitesimal symmetries of the action functional of a point-like particle propagating in an ambient metric space (M, g). The bracket was found in [33] and can be written for $\mathfrak{K}_i = ({}^M\mathcal{K}_i \oplus \kappa_i, {}^Q\mathcal{K}_i \oplus k_i, {}^{T_n}\mathcal{K}_i) \in \Gamma_\sigma(\mathsf{E}\mathcal{F}),\ i \in \{1,2\}$ in terms of its restrictions

$$ \begin{aligned} [\![\mathfrak{K}_1, \mathfrak{K}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_M &= [{}^M\mathcal{K}_1 \oplus \kappa_1, {}^M\mathcal{K}_2 \oplus \kappa_2]^{\mathrm{H}}_{\mathrm{C}}\,, \\ [\![\mathfrak{K}_1, \mathfrak{K}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_Q &= [{}^Q\mathcal{K}_1, {}^Q\mathcal{K}_2] \oplus \big({}^Q\mathcal{K}_1 \lrcorner \mathsf{d}k_2 - {}^Q\mathcal{K}_2 \lrcorner \mathsf{d}k_1 + {}^Q\mathcal{K}_1 \lrcorner {}^Q\mathcal{K}_2 \lrcorner \omega \\ &\quad + \tfrac{1}{2}\,({}^Q\mathcal{K}_1 \lrcorner \Delta_Q \kappa_2 - {}^Q\mathcal{K}_2 \lrcorner \Delta_Q \kappa_1)\big)\,, \\ [\![\mathfrak{K}_1, \mathfrak{K}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}|_{T_n} &= [{}^{T_n}\mathcal{K}_1, {}^{T_n}\mathcal{K}_2] \end{aligned} $$

[20] In this section, we deliberately avoid specifying the sector of the state space.

to the components of the extended target space. The first of them is determined by the Courant bracket of [66] twisted — in the sense of [67] — by the gerbe curvature 3-form,

$$[{}^M\mathscr{K}_1 \oplus \kappa_1, {}^M\mathscr{K}_2 \oplus \kappa_2]_{\rm C}^{\rm H} := [{}^M\mathscr{K}_1, {}^M\mathscr{K}_2] \oplus \big(\mathscr{L}_{{}^M\mathscr{K}_1}\kappa_2 - \mathscr{L}_{{}^M\mathscr{K}_2}\kappa_1 - \tfrac{1}{2}\,\mathsf{d}\big({}^M\mathscr{K}_1 \lrcorner \kappa_2 - {}^M\mathscr{K}_2 \lrcorner \kappa_1\big) + {}^M\mathscr{K}_1 \lrcorner {}^M\mathscr{K}_2 \lrcorner {\rm H}\big)\,.$$

Its relevance to the two-dimensional field theory in hand was noted already in [65], and the relation to gerbes, based on the notion of a twisted generalised tangent bundle of [63], leads to the following conclusion: The canonical (twisted) Courant algebroid $(\mathsf{E}^{(1,1)}M, [\cdot,\cdot]_{\rm C}^{\rm H}, \alpha_{\mathsf{T}M})$ of Refs. [66, 68, 69] (with the additional ingredient — the anchor map $\alpha_{\mathsf{T}M} : \mathsf{E}^{(1,1)}M \to \mathsf{T}M$ — given by the canonical projection) is a natural generalisation of the standard structure $(\mathsf{T}M, [\cdot,\cdot])$ in the presence of a gerbe over M. Its naturality rests on the fact that it accommodates (in a 'minimal' fashion) infinitesimal automorphisms of the target $(M, {\rm g}, \mathcal{G})$ just as the Lie algebra of Killing vector fields accommodates infinitesimal isometries of $(M, {\rm g})$. When viewed from this perspective, the complete structure $\big(\Gamma_\sigma(\mathsf{E}\mathscr{F}), [\![\cdot,\cdot]\!]^{({\rm H},\omega;\Delta_Q)}, \alpha_{\mathsf{T}\mathscr{F}}\big)$ (with the self-explanatory definition of the anchor $\alpha_{\mathsf{T}\mathscr{F}} : \mathsf{E}\mathscr{F} \to \mathsf{T}\mathscr{F}$), dubbed the $({\rm H},\omega;\Delta_Q)$-twisted bracket structure on $\Gamma_\sigma(\mathsf{E}\mathscr{F})$ in [33], can be understood as a natural algebraic structure over the extended target $\mathscr{F}$ in the presence of the full 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$ that captures automorphisms of the latter and thus generalises the Lie algebra of infinitesimal isometries of the target space. This interpretation is further substantiated by an exhaustive study of the rôle of local (hypercohomological) data of $(\mathcal{G}, \mathcal{B}, \mathcal{J})$ in the Hitchin-dual (understood in the sense of [63]) description of the $({\rm H},\omega;\Delta_Q)$-twisted bracket structure on $\Gamma_\sigma(\mathsf{E}\mathscr{F})$, reported in [33]. As an additional consistency check, one verifies that $\mathcal{G}$-bi-brane data (global or local, depending on the mode of description of $\mathsf{E}^{(1,1)}M$) canonically define a morphism between the (twisted) Courant algebroids associated with the two phases of the σ-model separated by the defect carrying the data of the $\mathcal{G}$-bi-brane.

Aided by the intrinsically geometric nature of the σ-model, we lift the algebraic structure on $\Gamma_\sigma(\mathsf{E}\mathscr{F})$ to the state space of the theory, to wit,

Theorem 11 *[33] The $({\rm H},\omega;\Delta_Q)$-twisted bracket structure $(\Gamma_\sigma(\mathsf{E}\mathscr{F}), [\![\cdot,\cdot]\!]^{({\rm H},\omega;\Delta_Q)}, \alpha_{\mathsf{T}\mathscr{F}})$ induced over the extended target space $\mathscr{F}$ of the σ-model in the presence of the 2-category $\mathfrak{BGrb}^\nabla(\mathscr{F})$ is homomorphically mapped, via the canonical lift of the tensor fields, to the Poisson subalgebra, within $(C^\infty(\mathsf{P}_\sigma, \mathbb{R}), \{\cdot,\cdot\}_{\Omega_\sigma})$, of Noether charges associated with infinitesimal symmetries engendered by elements of $\Gamma_\sigma(\mathsf{E}\mathscr{F})$, i.e. we have, for any two σ-symmetric sections $\mathfrak{K}_i$, $i \in \{1,2\}$ and for the corresponding Noether charges $h_{\mathfrak{K}_i}$,*

$$\{h_{\mathfrak{K}_1}, h_{\mathfrak{K}_2}\}_{\Omega_\sigma} = h_{[\![\mathfrak{K}_1,\mathfrak{K}_2]\!]^{({\rm H},\omega;\Delta_Q)}}\,.$$

It can be further lifted homomorphically to the commutator subalgebra, within $(\mathrm{End}(\Gamma(\mathcal{L}_\sigma)), [\cdot,\cdot])$, *of the corresponding charge operators, whereby we obtain, for the charge operators* $\widehat{\mathcal{O}}_{h_{\mathfrak{K}_i}}$ *associated to the* $h_{\mathfrak{K}_i}$,

$$\left[\widehat{\mathcal{O}}_{h_{\mathfrak{K}_1}}, \widehat{\mathcal{O}}_{h_{\mathfrak{K}_2}}\right] = \widehat{\mathcal{O}}_{h_{[\![\mathfrak{K}_1,\mathfrak{K}_2]\!]^{(\mathrm{H},\omega;\Delta_Q)}}} \,.$$

The canonical interpretation carries over to the interaction picture in which the data of the background $\mathfrak{B}$, upon pullback to an arbitrary defect graph embedded in the world-sheet interaction geometry, are seen to give rise to an intertwiner between representations, furnished — respectively — by the in-coming and out-going state spaces under fusion, of the $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket algebra of infinitesimal symmetries. In other words, the symplectomorphism defined by the isotropic interaction subspace within the Cartesian product of phase spaces under fusion transforms the Noether charges acting on the in-coming states into their counterparts acting on the out-going states, and an analogous statement holds for the associated bundle map that relates sections of the product pre-quantum bundles restricted to the interaction subspace. An important example of the situation just described is the multi-phase WZW σ-model with the maximally symmetric defects of [27], transmissive to the full bi-chiral Kac–Moody symmetry algebra of the defect-free model. The complete structure of the relevant background, required to define the model in the presence of a self-intersecting defect graph, was discussed at length in [32] (*cf.* also the earlier paper [31] that deals with the case of the target group $\mathrm{SU}(2)$).

The infinitesimal action of the Lie algebra of the Killing vector fields on a given metric manifold integrates, at least locally, to the action of a (local) isometry group. Our hitherto findings prompt a natural question as to the integrability of the $(\mathrm{H},\omega;\Delta_Q)$-twisted bracket structure on $\Gamma_\sigma(\mathsf{E}\mathscr{F})$. On the present level of generality, the question is very hard to answer, *cf.*, *e.g.*, [70] for a discussion of this issue in the 'tamer' category of Lie algebroids. We may, on the other hand, specialise to circumstances of immediate physical relevance. Thus, whenever we can choose a basis $\{\mathfrak{K}_a\}_{a\in\overline{1,D}}$ in the $\mathbb{R}$-linear span of σ-symmetric sections of $\mathsf{E}^{(1,1)}M$ in which the (untwisted) Courant bracket assumes the simple form

$$[\mathfrak{K}_a, \mathfrak{K}_b]_{\mathrm{C}} = f_{abc}\,\mathfrak{K}_c$$

with f_{abc} the structure constants of the Lie subalgebra $\mathfrak{g}$ spanned by the corresponding Killing vector fields ${}^M\mathscr{K}_a \equiv \alpha_{\mathsf{T}M}(\mathfrak{K}_a)$, the integrated structure for the thus defined Lie algebroid $\mathfrak{g}\ltimes M$, known as the action algebroid, is that of the action groupoid $\mathrm{G}\ltimes M$ for G obtained through the exponentiation of the Lie algebra $\mathfrak{g}$, *cf.* [71], as well as Refs. [16, 33] for an adaptation of the general framework to the context of interest. The groupoid is just the small category with object and morphism sets

$$\mathrm{Ob}(\mathrm{G}\ltimes M) = M\,, \qquad \mathrm{Mor}(\mathrm{G}\ltimes M) = \mathrm{G}\times M\,,$$

with the unit map $\mathrm{id}_m = (e, m)$, with the source map $s(g, m) = m$ and the target map $t(g, m) = \ell_g(m) \equiv g.m$ determined by a (left) action $\ell. : \mathrm{G}\times M \to M$ (whence its name), and with the inverse map $(g, m) \mapsto (g, m)^{-1} = (g^{-1}, g.m)$. Finally, it is equipped with the composition $(h, g.m) \circ (g, m) = (h \cdot g, m)$ that encodes the group operation in G. It is an example of a Lie groupoid, with all structure maps smooth and the source and target maps given by surjective submersions. The action algebroid is the so-called tangent Lie algebroid of $\mathrm{G}\ltimes M$. Returning to the original problem, it seems natural to restrict our attention to a situation in which there exists a basis $\{\mathfrak{K}_a\}_{a\in\overline{1,D}}$ in $\Gamma_\sigma(\mathsf{E}\mathscr{F})$ with the property

$$[\![\mathfrak{K}_a, \mathfrak{K}_b]\!]^{(\mathrm{H},\omega;\Delta_Q)} = f_{abc}\,\mathfrak{K}_c\,,$$

so that the $(\mathrm{H}, \omega; \Delta_Q)$-twisted bracket structure can be homomorphically identified with the action algebroid $\mathfrak{g}\ltimes\mathscr{F}$. The significance of such a restriction follows from the analysis of the conditions of gaugeability of internal symmetries of the σ-model in the presence of defects, reported in [16] (*cf.* also [33]), to which we turn next.

One of the basic constructive methods of CFT (or any field theory, for that matter), and a key guiding principle in the exploration of the moduli space of consistent CFTs, is the Gauge Principle, which consists in promoting a rigid symmetry of the theory to the rank of a local one. In the case of a discrete symmetry, this boils down to identifying (locally) field configurations related by the action of the symmetry group and passing to a suitable quotient of the original state space, whereby a new class of states emerges, absent from the parent theory — the so-called twisted states. In the geometric setting of the σ-model, the nature of the twisted sector is most straightforward to elucidate: Twisted states are associated with those patch-wise smooth embeddings of the world-sheet in the parent target space whose discrete jumps, localised along arbitrarily dense networks of discontinuity (or defect) lines, are determined by the action of the discrete symmetry group, and hence become smooth in the quotient. The symmetry group is usually termed the orbifold group[21] in this context, *cf.* Refs. [60, 61]. In the case of a continuous symmetry group G, the situation is more subtle as the gauging entails the introduction of additional physical degrees of freedom

[21] Whenever a discrete group of internal symmetries is enhanced by the world-sheet parity group $\mathbb{Z}_2$, the ensuing product structure is referred to as the orientifold group, *cf.* Refs. [72, 73].

that mediate the symmetry, namely, those of the gauge field. The gauge field is a collection of locally smooth 1-forms[22] on the space-time Σ of the theory, induced (through local sections) from a connection on the principal G-bundle $\mathrm{G} \hookrightarrow \mathsf{P} \to \Sigma$, and the 'mediation' of the symmetry transformations is explicitly realised through the passage from the covariant configuration bundle $\Sigma \times \mathscr{F} \to \Sigma$ of the parent field theory, via its principal G-extension $\mathsf{P} \times \mathscr{F} \to \Sigma$, to the associated bundle $\mathsf{P} \times_{\mathrm{G}} \mathscr{F} \to \Sigma$. It is the smooth sections of the latter bundle, $\mathsf{P} \times_{\mathrm{G}} \mathscr{F} \cong \Sigma \times \mathscr{F}$, that are to be identified with fields of the gauged theory. In this language, the twisted sector is produced by topologically nontrivial principal G-bundles. Upon specialisation to the σ-model, we thus arrive at the chain of extended targets

$$\Sigma \times \mathscr{F} \longrightarrow \mathsf{P} \times \mathscr{F} \longrightarrow \mathsf{P} \times_{\mathrm{G}} \mathscr{F}$$

that have to be constructed prior to passing to the quotient geometry $\Sigma \times \mathscr{F}/\mathrm{G}$. The significance of topologically nontrivial gauge configurations in the construction of the quotient, or coset σ-model was first emphasised in [74].

Clearly, realising the purely geometric transition indicated above, in itself a fairly straightforward task, does not give the final answer to the field-theoretic problem in hand. The strategy of its resolution can be divided into the following three stages, which at the same time display the nature of the obstructions that can arise:

- finding a consistent coupling between the various components of the parent σ-model background and an arbitrary gauge field, with view to defining a suitable extension of the former over $\mathsf{P} \times \mathscr{F}$ — here, the standard minimal-coupling prescription fails in general, and so a more intrinsic principle has to be conceived;

- lifting the geometric action of the symmetry group from the base $\mathscr{F}$ of the parent background to the 2-categorial structure $(\mathcal{G}, \mathcal{B}, \mathcal{J})$ over it — this is tantamount to endowing $(\mathcal{G}, \mathcal{B}, \mathcal{J})$ with the so-called G-equivariant structure, a procedure that may turn out to be cohomologically obstructed;

- descending the coupled background–gauge-field complex to the associated bundle $\mathsf{P} \times_{\mathrm{G}} \mathscr{F} \cong \mathscr{F}$ and, subsequently, the whole σ-model to the coset $\mathscr{F}/\mathrm{G}$ — even in the case of a topologically trivial gauge field, this calls for the same structure on $(\mathcal{G}, \mathcal{B}, \mathcal{J})$ as in the previous point.

All these issues were examined in great detail in a series of papers by Gawędzki *et al.*, [15, 16, 18, 75, 76], and in the pioneering paper [19] on orientifold gerbe theory. In the remainder of this note, we review the main

[22] In general, they do not glue up to a globally defined 1-form. This happens solely for trivial principal G-bundles over the space-time.

results of Refs. [15, 16] as they are the first to discuss the gauging of continuous symmetries in the gerbe-theoretic language. The general methods developed therein readily encompass the simpler discrete cases. Moreover, the papers also contain a hands-on construction of the relevant objects in the setting of the WZW σ-model with a simple 1-connected Lie-group target G (and the non-anomalous Ad_G internal symmetry as the symmetry to be gauged).

The first stage of the gauging procedure is the simplest to accomplish, and hints can be extracted from the inspection of a G-invariant topologically trivial (tensorial) background and a topologically trivial (1-form) gauge field. In this distinguished case, the minimal-coupling prescription works, and can be employed to establish conditions of gaugeability of the infinitesimal variant of the symmetry, captured by the Lie algebra $\mathfrak{g}$ (with structure constants f_{abc}) of the symmetry group G. These are the conditions necessary and sufficient for the minimally coupled action functional of the σ-model to be invariant under the natural action of $\mathfrak{g}$. They take the form

$$\mathscr{L}_{{}^M\mathscr{K}_a}\kappa_b = f_{abc}\,\kappa_c\,, \qquad {}^M\mathscr{K}_a \lrcorner\, \kappa_b = -{}^M\mathscr{K}_b \lrcorner\, \kappa_a\,, \qquad \mathscr{L}_{{}^Q\mathscr{K}_a} k_b = f_{abc}\, k_c\,,$$

and are to be imposed on the smooth σ-symmetric sections $\mathfrak{K}_a,\ a \in \overline{1, \dim\mathfrak{g}}$ of $\mathsf{E}\mathscr{F}$ forming the basis of $\mathfrak{g}$, introduced in the context of integrability of infinitesimal rigid symmetries of the σ-model[23]. The first two of these conditions were obtained independently in Refs. [77, 78] in the setting of the defect-free σ-model, and the remaining one was derived in [79] for the special case of a boundary defect. They admit a straightforward interpretation in the framework of $\mathfrak{g}$-equivariant cohomology for $\mathscr{F}$ (*e.g.*, in the Cartan model thereof), *cf.* Refs. [79, 80, 81]. In generalised-geometric terms, they translate into the requirement that the $\mathbb{R}$-linear span of the $\mathfrak{K}_a$ form a Lie subalgebroid within $\big(\Gamma_\sigma(\mathsf{E}\mathscr{F}), [\![\cdot,\cdot]\!]^{(\mathrm{H},\omega;\Delta_Q)}, \alpha_{\mathsf{T}\mathscr{F}}\big)$ isomorphic with the action algebroid $\mathfrak{g}{\ltimes}\mathscr{F}$.

The main lesson that can be drawn from the analysis outlined contains an answer to the first point on our list: The imposition of the above gaugeability constraints ensures the invariance of the σ-model with an arbitrary (G-isometric) background $\mathfrak{B} = (\mathcal{M}, \mathcal{B}, \mathcal{J})$ under the infinitesimal transformations from $\mathfrak{g}$ iff the background is coupled to the topologically trivial gauge field[24] $A = A^a \otimes t_a \in \Omega^1(\Sigma) \otimes \mathfrak{g}$ as per

$$(\mathcal{M}, \mathcal{B}, \mathcal{J}) \mapsto (\mathcal{M}_A, \mathcal{B}_A, \mathcal{J}_A)\,,$$

where

$$\mathcal{M}_A := (\Sigma \times M, \mathrm{pr}_2^*\mathrm{g}, \mathrm{pr}_2^*\mathcal{G} \otimes I_{\rho_A})\,, \qquad \mathcal{B}_A := (\Gamma \times Q, \widetilde{\iota}_\alpha, \omega_A, \mathrm{pr}_2^*\Phi \otimes J_{\lambda_A})\,,$$

[23] The existence of the $\mathfrak{K}_a$ is taken as a prerequisite of the gauging procedure.

[24] The field is written in terms of the standard generators $t_a,\ a \in \overline{1, \dim\mathfrak{g}}$ of $\mathfrak{g}$, subject to the structure relations $[t_a, t_b] = f_{abc}\, t_c$.

$$\mathcal{J}_A := \Big(\mathfrak{V}_\Gamma \times T_n, \Big(\widetilde{\pi}_n^{k,k+1} \mid k \in \overline{1,n}\Big), \mathrm{pr}_2^*\varphi_n \mid n \in \mathbb{N}_{\geq 3}\Big) ,$$

with

$$\rho_A := \mathrm{pr}_2^*\kappa_a \wedge \mathrm{pr}_1^* A^a - \tfrac{1}{2}\,\mathrm{pr}_2^*\left({}^M\mathscr{K}_a \lrcorner\, \kappa_b\right)\,\mathrm{pr}_1^*\left(A^a \wedge A^b\right) ,$$

$$\omega_A := \mathrm{pr}_2^*\omega - \widetilde{\Delta}_Q \rho_A + \mathsf{d}\lambda_A , \qquad \lambda_A := -\mathrm{pr}_2^*(k_a)\,\mathrm{pr}_1^*(A^a)$$

and

$$\widetilde{\iota}_\alpha := (\mathrm{id}_\Gamma, \iota_\alpha) \;:\; \Gamma \times Q \to \Sigma \times M , \qquad \widetilde{\Delta}_Q := \widetilde{\iota}_2^* - \widetilde{\iota}_1^* ,$$

$$\widetilde{\pi}_n^{k,k+1} := \Big(\mathrm{id}_{\mathfrak{V}_\Gamma}, \pi_n^{k,k+1}\Big) \;:\; \mathfrak{V}_\Gamma \times T_n \to \Gamma \times Q ,$$

all written in terms of the canonical projections $\mathrm{pr}_1 \,:\, \Sigma \times \mathscr{F} \to \Sigma$ and $\mathrm{pr}_2 \,:\, \Sigma \times \mathscr{F} \to \mathscr{F}$ (restricted suitably), and for $\mathfrak{V}_\Gamma \subset \Gamma$ the set of vertices (*i.e.* junctions) of the embedded defect graph. Finally, J_{λ_A} is a trivial line bundle with the global connection 1-form λ_A. Extension to an arbitrary gauge field is now straightforward: It consists in replacing the covariant configuration bundle $\Sigma \times \mathscr{F}$ in the above formulæ with its extended version $\mathsf{P} \times \mathscr{F}$, the gauge field A with the pullback, along the canonical projection $\mathrm{pr}_1 \,:\, \mathsf{P} \times \mathscr{F} \to \mathsf{P}$ of the globally defined principal G-connection $\mathcal{A} = \mathcal{A}^a \otimes t_a \in \Omega^1(\mathsf{P}) \otimes \mathfrak{g}$, and the extended maps $\widetilde{\iota}_\alpha$ and $\widetilde{\pi}_n^{k,k+1}$ on $\Sigma \times \mathscr{F}$ by their counterparts defined on $\mathsf{P} \times \mathscr{F}$. This yields the desired principal G-extension of the parent background,

$$\widetilde{\mathfrak{B}}_{\mathcal{A}} \equiv \left(\widetilde{\mathcal{M}}_{\mathcal{A}}, \widetilde{\mathcal{B}}_{\mathcal{A}}, \widetilde{\mathcal{J}}_{\mathcal{A}}\right) ,$$

over the extended covariant configuration bundle $\mathsf{P} \times \mathscr{F}$.

The last Ansatz promotes us to the second stage in which the composite left G-action $\mathrm{G} \times (\mathsf{P} \times \mathscr{F}) \to (\mathsf{P} \times \mathscr{F}) \,:\, (p.g^{-1}, g.m)$, induced from the assumed left G-action on $\mathscr{F}$ and the canonical right G-action on P, is to be lifted to $\widetilde{\mathfrak{B}}_{\mathcal{A}}$. This is a *sine qua non* condition for a consistent descent from the extended covariant configuration bundle $\mathsf{P} \times \mathscr{F}$ to its quotient $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$. The condition can be met by endowing the parent background $\mathfrak{B}$ with the so-called G-equivariant structure. A detailed clarification of the latter concept falls beyond the scope of the present review, and so we shall content ourselves with its intuitive description. Thus, from the categorial vantage point, the said structure is a choice of distinguished 1- and 2-cells from the 2-category $\mathfrak{BGrb}^\nabla(\mathsf{N}^\bullet(\mathrm{G} \ltimes \mathscr{F}))$ of Abelian bundle gerbes with curving and Hermitian connection supported over the simplicial G-space $\mathsf{N}^\bullet(\mathrm{G} \ltimes \mathscr{F})$ given by the nerve of the small category $\mathrm{G} \ltimes \mathscr{F}$, understood in the sense of [82]. In short, the 1- and 2-cells define a self-coherent realisation of G on the $(\mathcal{G}, \Phi, \varphi_n)$ that is — on the one hand — compatible with the associative product structure on the set G, and — on the other hand — enforces the desired property of the (inter-)bi-brane, which is that of an 'intertwiner' between the said realisations on restrictions of the gerbe supported over neighbouring phases of the σ-model under consideration. From the cohomological vantage point, the G-equivariant structure is an extension of the Čech–Deligne data representing $(\mathcal{G}, \mathcal{B}, \mathcal{J})$ to a cochain from the tricomplex obtained from the Čech–Deligne bicomplex through a standard extension in the direction of G-cohomology (which requires working with certain natural simplicial refinements of the open covers of $\mathscr{F}$ entering the definition of the original Čech–Deligne data, *cf.* Refs. [16, 83]). The cohomological perspective is particularly suitable for the discussion of obstructions against the existence of a G-equivariant structure and for the enumeration of inequivalent such structures. An extensive treatment of this classificatory issue is presented in Refs. [15, 16]. In summary, we stress the crucial feature of the structure which is that it determines the sought-after lift of the action of the symmetry group to the extended background $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ obtained earlier. An alternative and independent justification for its introduction comes from the analysis of the descent of the σ-model coupled to a topologically trivial gauge field from $\mathsf{P} \times_{\mathrm{G}} \mathscr{F} \cong \Sigma \times \mathscr{F}$ to the coset space $\Sigma \times \mathscr{F}/\mathrm{G}$, *cf.* [16].

At this point, it remains to descend the G-equivariant extended background $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ to the smooth quotient $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$, that is to say to canonically induce, over $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$, another background — call it $\mathfrak{B}_{\mathcal{A}}$ — whose (hypercohomological) equivalence class is to be determined *uniquely* by the parent structure $\widetilde{\mathfrak{B}}_{\mathcal{A}}$. General conditions under which an induction effect of this kind is possible in the 2-category $\mathfrak{BGrb}^{\nabla}(\mathsf{Y}\mathscr{M})$ supported over a surjective submersion $\pi_{\mathsf{Y}\mathscr{M}} : \mathsf{Y}\mathscr{M} \to \mathscr{M}$ over a manifold $\mathscr{M}$ were found in [14]. They constitute the content of the principle of descent[25] which establishes an equivalence between the 2-category $\mathfrak{BGrb}^{\nabla}(\mathscr{M})$ and a distinguished 2-subcategory $\mathfrak{Desc}(\pi_{\mathsf{Y}\mathscr{M}})$ within $\mathfrak{BGrb}^{\nabla}(\mathsf{Y}\mathscr{M})$ termed the descent 2-category and consisting of (0-, 1- and 2-) cells that behave 'trivially' with respect to the pullback cohomology, of the kind considered in [7], defined by $\pi_{\mathsf{Y}\mathscr{M}}$ (in conjunction with the canonical projection maps) over the simplicial space $\mathsf{Y}^{[\bullet]}\mathscr{M}$ with components $\mathsf{Y}^{[n]}\mathscr{M}$ given by the n-fold Cartesian powers of $\mathsf{Y}\mathscr{M}$ fibred over $\mathscr{M}$. In the context of interest, which is that of the surjective submersion $\mathsf{Y}\mathscr{M} \equiv \mathsf{P} \times \mathscr{F} \to \mathsf{P} \times_{\mathrm{G}} \mathscr{F} \equiv \mathscr{M}$ and of the simplicial space $\mathsf{Y}^{[\bullet]}\mathscr{M}$ canonically G-equivariantly isomorphic with the nerve of the action groupoid $\mathrm{G} \ltimes (\mathsf{P} \times \mathscr{F})$, the Principle of Descent reduces to an explicit identification of the specific G-equivariant structure on a background $\widetilde{\mathfrak{B}}'$ over $\mathsf{P} \times \mathscr{F}$ that is necessary and sufficient for (the G-equivariant equivalence class of)

[25] *Cf.* [53] for a more recent account that offers a neat abstraction of the notion of a gerbe (and of a bundle, *etc.*).

$\widetilde{\mathfrak{B}}'$ to induce a unique (up to equivalence) background $\mathfrak{B}'$ over the quotient $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$. The truly remarkable feature of the gauging scheme outlined in the foregoing paragraphs is that the assumed existence of an arbitrary G-equivariant structure on $\mathfrak{B}$ ensures that $\widetilde{\mathfrak{B}}_{\mathcal{A}}$ is just of the desired type, *i.e.* it belongs to the relevant descent 2-subcategory over $\mathsf{P} \times \mathscr{F}$. In accordance with the previous considerations, this enables us to define the gauged σ-model in terms of[26] the induced background $\mathfrak{B}_{\mathcal{A}}$. Its action functional is now readily proven invariant under the action of the gauge group $\Gamma(\mathsf{P} \times_{\mathrm{Ad}_{\mathrm{G}}} \mathrm{G})$.

The very last step in the gauging procedure, that is the field-theoretic transition from the gauged σ-model with the covariant configuration bundle $\mathsf{P} \times_{\mathrm{G}} \mathscr{F}$ to the σ-model with the quotient covariant configuration bundle $\Sigma \times \mathscr{F}/\mathrm{G}$ is — arguably — the least understood one. It entails integrating out the gauge field, a procedure with no clear-cut status within the geometric framework advertised heretofore. The procedure may well necessitate a conceptual enhancement of the latter capable — among other things — of incorporating a new geometric degree of freedom induced in the process, to wit, the dilaton. This seems to call for a further 'generalisation' of Hitchin's generalised geometry that seeks to unify the metric and gerbe (resp. gerbe-related) structure present over the target space of the σ-model. Such a conceptual enhancement of gerbe theory appears to be inevitable also from the point of view of possible generalisations of the gauge principle to *bona fide* dualities of the σ-model as these are bound to mix the various components of the background, including the dilaton. It is tempting to try to transpose the highly structured approach to gauging laid out in Refs. [15, 16] into this — as yet — largely unexplored domain of loop physics, and thus, ultimately, pave the way to a rigorous study of non-geometric loop backgrounds. This temptation is certain to greatly stimulate further development of the theory of gerbes.

5. Summary and prospects

In the present note, intended as a modest and — unavoidably — limited review of the recent progress in the gerbe theory of the σ-model, I sought to emphasise the naturality and efficiency of the full-fledged 2-categorial gerbe-theoretic framework in the description of the loop dynamics of the two-dimensional non-linear σ-model on an arbitrary (defect-decorated) space-time, and in the study of the associated (generalised) geometry of its covariant configuration bundle, leading — via the concept of a σ-model duality — to a generalisation of the Riemannian paradigm of (globally) smooth geometry.

The discourse developed herein merely skimmed or even altogether failed to mention a vast number of exciting aspects of the theory that are currently

[26] The metric, coupled minimally to $\mathcal{A}$, descends to the quotient without further complications.

under investigation. These include:

- a generalisation of the gauge principle encompassing dualities of the σ-model (capturing, *e.g.*, the (non-)geometry of T-folds);
- an augmentation of the gerbe-theoretic framework accommodating other components of the σ-model background, such as, *e.g.*, the dilaton;
- the structure of the fusion algebra of defects, as encoded by the inter-bi-brane data (in particular, in the setting of the WZW σ-model in the presence of the maximally symmetric defects);
- a 'holographic' relation between gerbes and higher-categorial structures (*e.g.* those associated with certain topological field theories, such as, for instance, the three-dimensional Chern–Simons theory);
- the gerbe-theoretic aspect of the issue of criticality of the σ-model background (to be viewed in the context of the generalised Ricci flows of Refs. [84, 85]);
- incorporation of supersymmetry in the gerbe-theoretic picture of the loop dynamics;
- the rôle of the gerbe in determining the nature of the emergent spectral non-commutative geometry on the target space of the σ-model (as suggested by the preliminary findings of [86], obtained along the lines of the seminal paper [87]).

The sheer number and the diversity of the open questions listed makes it apparent that the gerbe theory of the σ-model is antipodally remote from being a dead end in the rigorous study of the loop dynamics.

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