QUANTUM DIRAC FIELD ON MOYAL–MINKOWSKI SPACETIME — ILLUSTRATING QUANTUM FIELD THEORY OVER LORENTZIAN SPECTRAL GEOMETRY *

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A sketch of an approach towards Lorentzian spectral geometry (based on joint work with Mario Paschke) is described, together with a general way to define abstractly the quantized Dirac field on such Lorentzian spectral geometries. Moyal–Minkowski spacetime serves as an example. The scattering of the quantized Dirac field by a non-commutative (Moyal-deformed) action of an external scalar potential is investigated. It is shown that differentiating the S-matrix with respect to the strength of the scattering potential gives rise to quantum field operators depending on elements of the non-commutative algebra entering the spectral geometry description of Moyal–Minkowski spacetime, in the spirit of "Bogoliubov's formula", in analogy to the situation found in external potential scattering by a usual scalar potential.

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1. Introduction

The reason why we entertain the idea that non-commutative (NC) geometry provides a description of spacetime structure which supersedes the picture of spacetime as a differentiable manifold resides in the expectation that, at extremely short distances/high energies the classical concept of "events" looses its meaning. Hence, the mathematical concept of "events" as points in a smooth manifold would no longer be appropriate. The general argument leading to this expectation roughly runs as follows. According to general relativity, the energy content of matter determines spacetime geometry. The energy content of matter at very high energies and short distances is described by quantum field theory. Thus, we should expect a "quantum

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description" of spacetime geometry (and eventually, a theory of quantum gravity). A characteristic feature of such a description would be uncertainty relations for spacetime localization of events (marking "placement of energy/matter") to avoid matter from undergoing gravitational collapse, which would preclude any information about matter distribution/geometry. This idea was made precise by Doplicher, Fredenhagen and Roberts [1]. The corresponding uncertainty relations for localization of events can be implemented by requiring commutation relations for their spacetime coordinates. In consequence, the commutative algebra of coordinate functions of a "classical" spacetime manifold is replaced by a non-commutative algebra, generated by a set of "non-commutative coordinates" and their commutation relations. Following this philosophy, one can think of quite a number of different ways to set up "NC spacetime coordinates". Here, is a sample of the more prominent approaches which have been proposed and investigated:

Moyal-deformed Minkowski spacetime (see [2] and references cited there)

$$[X^\mu, X^\nu] = i\lambda\theta^{\mu\nu} \qquad \text{with} \qquad \theta^{\mu\nu} \in \mathbb{R}\,.$$

Lie-algebra deformation of Minkowski spacetime

$$[X^\mu, X^\nu] = i\lambda\tau^{\mu\nu}{}_\varrho X^\varrho\,, \qquad \tau^{\mu\nu}{}_\varrho \in \mathbb{C}$$

(generalization: $[X^\mu, X^\nu] = F(X)$, where F is a sufficiently nice function, *e.g.* with

$$\begin{aligned} F(X) &= i\lambda\theta^{\mu\nu} + i\lambda\tau^{\mu\nu}{}_\varrho X^\varrho + i\lambda^2\sigma^{\mu\nu}{}_{\varrho\kappa}X^\varrho X^\kappa + O\left(\lambda^3, (X)^3\right) \quad \text{and/or} \\ \theta^{\mu\nu} &= \theta^{\mu\nu}(X)\,, \qquad \textit{etc.}) \end{aligned}$$

Quantum space or Hopf-algebraic deformed Minkowski spacetime

$$X^\mu X^\nu = \lambda\xi^{\mu\nu}{}_{\varrho\kappa}X^\varrho X^\kappa\,, \qquad \xi^{\mu\nu}_{\varrho\kappa} \in \mathbb{C}\,.$$

DFR-Minkowski spacetime [1]

$$[X^\mu, X^\nu] = i\lambda Q^{\mu\nu}\,, \qquad [X^\kappa, Q^{\mu\nu}] = 0\,.$$

In these relations, λ is some real parameter setting the scale at which non-commutativity is relevant (supposing the other quantities characterizing non-commutativity are roughly of the order of 1). Of course, this list is not meant to be complete. For all these various "models" of NC spaces, certain quantum field theoretic models have been studied on these NC spaces. The general observation drawn from those investigations is that the UV behaviour improves, but there are new types of IR problems. In some approaches, they can be cured. Some work points at possibility of constructing interacting quantum field theory (QFT) models *e.g.* on Moyal-deformed spaces [3, 4, 5]. While this surely opens some very promising perspectives, there are also some drawbacks and conceptual problems:

- The promising constructive work uses *Euclidean* NC Moyal spacetime. For this class of spacetimes (and other NC spacetimes) there is no counterpart to the Osterwalder–Schrader theorem which establishes a correspondence between QFTs on Euclidean space and QFT on Minkowski space.

- For Moyal deformed Minkowski spacetime, Lorentz covariance is broken (to a smaller covariance group). This is conceptually unsatisfactory (although regarded by some as a sign that Lorentz covariance *is* broken in nature).

- The *operational significance* of NC spacetime in its relation to the QFT on it is often not very clear.

- What replaces the locality concept which is central to QFT in Minkowski spacetime on an NC spacetime?

- There are (more or less) good arguments for all of the various models of NC spaces (spacetimes). Which is the most appropriate (if any)? What conceptual and mathematical framework is needed to stage a systematic discussion of this question?

- What about general covariance? General relativity is one of the main motivations for considering NC spacetime. In QFT on classical spacetime, one can formulate general covariance for QFTs. This requires to consider not just a few particular spacetime models, but a whole class of spacetimes (abstractly characterized — "model independent").

- Actually, what is a QFT on an NC spacetime? What are its characterizing properties (needed for a sound physical interpretation)? Is there a model-independent framework — model-independent both on the NC geometry side *and* on the QFT side?

2. Lorentzian spectral geometry (but only some daring first steps into a vast jungle)

In an attempt to find answers to the list of questions displayed above, one may invoke a framework which unifies the general features of NC geometries as a starting point. In fact, there is a model-independent approach to (compact) Riemannian NC geometry — the spectral geometry approach developed by Connes [6,7,8,9]. The mainstream opinion, at least among connoisseurs of the spectral geometry approach, is that most of the examples of NC spaces usually considered (when they correspond to NC generalizations of Riemannian geometries) fulfill the conditions of spectral geometry. As it stands, this statement is not fully correct, *e.g.* Euclidean Moyal space does not correspond to an NC compact spectral geometry (since Euclidean space is non-compact), and the setting of Connes needs to be adapted to this case (see, *e.g.*, [10]). Thus, this mainstream opinion is subject to making adaptations to the original setting of Connes, and to be fair, I am unaware of any systematic investigation that would substantially support the stated opinion (and clarifies which adaptations have to be made in detail in the various cases). However, we take it, for the time being, as working hypothesis. The strength of the spectral geometry approach is based on

- "naturality" of the axioms;
- structural theorems, including "reconstruction" of a compact Riemannian manifold with spin structure in the "classical case".

Up to now, it remains unclear if a spectral geometry approach of comparable strength can be developed for the case of semi-Riemannian NC geometries. There are, however, some approaches [11, 12, 13, 14, 15]. We will sketch here the approach outlined in [15] (which draws partially on [13] and [14]) and set forth in [16]; it is developed with a view on "general covariance" as a central principle for quantum field theories on (NC) manifolds (*cf.* [15, 17] for discussion, see also discussion below). That approach has largely been developed by Mario Paschke, together with the present author, but as yet, it is tentative and unfinished. It should be seen as a proposal in which direction a generalization of compact Riemannian spectral geometry could proceed. Some structural elements can be generalized quite straightforwardly, others are less clear, and out of the various possibilities of generalization one has to make choices. Let us briefly remind the reader of the spectral geometry setting generalizing compact Riemannian spin manifolds. The central structure is called a *spectral triple*, since initially the emphasis was on a collection of three objects, but nowadays it has become customary to list in fact five objects, yet still referring to their collection as a spectral triple. This understood, a spectral triple consists of a collection $(\mathcal{A}, \mathcal{H}, D, \mathring{\gamma}, J)$ where $\mathcal{H}$ is a Hilbert space, $\mathcal{A}$ is a unital $*$-algebra of bounded linear operators on $\mathcal{H}$, D is an unbounded selfadjoint operator on a suitable dense domain in $\mathcal{H}$, $\mathring{\gamma}$ (often denoted simply as γ) is a bounded operator on $\mathcal{H}$ while J (often denoted as C) is a conjugation on $\mathcal{H}$. These objects are interrelated by a list of relations and regularity conditions (see [6]). It turns out that, in the case that $\mathcal{A}$ is Abelian, the spectral triple is equivalent to a compact Riemannian spin manifold, where $\mathcal{A}$ corresponds to the algebra of scalar C^∞ functions on the manifold, $\mathcal{H}$ is the Hilbert space of L^2 sections of the spinor boundle, D is a Dirac operator (the principal symbol is unique), $\mathring{\gamma}$ corresponds to an orientation and J is charge conjugation on the spinors [6]. (For an alternative approach, which is not related to spin structure and Dirac operators, see again [6].) When trying to generalize the structure to the Lorentzian case, there are a number of difficulties. First, causally well-behaved Lorentzian manifolds (*i.e.* spacetimes) are non-compact in timelike directions, so as to avoid closed causal curves. This makes it necessary to work with some non-unital algebras in place of $\mathcal{A}$, and this entails other difficulties. (Moreover, one needs to unitalize some of these algebras in the end, and there is no unique way of doing this, so some choice is involved here). Secondly, in the Riemannian case the Dirac operator on a spin manifold D is elliptic, which is quite important in the spectral geometry setting, but this is clearly not the case of a Lorentzian spin manifold. So, for the Lorentzian setting one needs a way of gaining an elliptic operator out of the Dirac operator. Moreover, on a Lorentzian spin manifold there is no canonical (or covariant) scalar product on the sections of the spinor bundle and thus no natural L^2 Hilbert space structure. However, there is a canonical sesquilinear form $\langle f, h\rangle$ on the C_0^∞ sections f, h of the spinor bundle, induced by the operation of taking the Dirac adjoint of a spinor, and the Dirac operator of the Lorentzian spin manifold, which we will denote here by $\nabla\!\!\!\!/$ (following physicists' notation), is symmetric with respect to this sesquilinear form on C_0^∞ sections. When one takes a future pointing unit vector field, n, on the Lorentzian spin manifold, then

$$(f, h) = \langle f, \boldsymbol{\gamma}(n)h\rangle \tag{1}$$

yields a scalar product (or a negative definite inner product, depending on choice of metric signature) on the C_0^∞ sections f, h of the spinor bundle, where $\boldsymbol{\gamma}(n)$ denotes the Clifford action of n on the spinors [18, 19]. (In physicists' abstract index notation, $\boldsymbol{\gamma}(n) = n^a {\boldsymbol{\gamma}_a}^A{}_B$, or $\boldsymbol{\gamma}(n) = n\!\!\!/$, *cf.* [18].) Note that $\nabla\!\!\!\!/$ is no longer symmetric with respect to that scalar product. Up to sign (again depending on metric signature), $\langle f, h\rangle$ can be regained from (f, h) as $\langle f, h\rangle = (f, \boldsymbol{\gamma}(n)h)$. Thus, in setting up a framework for Lorentzian spectral geometry, it is suggestive to add as another element of structure an operator β which, in the case of a Lorentzian spin manifold, plays the role of $\boldsymbol{\gamma}(n)$ for some timelike, normalized vector field n. This also induces the scalar product (1). When we write L^2 space of spinors below, we are referring to this — n-dependent — scalar product on the spinors. (Alternatively, one could work with the indefinite inner product corresponding to $\langle f, h\rangle$ for a Lorentzian manifold; this route is taken in [13].) With these remarks in mind, we present our proposal for the structure of Lorentzian spectral geometry. We proceed in such a way that we put side to side the objects forming what we call a Lorentzian spectral triple (left column) and what they correspond to in the "classical case", *i.e.* for a given Lorentzian spin manifold (right column).

A *Lorentzian Spectral Triple* (LOST) is a collection objects as follows:

$$\left(\mathcal{A}_0 \subset \mathcal{A}_2 \subset \mathcal{A}_b, \mathcal{H}, D, \beta, \overset{\circ}{\gamma}, J\right)$$

with the properties:

General (NC)	**Classical**
$\mathcal{H}$ is a Hilbert space	$\mathcal{H} = L^2$ spinors on a Lorentzian manifold M with spin structure
$\mathcal{A}_0$ is a pre-C^*-algebra of bounded linear operators on $\mathcal{H}$, $\mathcal{A}_b$ is a preferred unitalization	$\mathcal{A}_0 = C_0^\infty(M)$, $\mathcal{A}_b = C_b^\infty(M)$
D is a linear operator with dense C^∞ domain $\mathcal{H}^\infty = \mathcal{A}_2\mathcal{E}$ with a finitely generated $\mathcal{A}_b$-module $\mathcal{E}$	$D = \nabla\!\!\!\!/ =$ Dirac-operator, with C^∞ domain of smooth sections in the spinor bundle where all D-derivatives are L^2
$\beta, \overset{\circ}{\gamma}$ are bounded operators on $\mathcal{H}$, J is anti-unitary, with relations: $\beta^* = -\beta$, $\beta^2 = -1$, $D^* = \beta D \beta$, $[JaJ, b] = 0$ $(a, b \in \mathcal{A}_0)$, $[X_1, [X_2, ...[X_n, a]...]]$ are bounded for $a \in \mathcal{A}_0$ and $X_j = D$ or D^* and several other relations	$\beta = \boldsymbol{\gamma}(n)$, where n a timelike vectorfield on M, $\boldsymbol{\gamma}(\,.\,)$ is the Clifford algebra action on the spinor bundle, $\overset{\circ}{\gamma} = \boldsymbol{\gamma}(e_0) \cdots \boldsymbol{\gamma}(e_m)$ with an orthonormal frame $(e_0, \ldots, e_m)$, J corresponds to charge conjugation on the spinors D is a first order PDO
Setting $\langle D \rangle = \sqrt{D^*D + DD^*}$, $a(1 - \langle D \rangle)^{-1}$ is compact for $a \in \mathcal{A}_0$. There is a minimal $m \in \mathbb{N}$ so that the Dixmier-trace of $a\langle D \rangle^{-m}$ is finite and non-vanishing for all $a \in \mathcal{A}_0$	$\langle D \rangle$ is elliptic, m is the (spectral) dimension of M
Plus 3 more conditions: $\overset{\circ}{\gamma}$ is the image of a Hochschild cycle, β belongs to the 1-forms of $\mathcal{A}_b$, and Poincaré duality (alternatively, closedness and Morita-equivalence of $\mathcal{A}_b$ via $\mathcal{E}$)	Essentially: orientability and Hodge-duality

We remark that this list of conditions on the objects of a LOST, as given here, is incomplete, and there are also some open questions, *e.g.* related to precise domain conditions for D and D^* in relation to the inclusion of algebras $\mathcal{A}_0 \subset \mathcal{A}_2 \subset \mathcal{A}_b$. In the case of a Lorentzian spin manifold, one expects that one needs to impose regularity conditions on the timelike vector field n entering the definition of $\beta = \boldsymbol{\gamma}(n)$ in order that good domain conditions — meaning that they lead to a reconstruction theorem of a Lorentzian spin manifold in the case of an Abelian $\mathcal{A}_b$ as we will formulate it below — can be obtained. As was emphasized, the structure of a LOST makes reference to a distinguished normalized, timelike vector field in the "classical" case. However, the structure of a Lorentzian spin manifold does not single out any preferred timelike vector field (except for special cases). In other words, all LOSTs leading to the same (or isomorphic) Lorentzian spin manifolds are to be viewed as equivalent. The following definition provides, in this sense, the concept of equivalent LOSTs (where we use the symbol $\mathcal{A}$ as abbreviation of the inclusions $\mathcal{A}_0 \subset \mathcal{A}_2 \subset \mathcal{A}_b$):

$$\text{Let} \quad \left(\mathcal{A}, \mathcal{H}, D, \beta, \overset{\circ}{\gamma}, J\right) \quad \text{and} \quad \left(\widetilde{\mathcal{A}}, \widetilde{\mathcal{H}}, \widetilde{D}, \widetilde{\beta}, \widetilde{\overset{\circ}{\gamma}}, \widetilde{J}\right) \quad \text{be two LOSTs}\,.$$

They are called *equivalent* if there is a unitary $U : \mathcal{H} \to \widetilde{\mathcal{H}}$ so that

$$UXU^{-1} = \widetilde{X} \quad \text{for} \quad X = \mathcal{A},\ \beta,\ \overset{\circ}{\gamma},\ J$$

and

$$\left[\widetilde{D}, U \,.\, U^{-1}\right] = U[D,\,.\,]U^{-1}\,.$$

We expect that the open points in the definition of a LOST, which we alluded to above, can be filled in such a way that the following conjecture can eventually be established as a rigorous result.

Conjecture

To each Lorentzian manifold with spin structure there corresponds a LOST with commutative $\mathcal{A}$ and $D = \nabla\!\!\!\!/ =$ Dirac operator. The LOSTs corresponding to isometric Lorentzian manifolds with equivalent spin structures are equivalent.

Conversely, if a LOST has Abelian $\mathcal{A}$, then it derives from a Lorentzian manifold with spin structure. The Lorentzian spin manifolds deriving from equivalent LOSTs are isometric and have equivalent spin structures.

The latter statement would amount to a Lorentzian version of Connes' reconstruction theorem of Riemannian spin manifolds for Abelian $\mathcal{A}$. The idea is, of course, to use the structure of a LOST to derive from it a Riemannian spectral triple from which the manifold structure can be constructed as in [6]. However, if this can be achieved is, as yet, still open.

3. GHYSTs and quantum field theory

The LOST setting has the potential to describe very general Lorentzian spin manifolds and their NC generalizations. According to our present understanding, however, it is most difficult to set up a consistent framework for quantum field theory on Lorentzian spin manifolds which are not globally hyperbolic [20,21,22]. Therefore, in order to achieve a promising framework for quantum field theory on LOSTs, a first step consists in characterizing the counterpart of global hyperbolicity at the LOST level. A Lorentzian spin manifold M is globally hyperbolic if and only if the Dirac operator $\nabla\!\!\!/$ defined on it possesses unique advanced and retarded fundamental solutions, R_+ and R_-, taking C_0^∞ sections in the spinor bundle to C^∞ sections. They are characterized by

$$R_\pm \nabla\!\!\!/ f = f = \nabla\!\!\!/ R_\pm f \tag{2}$$

for all C_0^∞ spinor sections f, and by

$$\mathrm{supp}(R_\pm f) \subset J^\pm(\mathrm{supp}(f))\,, \tag{3}$$

where $J^\pm(S)$ denotes the causal future $(+)$/causal past $(-)$ set of a subset S of the Lorentzian spin manifold. This means, $J^\pm(S)$ is the set of points in M which can be reached by all future $(+)$/past $(-)$ directed causal curves emanating from S. Therefore, to characterize globally hyperbolic LOSTs — which will be referred to as GHYSTs, short for *globally hyperbolic spectral triples* — one would have to formulate conditions characterizing advanced and retarded fundamental solutions for the operator D in the LOST setting, *i.e.* using only the objects forming a LOST. Clearly, condition (2) can readily be generalized to the LOST setting. But condition (3) uses the localization concept of a "classical" differentiable manifold and this is not at hand within the LOST setting. Hence, it is unclear how condition (3) should be generalized to the LOST setting, and how to characterize advanced and retarded fundamental solutions of D in this setting. Nevertheless, let us for the moment proceed under the hypotheses that a suitable characterization of advanced and retarded fundamental solutions of D in the LOST setting can be given. At the level of concrete examples, there are situations where there are obvious candidates for advanced and retarded fundamental solutions: Drawing largely on results of [10], it can be shown that Moyal–Minkowski spacetime (a description of whose basic elements will be given in the next section) is an example of a LOST (M. Paschke, unpublished), and in that particular case, D is just the usual Dirac operator on Minkowski spacetime which has unique advanced and retarded fundamental solutions in the "classical" sense. In fact, as a consistency requirement, the concept of advanced and retarded fundamental solutions of D in the LOST setting should coincide with the "classical" concept whenever a LOST describes a Lorentzian spin manifold. In the following, we shall take it for granted (more appropriately, take as a working hypothesis) that Moyal–Minkowski spacetime is a GHYST. Now suppose we have a GHYST

$$\boldsymbol{G} = \left(\mathcal{A}, \mathcal{H}, D, \beta, \mathring{\gamma}, J, R_\pm\right),$$

where $R_\pm$ are the advanced and retarded fundamental solutions of D. Furthermore, setting $R = R_+ - R_-$ suppose — as is the case for a globally hyperbolic spin manifold — that

$$(f, h)_{(R)} = (f, \beta R h)$$

for f, h in a suitable dense domain $\mathcal{H}_{(R)}$ in $\mathcal{H}$ is positive semi-definite (possibly up to a constant overall phase). We denote by $\mathcal{K}$ the completion of $\mathcal{H}_{(R)}$ factorized by the kernel of $(. \, , \, .)_{(R)}$ with respect to the scalar product $(. \, , \, .)_{\mathcal{K}}$ induced by $(. \, , \, .)_{(R)}$. One can show that J furnishes a conjugation on $\mathcal{K}$ (again denoted by J). Moreover, $R : f \mapsto Rf$ is, under suitable identification, equivalent to the canonical surjection $\mathcal{H}_{(R)} \to \mathcal{K}$. Thus, one can invoke the abstract CAR quantization procedure [23] to associate to $\boldsymbol{G}$ an abstract C^*-"Dirac field"-algebra $\mathsf{F}(\boldsymbol{G})$ which is generated by a family $B(\chi)$, $\chi \in \mathcal{K}$, subject to the conditions

- $\chi \mapsto B(\chi)$ is $\mathbb{C}$-linear ,
- $B(\chi)^* = B(J\chi)$,
- $B(\chi)^* B(\xi) + B(\xi) B(\chi)^* = (\chi, \xi)_{\mathcal{K}} \mathbf{1}$,

where $\mathbf{1}$ denotes the unit element in the C^*-algebra $\mathsf{F}(\boldsymbol{G})$. Upon setting $\Psi(f) = B(Rf)$ for $f \in \mathcal{H}_{(R)}$ (identifying R with the canonical surjection), one obtains an "abstract Dirac field" over $\boldsymbol{G}$ with the characteristic conditions

$$\Psi(f)^* \Psi(h) + \Psi(h) \Psi(f)^* = (f, h)_{(R)} \mathbf{1} \quad \left(f, h \in \mathcal{H}_{(R)}\right) \quad \text{and}$$
$$\Psi(Df) = 0 .$$

The latter equation corresponds to an "abstract Dirac equation" associated to the underlying GHYST $\boldsymbol{G}$. Finally, the assignment $\boldsymbol{G} \to \mathsf{F}(\boldsymbol{G})$ of GHYSTs to C^*-CAR algebras is functorial in the following sense. Let us call a unitary equivalence

$$\boldsymbol{G} \xrightarrow{U} \widetilde{\boldsymbol{G}}$$

rigid if $UDU^* = \widetilde{D}$. Then there is a canonical C^* algebraic morphism

$$\mathsf{F}(\boldsymbol{G}) \xrightarrow{\alpha_U} \mathsf{F}\left(\widetilde{\boldsymbol{G}}\right)$$

which is induced by $\alpha_U(\Psi(f)) = \widetilde{\Psi}(Uf)$ in obvious notation. This implies the covariance property $\alpha_{U_2} \circ \alpha_{U_1} = \alpha_{U_2 U_1}$ for rigid unitary equivalences. This functorial structure corresponds to the "global" covariance of the quantized Dirac field on globally hyperbolic spacetimes which was first brought to the fore by Dimock [18]. The Dirac field fulfills also a stronger, "local" version of covariance [17, 19, 24, 25] which induces essentially the local and causal structure of the quantum field theory. However, this "local covariance", which to a large part also determines the interpretation of the quantum field theory (derived from the net of local observable algebras *cf.* [26]), crucially depends on the localization concept of classical differentiable manifolds, and that has no direct counterpart in the framework of LOSTs or GHYSTs. How, then, does one link the non-commutativity of the $\mathcal{A}$ algebra of $\boldsymbol{G}$ with the algebraic structure of $\mathsf{F}(\boldsymbol{G})$, and which quantum field operators associated to $\boldsymbol{G}$ carry a particular physical interpretation? While one can surely come up with many ideas for answers, we actually take a modest step and look at the simplest example of the "abstract" quantized Dirac field on a GHYST with NC $\mathcal{A}$ — corresponding to Moyal–Minkowski spacetime.

4. Dirac field on Moyal–Minkowski spacetime

Moyal–Minkowski spacetime is identical to Minkowski spacetime, except that the usual commutative pointwise product of (Schwarz-class) test-functions on Minkowski spacetime is replaced by the Moyal-product (or Rieffel-product). To set up matters more precisely, consider $n = 1 + d$ dimensional Minkowski spacetime $\mathbb{R}^{1,d}$. Let $\Theta = (\Theta_{\mu\nu})$ be an anti-symmetric real $n \times n$ matrix. Then one can define a deformed product of $\mathcal{S}(\mathbb{R}^n)$ by

$$f \star h(x) = (2\pi)^{-n} \int\int f\left(x - \tfrac{1}{2}\Theta u\right) h(x+v) e^{-iu\cdot v}\, d^n u\, d^n v\,, \tag{4}$$

where $u \cdot v$ is the standard Euclidean scalar product on $\mathbb{R}^n$. Usually, when n is even, one takes Θ to be the standard symplectic matrix times a positive scaling factor. In this case, and also in more general cases, one can show that the above product between test-function is associative. When Θ has non-zero entries it is, however, non-commutative. We now define an algebra $\mathcal{A}_0 = \mathcal{S}(\mathbb{R}^n)$ with the above Moyal product as algebra product. By the $\star$-product it acts naturally on the Hilbert-space $\mathcal{H} = L^2(\mathbb{R}^n, \mathbb{C}^N)$ where $N = N(n)$ is the lowest dimension for an irreducible, self-dual representation of the Clifford algebra $Cl(1, d)$ (requiring existence of such a representation puts restrictions on the dimension n, see [16] and references given there for details). The Clifford algebra generators are then represented by a set of "Gamma-matrices", $\gamma^0, \gamma^1, \ldots, \gamma^d$. When taking D as the usual Dirac-operator (with some arbitrary, but fixed mass term $m \leq 0$),

$$D = i\gamma^\mu \frac{\partial}{\partial x^\mu} + m$$

on the domain $\mathcal{S}(\mathbb{R}^n, \mathbb{C}^N)$, together with $\beta = \gamma_0$, $\overset{\circ}{\gamma} = \gamma_0 \cdots \gamma_d$, J as the charge-conjugation on $\mathcal{H}$, and taking $\mathcal{A}_2$ and $\mathcal{A}_b$ as in [10], then one can use much of the results of [10] to show that one has collected the data of a LOST, at least in the case of even n and with non-degenerate Θ (M. Paschke, unpublished — as there is no complete published proof, we now proceed under the fiction that these data indeed form the data of a LOST). This LOST is essentially just the LOST corresponding to n-dimensional "classical" Minkowski spacetime, but with the Moyal product instead of the commutative, pointwise product on the algebra of test-functions. It is even a GHYST (strictly, we assume the conditions on GHYSTs to be formulated so that this holds true, *cf.* our discussion above), since the Dirac operator D possesses unique advanced and retarded fundamental solutions (in the classical manifold sense, not (yet) expressed using only the data of the LOST). Using these advanced and retarded fundamental solutions $R_\pm$ and their difference $R = R_+ - R_-$, one can even set up the CAR algebra F_{MM} of the quantized Dirac field on Moyal–Minkowski spacetime. However, it is easy to see that this algebra is in no way different from the CAR algebra $\mathsf{F}_{\mathrm{Mink}}$ of the quantized Dirac field. The information about the non-commutativity of $\mathcal{A}_0$ is not directly visible in at the level of the generators of the algebras — *i.e.* in the quantization procedure, if one wishes to put it like that — but it is hidden somewhere else. How can we access this information? Which Dirac quantum field operators carry that information? Does it provide any link to the more customary approach of quantum field theory on Moyal deformed spaces which essentially replaces the "usual" products of Dirac quantum field operators by their Moyal–Rieffel products? Obviously, we need to look at some way the elements c of the algebra of test-functions can take action on the Dirac quantum field operators. Let $\Psi(f)$ denote the abstract quantum field operators ($f \in \mathcal{S}(\mathbb{R}^n, \mathbb{C}^N)$) generating both F_{MM} and $\mathsf{F}_{\mathrm{Mink}}$. On usual Minkowski spacetime, one can look at the map $\Psi(f) \mapsto \Psi(cRf)$. This map arises when scattering the quantized Dirac field by an external scalar potential c. In the next sections, we explain this, and explain how this potential scattering can be generalized to scattering by an NC potential.

5. Dirac field NC potential scattering — commutative time

In order to keep matters as simple as possible, we will, in the present section, specialize to the case $n = 3$ (implying $N = 4$), *i.e.* 3-dimensional Minkowski — respectively Moyal–Minkowski spacetime. However, most of our considerations apply to more general spacetime dimensions, see [16] for details. To begin, we need a bit of notation. We denote by $\mathcal{K}_+$ the positive frequency part of the one-particle Hilbert space of the Dirac field on 3-dimensional Minkowski spacetime. This corresponds to the subspace of positive frequency solutions of the solution space $\mathcal{K}$ (containing solutions χ, with Cauchy-data of Schwarz class, to the Dirac equation $(i\gamma^\mu \partial_{x^\mu} + m)\chi = 0$) [27]. Furthermore, we consider the quantized Dirac field on Minkowski spacetime in its usual vacuum representation. Consequently, we identify the "abstract Dirac field operators" $\Psi(f)$ with the represented operators $\psi(f)$ which are concretely given, as usual, in terms of annihilation and creation operators in the fermionic Fock space $F_+(\mathcal{K}_+)$ over the one-particle space $\mathcal{K}_+$. Now let c be a real-valued Schwarz function on $\mathbb{R}^3$. Then one can show [16] — and we believe it is well-known — that

$$i\left[: \psi^+ \psi : (c), \psi(f)\right] = \psi(cRf) \tag{5}$$

holds for all test spinors $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}^4)$. Here, $[X, Y] = XY - YX$ denotes the commutator, and $: \psi^+\psi : (c)$ is the normal-ordered coinciding-point-limit-product (Wick-product) of the Dirac-adjoint ψ^+ with ψ itself. This results in a scalar quantum field, which in the above formula is smeared with c as a test function (see [16] for further discussion). On the other hand, (5) is the result of differentiating the scattering transformation related to external potential scattering of the quantized Dirac field on Minkowski spacetime with respect to the potential strength. Let us explain this in a bit more detail. First, we put the free Dirac equation $(i\gamma^\mu \partial_{x^\mu} + m)\chi = 0$ into Hamiltonian form: We fix some inertial time coordinate t ($\equiv x^0$) and write $\chi_t(\,.\,) = \chi(t,\,.\,)$. Then the free Dirac equation is equivalent to

$$i\frac{d}{dt}\chi_t + H_0\chi_t = 0\,,$$

where the free Hamiltonian is a selfadjoint operator on a suitable dense domain in $L^2(\mathbb{R}^2, \mathbb{C}^4)$ which acts as

$$H_0 v(\underline{x}) = \left(i\gamma^0\gamma^k\partial_{x^k} + \gamma^0 m\right) v(\underline{x})\,,$$

where $\underline{x} = (x^k)_{k=1}^2$. Now let $c = c(t, \underline{x})$ be a real-valued Schwarz function, regarded as a time-dependent external scalar potential for the Dirac field. Then the Dirac equation

$$(D + \lambda c)\chi = (i\gamma^\mu\partial_{x^\mu} + m + \lambda c)\,\chi = 0$$

is equivalent to

$$H_\lambda(t)\chi_t = (H_0 + V_\lambda(t))\chi_t = 0 \tag{6}$$

with the time-dependent potential operator

$$V_\lambda(t)v(\underline{x}) = \lambda\gamma^0 c(t,\underline{x})v(\underline{x}) \tag{7}$$

defined on a suitable domain of $L^2(\mathbb{R}^2, \mathbb{C}^4)$; here we have introduced a positive real parameter λ scaling the strength of the interaction with the external potential. Assuming that appropriate self-adjointness and domain conditions are fulfilled (see [16] and literature cited there for full details), one can show that there is a two-parametric family of unitaries $U_\lambda(t,s)$ in $L^2(\mathbb{R}^2, \mathbb{C}^4)$ with $U_\lambda(t,r)U_\lambda(r,s) = U_\lambda(t,s)$, $U_\lambda(t,t) = \mathbf{1}$ and such that $\chi_t = U_\lambda(t,t_0)v$ is the unique solution to (6) with initial condition $\chi_{t_0}(\underline{x}) = v(\underline{x})$. Moreover, the one-particle scattering operator

$$s_\lambda = \lim_{\pm t_\pm \to \infty} e^{it_+ H_0} U_\lambda(t_+, t_-) e^{-it_- H_0}$$

exists and is a unitary on the space $L^2(\mathbb{R}^2, \mathbb{C}^4)$ of Cauchy-data for the free Dirac equation. The latter Hilbert space is canonically isomorphic to the solutions' Hilbert space $\mathcal{K}$. Remembering that $\psi(f)$ depends only on Rf, where R is the difference of advanced minus retarded fundamental solution of the free Dirac equation, one can define a re-labelled field operator $\check{\psi}(Rf) = \psi(f)$, and identifying s_λ with an operator in $\mathcal{K}$, one can actually show that

$$\left.\frac{d}{d\lambda}\right|_{\lambda=0} \psi(s_\lambda Rf) = \check{\psi}(cRf)\,.$$

Hence, using (5), one finds

$$[:\psi^+\psi:(c), \check{\psi}(Rf)] = \frac{1}{i}\left.\frac{d}{d\lambda}\right|_{\lambda=0} \check{\psi}(s_\lambda Rf)\,.$$

Moreover, the one-particle scattering transformation s_λ can be unitarily implemented in the vacuum representation of the free Dirac field, meaning that there is a unitary S_λ on $F_+(\mathcal{K}_+)$ (the S-matrix, or 2nd quantized scattering operator) such that

$$S_\lambda \check{\psi}(Rf) S_\lambda^* = \check{\psi}(s_\lambda Rf)\,.$$

Therefore, one has

$$[:\psi^+\psi:(c), \check{\psi}(Rf)] = \frac{1}{i}\left.\frac{d}{d\lambda}\right|_{\lambda=0} S_\lambda \check{\psi}(Rf) S_\lambda^*\,.$$

Now we will see that one obtains identical results when replacing the potential operator $V_\lambda(t)$ of (7) by a more general operator involving the Moyal product, at least as long as our underlying 3-dimensional Moyal–Minkowski spacetime still has "commutative time". (This restriction will be lifted in the next section.) As the matrix appearing in the definition of the Moyal product we choose

$$\Theta = (\Theta^{\mu\nu}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \theta \\ 0 & -\theta & 0 \end{pmatrix},$$

where θ is some positive parameter that will be kept fixed. Let us now choose some real-valued scalar test function c of the form

$$c(t, \underline{x}) = a(t)b(\underline{x}),$$

where a is C_0^∞ and b is Schwarz. Then we define the two interaction potentials

$$V_{\lambda\star}(t)v(\underline{x}) = \lambda a(t)\gamma^0(b \,\underline{\star}\, v(\underline{x}) + v \,\underline{\star}\, b(\underline{x})), \tag{8}$$

$$V_{\lambda\star\star}(t)v(\underline{x}) = \lambda a(t)^2\gamma^0(b \,\underline{\star}\, v \,\underline{\star}\, b(\underline{x})), \tag{9}$$

where $\underline{\star}$ denotes the Moyal product on Schwarz functions over $\mathbb{R}^2$, given by

$$b \,\underline{\star}\, g(\underline{x}) = (2\pi)^{-2} \int\int b\left(\underline{x} - \tfrac{1}{2}\underline{\Theta}\underline{y}\right) g(\underline{x} + \underline{q}) e^{-i\underline{y}\cdot\underline{q}}\, d^2\underline{y}\, d^2\underline{q} \tag{10}$$

with

$$\underline{\Theta} = \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix}.$$

Thus, if $\chi_t(\,.\,) = \chi(t,\,.\,)$ is a solution to

$$i\frac{d}{dt}\chi_t + V_{\lambda\#}(t)\chi_t = 0 \quad (\# = \star \text{ or } \star\star),$$

this is equivalent to

$$D\chi + \lambda(c \star \chi + \chi \star c) = 0 \quad \text{if } \# = \star, \text{ and} \tag{11}$$

$$D\chi + \lambda c \star \chi \star c = 0 \quad \text{if } \# = \star\star\,. \tag{12}$$

In [16], we have established the following results.

Theorem 5.1

- *The one-particle scattering operator* $s_{\lambda\#}$ *exists for the potentials* $V_{\lambda\#}(t)$ *defined above* ($\# = \star$ *or* $\# = \star\star$).
- *The one-particle scattering operator is unitarily implemented in the vacuum representation of the quantized Dirac field, i.e. there are unitary operators* $S_{\lambda\#}$ *on the Fock-space* $F_+(\mathcal{K}_{0+})$ *such that*

$$S_{\lambda\#}\check{\psi}(Rf)S^*_{\lambda\#} = \check{\psi}\left(s_{\lambda\#}Rf\right) .$$

- *There is an essentially self-adjoint operator* $\Phi_\#(c)$ *on the Wightman domain of Fock-space such that*

$$i[\Phi_\#(c), \psi(f)] = \left.\frac{d}{d\lambda}\right|_{\lambda=0} S_{\lambda\#}\psi(f)S^*_{\lambda\#} \tag{13}$$

$$= \begin{cases} \psi(c \star Rf + Rf \star c) & \textit{if } \# = \star \\ \psi(c \star Rf \star c) & \textit{if } \# = \star\star \end{cases} . \tag{14}$$

6. Dirac field NC potential scattering — the general case

In the present section, our aim is to generalize the findings on the scattering of the quantized Dirac field by an NC potential, but still keeping time "commutative", to the general case, where also time is turned into an NC "coordinate (operator)". To this end, we now consider 4-dimensional Moyal–Minkowski spacetime (but the discussion of this section can be generalized to other spacetime dimensions, see [28]. The Moyal-product of test-functions f and h on $\mathbb{R}^4$ is as in (4) for $n = 4$, with the matrix

$$\Theta = \theta \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} ,$$

where θ is some fixed positive parameter. For simplicity, we consider only one of the non-commutative potential terms from the two of the previous section — corresponding to the field equation

$$D\chi + \lambda c \star \chi \star c = 0 \tag{15}$$

for some real-valued Schwarz function on c on $\mathbb{R}^4$. (Again, λ is a positive parameter scaling the interaction coupling.) Now we face the problem that, due to the non-local action of the Moyal product with respect to the time-coordinate, the field equation is no longer equivalent to a time-dependent Hamiltonian equation where at each point in time the Hamilton operator acts only with respect to the spatial coordinates. Thus, we need another way of finding solutions to (15). The first step is to not consider (15) as it stands, but to replace it by a simpler form where the potential is made nicer by introducing suitable cut-offs, then to establish solutions to the cut-off dynamical equations, and finally to control the limit of such solutions as the cut-offs are being removed. In fact, we consider two cut-offs. Let $\tau > 0$ and let, with respect to a chosen time-coordinate t, $M_\tau = \{(t, x^1, x^2, x^3) \in \mathbb{R}^4 : -\tau < t < \tau\}$ be a slice of Minkowski-spacetime whose time-extension is controlled by τ. Then consider any non-negative C_0^∞ function ξ defined on $\mathbb{R}$ which is equal to 1 on the interval $[-\tau/2, \tau/2]$ and zero outside the interval $(-\tau/\sqrt{2}, \tau/\sqrt{2})$. Then we define the operator $D + V_\xi(\lambda)$ on the spinor fields $\mathcal{S}(M_\tau, \mathbb{C}^4)$ over M_τ (defined as having compact support in time) where the cut-off potential operator is given by

$$V_\xi(\lambda) f = \lambda \xi (c \star (\xi f) \star c) \,.$$

(Here, ξ acts as multiplication operator.) Then one can show that, provided λ is sufficiently small (depending on τ and ξ), there exist unique advanced and retarded fundamental solutions $\mathcal{R}^{\pm}_{\tau,\lambda,\xi}$ for the operator $D + V_\xi(\lambda)$ on the slice M_τ which can be gained as Neumann series [28],

$$\mathcal{R}^{\pm}_{\tau,\lambda,\xi} = R^{\pm} \left(1 + \lambda V_\xi(c) R^{\pm}\right)^{-1} = R^{\pm} \left(\sum_{j=0}^{\infty} (-1)^j \left(\lambda V_\xi(c) R^{\pm}\right)^j \right) ,$$

where $R^{\pm}$ denote the advanced/retarded fundamental solutions of the Dirac operator D on M_τ. Using these advanced/retarded fundamental solutions, it is possible to define a one-particle scattering operator $s = s(\tau, \lambda, \xi)$ on the solution space of the free Dirac equation on M_τ. Schematically, the action of this one-particle scattering operator can be described as follows. One chooses initial data for the free Dirac equation at $t = 0$, propagates those data forward in time with the dynamics of the free Dirac equation up to $t = \tau/1.25$; then, using $\mathcal{R}^{\pm}_{\tau,\lambda,\xi}$, one propagates the initial data backwards in time using the dynamics of $D + V_\xi(\lambda)$ up to $t = -\tau/1.25$, and then the resulting initial data are propagated forward in time back to $t = 0$, using the dynamics of the free Dirac equation. What we just described verbally is depicted in the diagram of Fig. 1, when following the arrows counterclockwise starting from the solid black line which represents the $t = 0$ hyperplane in M_τ. By standard arguments [23, 29], the one-particle scattering operator $s(\tau, \lambda, \xi)$ induces a C^*-algebraic Bogoliubov-transformation $\beta_{\tau,\lambda,\xi}$ on $\mathsf{F}(M_\tau)$,

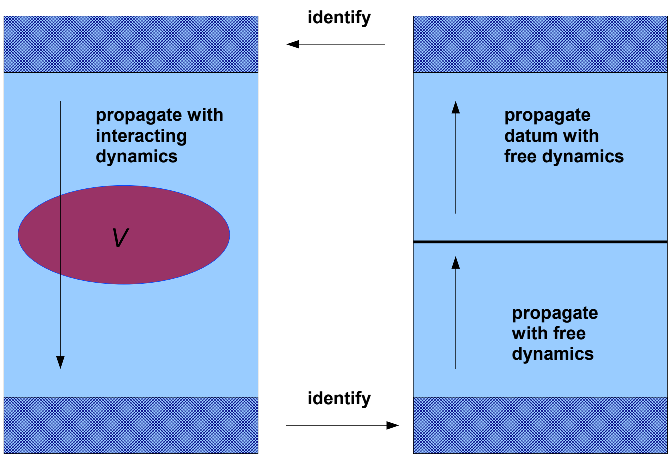

Fig. 1. Sketch of action of the one-particle scattering operator for the cut-off dynamics.

the CAR-algebra of the free Dirac field on M_τ (regarded as a spacetime in its own right), by setting

$$\beta_{\tau,\lambda,\xi}(\Psi(f)) = \check{\Psi}(s(\tau,\lambda,\xi)Rf)\,.$$

Differentiating with respect to λ, one obtains a derivation $\delta_{\tau,\xi}$ on $\mathsf{F}(M_\tau)$,

$$\begin{aligned}\delta_{\tau,\xi}(\Psi(f)) &= \frac{d}{d\lambda}\Big|_{\lambda=0}\beta_{\tau,\lambda,\xi}(\Psi(f))\\ &= \Psi(\xi(c\star(\xi Rf)\star c))\,.\end{aligned}$$

Finally, one can remove the cut-offs by letting $\tau\to\infty$ and $\xi\to 1$. The following result states that these limits are well-behaved.

Proposition 6.1 *(a) The limit of $\delta_{\tau,\xi}(\Psi(f))$ as $\tau\to\infty$, $\xi\to 1$ exists for each test-spinor $f\in\mathcal{S}(\mathbb{R}^4,\mathbb{C}^4)$. It defines a derivation δ on $\mathsf{F}(\mathbb{R}^4)$, the CAR-algebra of the free Dirac field on Minkowski spacetime, acting as*

$$\delta(\Psi(f)) = \Psi(c\star Rf\star c)\,.$$

(b) There is an essentially selfadjoint operator $Y(c)$ on the Wightman domain of the Fockspace $F_+(\mathcal{K}_+)$ such that

$$i[Y(c),\psi(f)] = \delta(\psi(f)) = \psi(c\star Rf\star c)\,,$$

where $\psi(f)$ is the vacuum representation of $\Psi(f)$ on the Fockspace $F_+(\mathcal{K}_+)$ in terms of creation and annihilation operators.

The proof of this proposition will appear in [28].

7. An answer to a previous question and some discussion

At the end of Sec. 4 we posed the question how one can retrieve information about the non-commutativity of the underlying Moyal–Minkowski spacetime from the quantized Dirac field constructed on it by abstract CAR quantization (of the GHYST describing Moyal–Minkowski spacetime), as apparently the construction of the quantized Dirac field is no other than in the case of usual Minkowski spacetime. In Sec. 5 we have seen that for the case of usual Minkowski-spacetime, for a scalar test-function c the passage from $\psi(f)$ to $\psi(cRf)$ is given by a derivation on the CAR algebra of the (free) Dirac field in vacuum representation,

$$\psi(cRf) = i\left[: \psi^+\psi : (c), \psi(f)\right] .$$

To be noted is, first, that the derivation is induced by a selfadjoint operator $: \psi^+\psi : (c)$, the quantized counterpart of the absolute square of field strength of the Dirac field, smeared with c. This operator thus gives a measure of the localization and strength of the external field inducing the scattering process. Secondly, the derivation is obtained by differentiating the S-matrix with respect to the field strength scaling parameter λ, and following the idea of Bogoliubov's formula [30], diffentiating an S-matrix of an interaction with respect to the interaction coupling strength is a general method of obtaining the observable quantum fields of a quantum field theory. For usual Minkowski spacetime, we view the test function c as an element of the commutative algebra $\mathcal{A}_0$ entering the data of the GHYST corresponding to Minkowski spacetime, and therefore, we view $Rf \mapsto cRf$ as the algebraic action of that algebra on a suitable module. This point of view we carried over, in Sec. 5, to the case of Moyal–Minkowski space (with commutative time): Here, the algebra $\mathcal{A}_0$ are the test-functions with the non-commutative Moyal product. The "module actions" are, therefore, modified to $Rf \mapsto c \star Rf + Rf \star c$ or $Rf \mapsto c \star Rf \star c$. (Strictly speaking, these are not module actions; the symmetrized form here is needed to ensure J-invariance of the resulting potential term in order to be able to obtain Bogoliubov transformations on the CAR algebra.) In the case of commutative time studied in Sec. 5 we could use a variation of the methods used to solve the scattering problem for a usual scalar potential to obtain a solution to the scattering problem for the NC scalar potential. This also leads to an S-matrix, and differentiating with respect to the field strength gives derivations induced by operators $\Phi_{\#}(c)$ such that

$$i[\Phi_{\star}(c), \psi(f)] = \psi(c \star Rf + Rf \star c)\,,$$
$$i[\Phi_{\star\star}(c), \psi(f)] = \psi(c \star Rf \star c)\,.$$

It is important here that the c appearing in the argument of $\Phi_{\#}(c)$ is to be viewed not just as a test-function, but as an element of the non-commutative algebra $\mathcal{A}_0$ of test-functions endowed with the Moyal product. Again, $\Phi_{\#}(c)$ is an observable measuring strength and localization of the external — and now, non-commutative — potential, where the localization is, due to the non-local action of the Moyal product, no longer as sharp as in the sense of localization on a usual differentiable manifold. Furthermore, the penultimate equation furnishes a link to the more heuristic approach to quantum field theory on Moyal–Minkowski spacetime where the usual product AB of quantum field operators is replaced by the Rieffel–Moyal product [31,32],

$$A \star_{\Theta} B = (2\pi)^{-n} \int\int \alpha_{\frac{1}{2}\Theta u}(A)\alpha_{-v}(B)e^{-iu\cdot v}\, d^n u\, d^n v\,,$$

where $\alpha.$ denotes the automorphic action $\alpha_y(\psi(f)) = \psi(f_y)$, $f_y(x) = f(x-y)$ of the translations on the operator algebra generated by the Dirac field in n-dimensional Minkowski spacetime. For the sake of simplicity, let us assume that the parameter θ appearing in the definition of Θ is equal to 2. Then we have, formally,

$$\psi\,(c \star Rf + Rf \star c) = i\left[:\psi^{+}\psi:(c), \psi(f)\right]_{\Theta} + i\left[:\psi^{+}\psi:(c), \psi(f)\right]_{-\Theta}\,.$$

The notation means that in the first commutator, the operator product is replaced by the product $\star_{\Theta}$, while in the second commutator, the operator product is replaced by the product $\star_{-\Theta}$. As the equation stands, it is only formal in nature because one cannot rely on the theorems in [31,32] for the existence of the product $\star_{\Theta}$ owing to the fact that $:\psi^{+}\psi:(c)$ is an unbounded operator and Θ is degenerate in the case of commutative time, so one would have to specify very carefully the domain on which the equality is valid. Relegating this technical question elsewhere, one can see that the Rieffel–Moyal product between quantum field operators appears naturally in the present setting, too. The results of Sec. 6 show that the vantage point just described can also be maintained in the case of non-commutative time. The central difficulty here is to define a dynamics for the interaction potential which now is non-commutative and hence, non-local in time, so that it cannot be formulated as a time-dependent Hamiltonian dynamics as in Sec. 5. Nevertheless, interpreting the dynamical problem in terms of a family of cut-off dynamics, one again obtains an operator $Y(c)$ which basically can be seen as the result of differentiating the S-matrix with respect to the interaction coupling strength of the non-commutative potential. Thus, it appears that in the case of Moyal–Minkowski spacetime (as a — strictly speaking, hypothetical — model for a GHYST) the CAR-quantization together with external scattering by a non-commutative potential and "Bogoliubov's formula" yields a correspondence between (Hermitean) elements c in the algebra $\mathcal{A}_0$ and observables $\Phi_{\#}(c)$ or $Y(c)$ of the quantized Dirac field, also establishing a relation to the Rieffel–Moyal product between quantum field operators. Despite the fact that in the construction of the S-matrix, or the related derivations with respect to the potential coupling strength, we have used some properties which are not direct consequences of the structure of a GHYST, like commutative time in Sec. 5 or the cut-off dynamics localized in time in Sec. 6, we are confident that our construction of a relation between elements of $\mathcal{A}_0$ and observable quantum field operators can in principle be extended to more general GHYSTs. This, of course, requires a better understanding of the structure of GHYSTs, and in particular, of concepts of localization in non-commutative geometry.

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