YANG–MILLS SPECTRUM WITH AN ARBITRARY SIMPLE GAUGE ALGEBRA*

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The mass spectrum of pure Yang–Mills theory in $3+1$ dimensions is discussed for an arbitrary simple gauge algebra within a quasigluon picture. The general structure of the low-lying gluelump and glueball spectrum is shown to be common to all algebras, excepted the lightest $C=-$ glueballs that only exist when the gauge algebra is $A_{r\geq 2}$. The shape of the static energy between adjoint sources is also discussed assuming the Casimir scaling hypothesis and finally, the obtained results are shown to be consistent with existing lattice data in the large-N limit of an $\mathfrak{su}(N)$ gauge algebra.

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1. Generalities

Although pure Yang–Mills (YM) theory with the gauge algebra $\mathfrak{su}(3)$ has logically been the most studied case so far, it can be in principle formulated for any gauge algebra. When this algebra is simple, YM theory exhibits asymptotic freedom, and a relevant question is: What is the global structure of the low-lying YM spectrum and does it strongly depend on the considered gauge algebra? This problem has been addressed within a quasigluon picture in [1], whose results are summarized hereafter. Note that such a framework has already proven to be remarkably successful in describing the low-lying glueball spectrum in pure gauge QCD [2].

The YM field is defined as $A^\mu = A^\mu_a T^a_{(r)}$, where $T^a_{(r)}$ denotes the generators of an arbitrary simple Lie algebra in the representation r. The charge-conjugated gluon field is $A^{\mathcal{C}}{}_\mu = -A^{\rm T}_\mu$, where T denotes the transposition of the $T^a_{(r)}$ matrices. In $3+1$ dimensions, the YM theory thus contains dim adj

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transverse massless vector particles, or gluons, where adj denotes the adjoint representation of the considered algebra. Then the quantum state of a given glueball made of n_g quasigluons reads $|n_g;\, J^{\rm PC}\rangle = |\text{colour}\rangle \otimes |\text{spin-space}\rangle$, the colour part being responsible for the charge conjugation and the spin-space part being responsible for the total spin and parity. Let us recall that the spin-space part of a given glueball state can be obtained and eventually expressed in a LS-basis by resorting to the helicity formalism [3].

The emergence of a constituent picture appears naturally in Coulomb gauge QCD [4], where the QCD Hamiltonian is written with the gluonic field in the Coulomb gauge, for which the elimination of the non-dynamical degrees of freedom generates an instantaneous non-perturbative interaction. One can show that the Coulomb gauge gluon mass gap equation leads to a gluon dispersion relation of the form $\omega^2(q) = q^2 + m_g(q)^2$, in which the dynamically generated mass is qualitatively independent of the gauge algebra and such that $m_g(0)$ is finite and positive [1]. The existence of such a mass justifies a Fock space expansion of gluonic states in terms of states with a given number of quasigluons (gluons with a dynamically generated mass).

2. Gluelumps and glueballs

Gluelumps are colour singlet bound states of the YM field in a static adjoint source defined as $\phi = \phi_a\, T^a_{(r)}$. Although not "physical" in the sense that they require the presence of an extra static source, gluelumps are nevertheless worth of interest since in QCD, ϕ can be thought as a pointlike, adjoint, static quark–antiquark pair [5]. So, the gluelump mass is *grosso modo* the binding energy of a heavy hybrid meson, and the lowest-lying gluelumps should be one-quasigluon states, the presence of the static source allowing to build a colour singlet. Indeed, the tensor product of the adjoint representation by itself has schematically the following structure for any gauge algebra

$$\text{adj} \otimes \text{adj} = \bullet^{\rm S} \oplus \text{adj}^{\rm A} \oplus \dots\,, \tag{1}$$

where the S (A) superscript denotes a(n) (anti)symmetric colour configuration, and where the singlet is represented by $\bullet$. The $\bullet^{\rm S}$ channel corresponds to the configuration $\delta_{ab}\phi^a A^b_\mu \propto {\rm Tr}(\phi A_\mu) = -{\rm Tr}(\phi^{\mathcal{C}} A^{\mathcal{C}}_\mu)$ and has always a negative charge conjugation. The helicity formalism imposes that $J \geq 1$ for gluelumps, among which the lightest states are obviously those with $J = 1$: They have the minimal rotational energy. The 1^{+-} gluelump, being dominated by a S-wave component, will be the lightest one, while the 1^{--} state is a pure P-wave and will be heavier, as already found in $\mathfrak{su}(3)$ lattice computations [5, 6]. According to these last works, the typical gluelump mass scale, *i.e.* $M_{\rm l} = (M_{1^{+-}} + M_{1^{--}})/2$, is roughly equal to 1 GeV.

We now turn to glueballs, which are colour singlet states made of quasigluons only. As shown by (1), one can build from two quasigluons the symmetric colour singlet $\delta_{ab}A^a_\mu A^b_\nu \propto \mathrm{Tr}(A_\mu A_\nu) = \mathrm{Tr}(A^{\mathcal{C}}_\mu A^{\mathcal{C}}_\nu)$, whose charge conjugation is positive. One finds four families of symmetrized two-quasigluon helicity states that will not be written here for the sake of simplicity. It is nevertheless worth saying that a look at their decomposition in a LS-basis immediately suggests the mass ordering $M_{0^{++}} < M_{2^{++}}, M_{0^{-+}}$, in agreement with what has been found in $\mathfrak{su}(3)$ lattice QCD [7]. Thus, any gauge algebra allows the existence of two-quasigluon glueballs with $C = +$, that should stand at the bottom of the glueball spectrum with a typical mass of $2\,M_{\mathrm{l}}$. This estimate is an immediate consequence of the quasigluon picture. As an illustration, it can be remarked that the lightest glueball masses are $M_{0^{++}} = 1.730(50)(80)$ GeV and $M_{2^{++}} = 2.400(25)(120)$ GeV, around $2\,M_{\mathrm{l}} \approx 2$ GeV. Notice that no 1^{P+} glueball is expected around $2\,M_{\mathrm{l}}$, regardless of the considered gauge algebra.

It is worth going a step further and discuss the properties of glueballs made of three quasigluons. The tensor product

$$\mathrm{adj} \otimes \mathrm{adj} \otimes \mathrm{adj} = \begin{cases} \bullet^{\mathrm{A}} & \oplus \quad \dots \\ \bullet^{\mathrm{A}} & \oplus \quad \bullet^{\mathrm{S}} \quad \oplus \dots \qquad \text{only for } A_{r\geq 2} \end{cases} \tag{2}$$

means that a totally antisymmetric colour singlet can always be formed, typically by using the structure constants. One gets the colour structure $f^{abc}A^\mu_a A^\nu_b A^\rho_c$ that can be shown to have $C = +$. A peculiar feature of the algebras $A_{r\geq 2}$ is that they possess a totally symmetric invariant tensor, generally denoted d^{abc}, that allows to build the totally symmetric colour singlet $d^{abc}A^\mu_a A^\nu_b A^\rho_c$, with $C = -$. The phenomenological relevance of this result is considerable since A_r in its compact real form is the algebra $\mathfrak{su}(r+1)$. The lowest-lying three-quasigluon states should have a mass around $3\,M_{\mathrm{l}}$. At such a mass scale one expects both excited two-gluon and low-lying three-gluon states to coexist (and probably significantly mix) in the $C = +$ sector. There, only 1^{P+} states could safely be interpreted as three-gluon ones. No glueball state around $3\,M_{\mathrm{l}}$ is present in the $C = -$ sector excepted when the gauge algebra is $A_{r\geq 2}$. A check of that result is that 1^{+-} and 3^{+-} states have been observed in $\mathfrak{su}(3)$ lattice QCD with a mass of 2.940(30)(140) GeV and 3.550(40)(170) GeV respectively [7], while no such states exist when $\mathfrak{su}(2)$ is used [8]. A summary plot is shown in Fig. 1.

3. Static energy

The static energy between coloured sources is an observable that is both accurately computable on the lattice and relevant in view of understanding the structure of confinement. Let us first focus on the potential energy

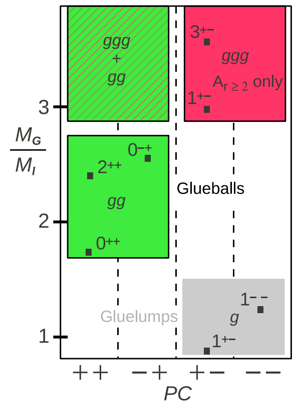

Fig. 1. (Colour online) Schematic representation of the YM spectrum for arbitrary simple gauge algebras. $M_{\rm G}$ denotes the mass of a given state, while $M_{\rm l}$ is the typical mass of the lightest gluelumps. Results from $\mathfrak{su}(3)$ lattice QCD have been indicated for comparison (squares) [5, 7]; the error bars are not shown for clarity.

between two adjoint sources. Since YM theory is confining, the long-range part of the potential energy should be given by a linear confinement (the QCD string), in which we assume the string tension to follow the Casimir scaling. The short-range part, however, should be dominated by one-gluon-exchange effects, and renormalization theory tells us that the strong coupling constant reads $\alpha_{\rm s} = \alpha_0/C_2^{(\rm adj)}$, where $C_2^{(\rm adj)}$ is the quadratic Casimir in the adjoint representation. The static energy is then expected to behave like $V_{2g}(R) = \sigma_0 R - \alpha_0/R$, where the values of σ_0 and α_0 could be measured on the lattice for any gauge algebras.

A particularly interesting case is that of the static energy between three adjoint sources. It can be read from (1) and (2) that the following colour structure exists for any algebra: $[[\mathrm{adj}, \mathrm{adj}]^{\mathrm{adj}^{\rm A}}, \mathrm{adj}]^{\bullet^{\rm A}}$, *i.e.* each pair is in the adjoint representation, while the three sources are in an antisymmetric colour singlet. Assuming that each source generates an adjoint flux tube, the long-range part of the static energy corresponding to the above colour configuration should be given by the so-called Y-junction potential $V_Y = \sigma_0 \sum_{i=1}^{3} |\boldsymbol{r}_i - \boldsymbol{Y}|$, where the source's positions are denoted $\boldsymbol{r}_i$. The point $\boldsymbol{Y}$ where the flux tubes meet is such that the sum $\sum_{i=1}^{3} |\boldsymbol{r}_i - \boldsymbol{Y}|$ is minimal. But, the Y-junction potential is not the only allowed possibility. Excepted for E_8 indeed, the adjoint representation is not the lowest-dimensional one, that is here called fundamental and denoted f. It can be checked that the adjoint representation appears in the tensor product $f \otimes f$ when the algebra is self-dual, and in the tensor product $f \otimes \bar{f}$ when the algebra is not self-dual, that is for $A_{r\geq 2}$ and E_6. This means that an adjoint source can always generate two fundamental (or a fundamental and a conjugate) flux tubes instead of an adjoint one, E_8 excepted. In the case where fundamental flux tubes are present, the long-range potential will be referred to as a Δ-potential, whose form is $V_\Delta = (C_2^{(f)}/C_2^{(\mathrm{adj})})\sigma_0 \sum_{i<j=1}^{3} |\boldsymbol{r}_i - \boldsymbol{r}_j|$.

When the three adjoint sources are located on the apices of an equilateral triangle, one can compute that $V_Y/V_\Delta = C_2^{(\mathrm{adj})}/(\sqrt{3}C_2^{(f)})$. If this ratio is > 1 (< 1), V_Δ (V_Y) is energetically favoured. For example, $V_Y/V_\Delta > 1$ for the A_r-family, and in particular for the gauge algebra $\mathfrak{su}(3)$ in agreement with the results of [9]. It appears that the gauge algebras for which the Y-junction is maximally favoured are E_7 and E_8, while a Δ-shape is maximally favoured in the case of A_1, A_2 and C_2.

4. Concluding comments: the large-N limit

As an application of the above discussion, the special case of the gauge algebra $\mathfrak{su}(N)$ (A_{N-1}) is worth being discussed since it is related to the large-N limit of QCD. In the recent work [10], it has been checked that the glueball spectrum obtained on the lattice is accurately described by $M_\mathrm{G}(N) = M_\mathrm{G}(\infty) + c_\mathrm{G}/N^2$ with c_G compatible with zero up to a few exceptions. In our quasigluon approach, both σ_0 and α_0 are independent of N by definition of the 't Hooft limit [1]. Consequently, nothing in the two-body part of an explicit Hamiltonian would depend on N, in agreement with a value of c_G compatible with zero. One can moreover check the remarkable independence of N for the masses of the lightest scalar, pseudoscalar and tensor glueballs in [10]. Another check of the quasigluon picture in the two-quasigluon sector is provided by the earlier data of [11]. In this last work, the scalar and tensor glueball masses are computed for different values of N but normalized to $\sqrt{\sigma^{(f)}}$. Since the 0^{++} and 2^{++} masses should be independent of N in a quasigluon picture, all the N-dependence will be contained in the normalization factor. Following the Casimir scaling hypothesis, one expects, for a two-quasigluon state,

$$\frac{M_{gg}}{\sqrt{\sigma^{(f)}}} = \sqrt{\frac{2N^2}{N^2-1}}\,\theta_{gg}\,. \tag{3}$$

The above formula compares favourably to the lattice data, see Fig. 2.

In the case of a three-quasigluon glueball, the static potential between three quasigluons can be either N-independent, V_Y, or N-dependent, V_Δ. The dependence (or not) on N arises at the level of the confining term, which contains the only dimensioned parameter of the system, that is the string tension. So the mass of a three-quasigluon state should be either constant if V_Y is used, or of the form

$$M^\Delta_{ggg} = \sqrt{\frac{N^2-1}{2N^2}}\theta_{ggg} \tag{4}$$

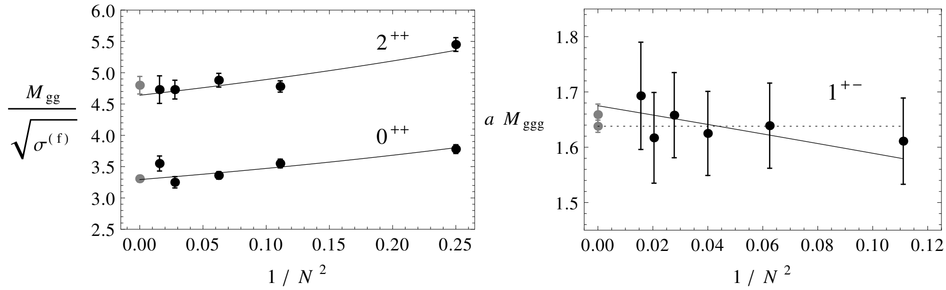

Fig. 2. Left panel: 0^{++} and 2^{++} masses normalized to the fundamental string tension computed on the lattice (black and gray points) for various N [11]. The lattice data are compared to formula (3) with $\theta_{gg} = 2.33$ and 3.28 for the 0^{++} and 2^{++} glueballs, respectively. Right panel: Lightest 1^{+-} mass in units of the lattice spacing computed on the lattice (black and gray points) for various N [10]. The lattice data are compared either to a constant mass $a\,M_{ggg} = 1.64$ (dotted line) or to formula (4) with $a\,\theta_{ggg} = 2.37$ (solid line).

if V_Δ is used. This last case is *a priori* favoured for the gauge algebra $\mathfrak{su}(N)$, as discussed in the previous section. The evolution of the 1^{+-} glueball mass *versus* N has been computed in [10]; it can be seen in Fig. 2 that both a constant mass and formula (4) are compatible with the current error bars. Nevertheless, this in an indication that further lattice calculations in the $C = -$ glueball sector could be very useful in order to disentangle the different models of confinement.

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