THERMODYNAMICS AT STRONG COUPLING FROM HOLOGRAPHIC QCD*

Francesco Nitti

Laboratoire APC, Université Paris VII
10 rue Alice Domon et Léonie Duquet, 75205 Paris, France

(Received July 26, 2011)

In the context of the AdS/CFT holographic correspondence, the thermodynamics of the deconfined phase of four-dimensional gauge theories is mapped to the thermodynamics of higher-dimensional black hole solutions of a dual gravity model. Here, I review the basic ingredients of the correspondence, and how one can construct simple semi-realistic holographic models that give a quantitatively good description of Yang–Mills thermodynamics.

DOI:10.5506/APhysPolBSupp.4.661
PACS numbers: 11.25.Tq, 11.10.Kk, 11.15.Tk, 12.38.Mh

1. Introduction

In the past ten years, the holographic gauge/gravity duality conjecture [1] has provided a new tool for the description of the nonperturbative physics of strongly coupled gauge theories. In this context, the nonperturbative gauge theory dynamics can be rephrased in terms of the dynamics of a gravitational system in a higher-dimensional curved space-time (the *gravity dual* description). Here, I will focus on the thermodynamics of the system at finite temperature. The thermodynamics of the gauge theory is mapped to the black hole thermodynamics in the gravity dual. One important goal is to give a dual description of the phase transition (or cross-over) to a high temperature, strongly coupled deconfined phase, such as the quark-gluon plasma. As I will review, it is possible to construct phenomenological holographic models which provide a quantitatively accurate description of the deconfining transition and the thermodynamics of the high temperature phase of the pure Yang–Mills theory[1].

* Presented at the Workshop "Excited QCD 2011", Les Houches, France, February 20–25, 2011.

[1] This approach may be considered of the "third kind" (or *creative*), according to the classification given in [2] of the possible ways to tackle non-perturbative aspects of QCD.

2. Lightning review of gauge/gravity duality

The *gauge/gravity duality* (or AdS/CFT correspondence) is the conjectured equivalence between a gauge theory in D space-time dimensions and a gravitational theory in a $D+p$-dimensional curved space-time. The first and best understood example is the conjectured equivalence [3] between large-N, maximally Supersymmetric SU(N) Yang–Mills theory ($\mathcal{N}=4$ SYM) in four dimensions, and Type IIB string theory on the ten-dimensional space-time given by the product of five-dimensional anti-de Sitter and a five-sphere ($\mathrm{AdS}_5 \times S^5$), whose metric is

$$ds^2 = \left(\frac{\ell}{r}\right)^2 \left(dr^2 + \eta_{\mu\nu} dx^\mu dx^\nu\right) + \ell^2 d\Omega_5^2 \,. \tag{1}$$

The parameter ℓ sets the curvature of AdS and the radius of the S^5. The boundary of this space-time at $r=0$ is conformal to the flat 4D Minkowski space-time, parametrized by the coordinates x^μ, where the SYM theory is defined[2]. The non-compact coordinate r is dual to the renormalization group energy scale in the gauge theory, the UV being mapped to the $r \sim 0$ region. The fact that the metric (1) has the scaling isometry $(r, x^\mu) \to (\alpha r, \alpha x^\mu)$ reflects the exact conformal invariance of the $\mathcal{N}=4$ SYM theory (hence the term AdS/CFT correspondence). The usefulness of the duality lies in the fact that, when the SYM coupling is large, string theory reduces to classical supergravity (the AdS curvature is small in string and 10D Planck units). This allows to compute strong coupling observables using classical general relativity.

Many generalizations of these correspondence have been studied, involving less supersymmetric and non-conformal theories. In the following, I will concentrate on the non-compact, 5D part of the bulk space-time, parametrized by (r, x^μ), and I will consider non-conformal situations that allow to get closer to real QCD.

The ingredient that makes the correspondence a calculational tool is the *field/operator correspondence:* to any field $\Phi(r, x^\mu)$ propagating in the bulk, one can associate a gauge-invariant operator $O(x^\mu)$ in the boundary field theory[3]. The boundary value of Φ is interpreted as an external source for O, in the sense that the QFT action gets deformed by a term $\int d^4x \bar{\Phi}(x) O(x)$, where $\bar{\Phi}(x) \sim \Phi(x, r=0)$. Correlation functions for the operator $O(x)$ can be then calculated by using the GKPW prescription [4]

[2] This is at the origin of the term *holographic correspondence:* the 10D bulk dynamics is completely encoded in the lower-dimensional boundary field theory.

[3] This applies in particular to the bulk metric, whose associated operator is the QFT stress-tensor.

$$\mathcal{Z}_{\mathrm{QFT}}\left[\bar{\Phi}\right] = \exp iS_{\mathrm{grav}}[\Phi_{\mathrm{cl}}(x)]\Big|_{\Phi_{\mathrm{cl}}\to\bar{\Phi}} . \tag{2}$$

The left-hand side is the generating functional of correlation functions for $O(x)$, while the right-hand side contains the action S_{grav} that gives the bulk dynamics of field $\Phi(x,r)$, evaluated on a solution of the bulk field equations with fixed boundary condition specified by $\bar{\Phi}(x)$ (thus both sides are functionals of $\bar{\Phi}(x)$).

3. Holographic phase transitions

To go to finite temperature, the standard procedure is to pass to Euclidean time, compactified with period $\beta = 1/T$. Then, the relevant quantity becomes the partition function $\mathcal{Z}(\beta)$, and the relation (2) becomes

$$\mathcal{Z}(\beta) = \exp\left[-\beta\mathcal{F}\right] , \qquad \beta\mathcal{F} = S_{\mathrm{grav}}\left[g^0_{ab}, \Phi^0\right] , \tag{3}$$

where now the r.h.s. is written in terms of the Euclidean action evaluated on a solution of Einstein's equations (g^0_{ab}, Φ^0) that satisfies the appropriate periodicity in time (here, I made explicit the bulk metric g_{ab} as part of the bulk fields). The quantity $\mathcal{F}(T)$ is interpreted as the free energy of the thermal equilibrium state in the QFT that corresponds to that solution.

It may happen that, for a given inverse temperature β, there are several classical solutions $(g^0_{ab,i}, \Phi^0_i)$ of the gravitational field equation. In this case, the r.h.s. must be replaced, in the semiclassical limit, by a sum over saddle points of the action

$$\mathcal{Z}(\beta) = e^{-\beta\mathcal{F}_1} + e^{-\beta\mathcal{F}_2} + \dots , \qquad \beta\mathcal{F}_i(T) \equiv S_{\mathrm{grav}}\left[g^0_{ab,i}, \Phi^0_i\right] . \tag{4}$$

Each solution corresponds to a different thermal equilibrium state in the dual QFT, and one can have first order phase transitions if it happens that $\mathcal{F}_1(T) - \mathcal{F}_2(T)$ changes sign at some critical temperature T_{c}. Also, from the free energy we can compute other equilibrium quantities (such as entropy, susceptibilities, *etc.*) by using standard thermodynamical formulae[4].

Among the solutions on the gravity side, a particular role is played by black hole solutions. They are thermal equilibrium solutions[5] that come with a Hawking temperature T_{H} which can be naturally interpreted as the equilibrium temperature on the field theory side. Moreover, they correspond, on the field theory side, to a *deconfined phase*, as can be seen *e.g.* by computing holographically the v.e.v. of the Polyakov loop operator [5], and by the

---

[4] The reason that this works, is ultimately insured by the fact that gravitational solutions such as black holes are thermodynamical objects.

[5] Black holes in asymptotically AdS space-times can be thermodynamically stable, unlike asymptotically flat Schwarzschild black holes.

fact that they display a spectrum of quasi-normal modes, analogous to what one expects in a deconfined plasma. Thus, a deconfinement phase transition appears in the holographic context as the transition between black hole solutions that dominate at high temperature, and low temperature solutions which do not contain black holes.

In the following subsections, I will describe several examples of such phase transitions in theories of pure gravity; in Sec. 4, I will discuss the phenomenological Einstein-dilaton model we studied in [6] and used in [7] to match the thermodynamics of the Yang–Mills plasma.

3.1. AdS in Poincaré coordinates

Let us first consider the case of pure 5D gravity with a negative cosmological constant

$$S_{\rm grav} = -M_p^3 \int d^5x\, \sqrt{g} \left(R + \frac{12}{\ell^2} \right) . \tag{5}$$

We are interested in Euclidean solutions that preserve 3D spatial rotations and translations: this is the symmetry of the field theory, the full 4D symmetry being broken by temperature. There are two kinds of such solutions:

- Euclidean AdS in Poincaré coordinates, whose metric is

$$ds^2 = \left(\frac{\ell}{r} \right)^2 \left(dr^2 + d\tau^2 + d\vec{x}^2 \right) . \tag{6}$$

  At finite temperature this represents a thermal gas of the excitations over the vacuum state, the latter being represented by the same metric but with $\beta \to \infty$ (τ decompactified).

- AdS black holes, with metric

$$ds^2 = \left(\frac{\ell}{r} \right)^2 \left(\frac{dr^2}{f(r)} + f(r)d\tau^2 + d\vec{x}^2 \right) , \qquad f(r) = 1 - (\pi T r)^4 . \tag{7}$$

  The black hole horizon is related to the temperature T as $T = (\pi r_{\rm h})^{-1}$, and the entropy per unit 3D volume is proportional to $(\ell/r_{\rm h})^3$. This solution is holographically dual to a deconfined fluid.

To know whether there is a phase transition between the two classes of solutions, one could compute the free energy difference by evaluating the action (5) on the AdS and black hole solutions at a given temperature. However, there is a shortcut to avoid such calculation [6]. By integrating the thermodynamic relation between free energy and entropy, $\mathcal{F} = -\partial S/\partial T$, one can show that in general, if the temperature is monotonic as a function of the horizon position r_h, *and* the family of black hole solutions smoothly connects to the solution without a horizon, (as in this case, by taking $T \to 0$), then the black hole free energy cannot change sign. Thus, in the case at hand, there cannot be a first order phase transition at a finite T_c (this can be seen as a consequence of the exact conformal symmetry of the zero-temperature theory). The theory always prefers to be in the black hole state, and the free energy of this phase is that of a conformal gas, $\mathcal{F} = -cT^4$, where the constant c equals 3/4 of the value for a free relativistic gas.

3.2. Global AdS

To see an example of a theory that does display a phase transition, consider now the same model, (5), but writing the AdS and metric in global coordinates. The AdS and black hole solutions are now

$$ds^2 = \left(\frac{\ell^2}{r^2}\right)\left(f(r)d\tau^2 + \frac{dr^2}{f(r)} + \ell^2 d\Omega_3^2\right), \qquad f(r) = 1 + \frac{r^2}{\ell^2} - \frac{G_5 M}{\ell^6} r^4 , \tag{8}$$

where G_5 is the 5D Newton constant. For $M = 0$ this metric describes AdS space in global coordinates. This is locally the same solution as (6), but now the boundary $r = 0$ is conformal to $S^1_\beta \times S^3$, rather than $S^1_\beta \times R^3$. This means that the dual field theory is still $\mathcal{N} = 4$ SYM, but living on a spatial 3-sphere[6] of radius ℓ, rather than in flat space.

The introduction of a scale (the 3-sphere radius) allows the emergence of a deconfining phase transition. In fact, now the relation between temperature and horizon position is not one-to-one, as we have

$$T(r_h) = \frac{1}{\pi r_h} + \frac{r_h}{2\pi\ell^2} . \tag{9}$$

The above equation shows that there are no black holes below a certain minimal temperature $T_{\min}$ and for $T > T_{\min}$ there are always two black holes, with different horizon radius (thus different entropy and free energy). By the same general argument mentioned in the previous subsections, it follows that in such a situation one *does* find a first order phase transition when the temperature is a non-monotonic function of r_h, with a critical temperature $T_c > T_{\min}$. One can check this by explicit calculation of the free energy, as was done originally by Hawking and Page [8]. Also, this is indeed the expected behavior of $\mathcal{N} = 4$ SYM on a S^3.

[6] Unlike the R^3 counterpart, at $T = 0$ this theory is in fact confining, for topological reasons.

3.3. Hard wall AdS/QCD

In the previous example, we saw that introducing a scale in the theory (in this case, changing the geometry of space) can lead to a phase transition. One may ask whether we can get a phase transitions, when the gauge theory side is in *flat* space. To achieve this, we must find another way to introduce a scale in the theory.

The simplest way to do this while keeping a flat spatial geometry, is to consider what is called the AdS/QCD hard wall model [9]. Here the metric is again AdS in Poincaré coordinates, (6), but this time we restrict the range of r to the interval $(0, r_{\rm IR})$. In other words, the scale is introduced by cutting-off the bulk in the IR. This makes the theory confining at $T = 0$.

At finite T, no static black hole solutions can be found for $r_{\rm h} > r_{\rm IR}$ (or for $T < T_{\rm min} = (\pi r_{\rm IR})^{-1}$) . So black holes have a minimum temperature, as in the global AdS case. For $T > T_{\rm min}$ ($r_{\rm h} < r_{\rm IR}$), one still finds a single black hole (whose metric is exactly the same as (7), with $r_{\rm h} < r_{\rm IR}$.).

A direct calculation[7] of the black hole free energy [10] shows that the latter takes the form $\mathcal{F} = \mathcal{F}_0 - cT^4$, where c is the same constant as for Poincaré AdS, and $\mathcal{F}_0 \propto T^4_{\rm min}$. The additive term comes from an explicit contribution in the IR. This means that we have a first-order phase transition at $T_{\rm c} = (\mathcal{F}_0/c)^{1/4}$.

Although interesting, there is a number of drawbacks in this model: (1) the cut-off is introduced by hand, and not dynamically; (2) the thermodynamics is still too close to that of a conformal field theory (*e.g.* the entropy density is $\sim$ constant $\times\, T^3$), thus not a good match for lattice Yang–Mills theory; (3) the existence of the phase transition depends crucially on the IR boundary conditions. If those are modified, the transition may disappear.

We will see in the next section how to construct holographic models that are immune to these problems.

4. Phase transitions in 5D Einstein-dilaton gravity

From the previous sections we have learned that to have a first order phase transition, the theory must have an infrared scale. But in order to get closer to the thermodynamics of QCD (more simply Yang–Mills theory) in flat space, we need to complicate the model beyond pure gravity. This is to be expected, since 4D YM contains at least another dimension 4 gauge-invariant operator, namely $\mathrm{Tr}F^2$, beyond the stress tensor. This means that the holographic dual should have at least one scalar field which couples to this operator. Moreover, a non-trivial profile for this scalar field will break

[7] Since the black holes do not connect smoothly to the zero-temperature situation, the argument based on black hole thermodynamics does not apply here.

scale-invariance, and, in fact, it should correspond holographically to the Yang–Mills coupling constant, whose running breaks scale invariance in the gauge theory already at the perturbative level.

If we want to keep the model minimal, we are led to consider the following 5D action for gravity coupled to a scalar field[8]

$$S = -M_p^3 \int d^5x \left[R - \tfrac{4}{3}(\partial\Phi)^2 + V(\Phi)\right] . \tag{10}$$

As shown in [12], in such a theory one can naturally incorporate the holographic realisation of asymptotic freedom and confinement, by choosing $V(\Phi)$ appropriately.

At finite temperature, the general solutions of the coupled Einstein-scalar field equations, preserving the 3D spatial Euclidean symmetry are of two types (up to diffeomorphisms):

- the thermal gas solution,

$$ds^2 = b_o(r)\left(dr^2 + d\tau^2 + d\vec{x}^2\right) , \qquad \Phi(r, x^\mu) = \Phi_o(r) , \tag{11}$$

  which corresponds to a thermally excited confined phase;

- the black hole solutions,

$$ds^2 = b(r)\left(\frac{dr^2}{f(r)} + f(r)d\tau^2 + d\vec{x}^2\right) , \qquad \Phi(r, x^\mu) = \Phi(r) , \tag{12}$$

  with $f(r)$ such that $f(r_{\mathrm{h}}) = 0$, which correspond to a deconfined phase.

We demand that in the UV, both types of metric become equal, and are asymptotically AdS, *i.e.*

$$b(r) \sim b_0(r) \to \ell/r , \qquad f(r) \to 1 , \qquad r \to 0 . \tag{13}$$

This is guaranteed if we demand that $V(\Phi)$ has a regular asymptotic expansion near $e^\Phi \simeq 0$, *i.e.* $V \simeq V_0 + V_1 e^\Phi + \ldots$ for $\Phi \to -\infty$. In this case, $\Phi \to -\infty$ maps to the high energy regime. The asymptotic AdS length ℓ is then given by $\ell = \sqrt{12/V_0}$. The non-trivial potential implies a departure of the metric from AdS, thus a breaking of scale invariance (which is however recovered in the UV, as in QCD).

[8] For a review of the holographic properties of these models, see [11].The non-canonical normalisation of the scalar kinetic term is a matter of convention.

The nonperturbative behavior, including the presence of a deconfining phase transition, is determined instead by the form of the potential at large Φ, *i.e.* in the IR. The most interesting case consists of potentials $V(\Phi)$ that for large Φ behave as

$$V(\Phi) \sim \Phi^{(\alpha-1)/\alpha} e^{4/3\Phi} . \tag{14}$$

It turns out [12] the zero-temperature theory is confining if and only if $\alpha \geq 1$. Remarkably, as shown in [6], the phase diagram of black hole solutions exhibits qualitatively different behavior according precisely to the same criterion:

- If $\alpha < 1$ there is a single black hole for any temperature, and $T(r_{\rm h})$ is a monotonic function. The situation is like in Poincaré-AdS, and there is no phase transition.
- If $\alpha > 1$ there are two black hole solutions above a (non-zero) minimum temperature $T_{\rm min}$, and none below. The situation is like in AdS in global coordinates: one finds a Hawking–Page-like phase transition at a finite $T_{\rm c} > T_{\rm min}$, but this time the spatial geometry is flat[9].

Thus, in this class of models, we have established that confinement at zero temperature is always associated with a deconfining first order phase transition. Moreover, the thermodynamics at high temperature reduces to that of AdS black holes, due to the form of the metric in the UV. Thus these models have all the qualitative thermodynamic feature to interpolate between the region close to the deconfining transition and the correct high-temperature regime. This agreement can be made quantitative by an appropriate choice of the potential $V(\Phi)$ [7].

5. Final remarks

The holographic setups I discussed here are examples of *phenomenological holographic models:* they are designed to capture *some* features of 4D QCD, while staying reasonably simple and calculable. On the other hand, to *truly* describe QCD holographically, one should need a full fledged string theory, not limited to just a 2-derivative action. The reason is that there is a single scale in QCD, thus in the holographic dual we cannot expect the massive string excitations to be parametrically heavier (thus decoupled) from the lowest lying gravity states.

In this respect, the models described here should be thought of effective models that provide an alternative approach to non-perturbative physics, and contain some adjustable quantities (*e.g.* the potential $V(\Phi)$ in the

[9] In the limiting $\alpha = 1$ case there is a single black hole solution above $T_{\rm min}$, but the transition is still present.

Einstein-scalar models) to be fixed phenomenologically. If we take this point of view, the fact that some of these models reproduce *e.g.* the Yang–Mills thermodynamics, is not really the point: lattice methods can already achieve the same. Rather, what is interesting is that, once a reasonably realistic holographic model has been developed, it can be used to describe or predict regimes which are out of the reach of other methods. The most important extension is to hydrodynamics and, more generally, out of equilibrium properties of the deconfined phase, which are very hard to describe on the lattice, but to which the holographic approach is easily applicable.

REFERENCES

[1] O. Aharony *et al.*, *Phys. Rep.* **323**, 183 (2000) [arXiv:hep-th/9905111].

[2] M. Nahrgang, *Acta Phys. Pol. B Proc. Suppl.* **4**, 609 (2011), this issue.

[3] J. M. Maldacena, *Adv. Theor. Math. Phys.* **2**, 231 (1998) [arXiv:hep-th/9711200].

[4] S.S. Gubser, I.R. Klebanov, A.M. Polyakov, *Phys. Lett.* **B428**, 105 (1998) [arXiv:hep-th/9802109]; E. Witten, *Adv. Theor. Math. Phys.* **2**, 253 (1998) [arXiv:hep-th/9802150].

[5] E. Witten, *Adv. Theor. Math. Phys.* **2**, 505 (1998) [arXiv:hep-th/9803131].

[6] U. Gursoy, E. Kiritsis, L. Mazzanti, F. Nitti, *Phys. Rev. Lett.* **101**, 181601 (2008) [arXiv:0804.0899 [hep-th]]; *J. High Energy Phys.* **0905**, 033 (2009) [arXiv:0812.0792 [hep-th]].

[7] U. Gursoy, E. Kiritsis, L. Mazzanti, F. Nitti, *Nucl. Phys.* **B820**, 148 (2009) [arXiv:0903.2859 [hep-th]].

[8] S.W. Hawking, D.N. Page, *Commun. Math. Phys.* **87**, 577 (1983).

[9] J. Erlich, E. Katz, D.T. Son, M.A. Stephanov, *Phys. Rev. Lett.* **95**, 261602 (2005) [arXiv:hep-ph/0501128]; L. Da Rold, A. Pomarol, *Nucl. Phys.* **721**, 79 (2005) [arXiv:hep-ph/0501218].

[10] C.P. Herzog, *Phys. Rev. Lett.* **98**, 091601 (2007) [arXiv:hep-th/0608151].

[11] U. Gursoy *et al.*, *Lect. Notes Phys.* **828**, 79 (2011) [arXiv:1006.5461 [hep-th]].

[12] U. Gursoy, E. Kiritsis, *J. High Energy Phys.* **0802**, 032 (2008) [arXiv:0707.1324 [hep-th]]; U. Gursoy, E. Kiritsis, F. Nitti, *J. High Energy Phys.* **0802**, 019 (2008) [arXiv:0707.1349 [hep-th]].