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STRUCTURAL STOCHASTIC MULTIRESONANCE IN THE ISING MODEL ON TWO COUPLED SCALE-FREE NETWORKS*

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The phenomenon of structural stochastic multiresonance is studied in the Ising model on a composite network consisting of two coupled scale-free subnetworks with, possibly, different critical temperatures for the ferromagnetic transition, driven by a weak, slowly oscillating magnetic field. Theoretical results obtained from the linear response theory in the mean-field approximation and numerical results from Monte Carlo simulations yield qualitatively similar results. The spectral power amplification, evaluated from the time-dependent order parameter, as a function of temperature exhibits two or, possibly, three maxima, depending on the exponent in the power-law tails of the degree distributions of the subnetworks and the fraction of inter-network edges. For small to moderate fraction of inter-network edges sharp maxima occur at temperatures close to the critical ones for the Ising model on the individual subnetworks. For densely coupled networks the spectral power amplification is determined mainly by the response of the spins on the subnetwork with higher critical temperature to the oscillating magnetic field.

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1. Introduction

Stochastic resonance (SR) [1, 2, 3] is a phenomenon where noise plays a constructive role by enhancing response of a nonlinear system to a periodic signal (for review see Ref. [4, 5, 6]). This response can be characterized, *e.g.*, by the spectral power amplification (SPA), defined as the strength of the Fourier component of the output signal at the frequency of the input signal

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divided by the strength of the input signal, which exhibits a maximum at non-zero noise intensity. An interesting extension of SR is stochastic multiresonance (SMR), where the response to the periodic signal is enhanced for many different values of the noise intensity, which results in multiple maxima of the SPA [7, 8, 9]. In particular, SR was studied in complex systems, including the Ising model, with a weak periodic magnetic field as the input signal, time-dependent magnetization as the output signal, and thermal fluctuations playing the role of noise. Typically, the response of the Ising model on regular arrays [10, 11, 12] to the oscillating magnetic field was maximum in the vicinity of the critical temperature for the ferromagnetic transition due to the divergence of the magnetic susceptibility, while that of the Ising model on complex networks [13, 14, 15, 16, 17] in some cases could be strong at temperatures much below the critical one, in the ferromagnetic phase.

It is known that many weblike structures as the Internet, world-wide web, power supply networks, *etc.*, which are of high importance for the modern society, have scale-free (SF) topology, *i.e.*, their degree distribution (distribution of the number of edges, or connections, per node) obeys a power scaling law $p_k \propto k^{-\gamma}$, usually with $\gamma > 2$. SF networks belong to a general class of complex networks whose study is a rapidly developing area in statistical physics (for review see Ref. [18,19,20]). Namely in the Ising model on SF networks with $2 < \gamma < 3$ under certain assumptions strong response to the oscillating magnetic field is observed both at the critical temperature for the ferromagnetic transition and at lower temperature in the ordered phase, which results in the occurrence of two maxima of the SPA evaluated from the time-dependent order parameter as a function of temperature [16, 17]. This is an example of so-called structural SMR, since the origin of the two maxima of the SPA can be traced back to the topological properties of the SF network.

In this paper, properties of SR in the Ising model on a complex composite network which consists of two coupled SF subnetworks are studied; the subnetworks can have different numbers of nodes and scaling exponents γ, thus, possibly, different critical temperatures for the Ising model defined on them. Such a composite network can be treated as the simplest example of a modular network, with dense links between nodes belonging to the same subnetwork, and more sparse inter-network connections. Structural SMR is again observed, with two or, possibly, three maxima of the curve SPA *vs.* temperature. It is shown that the response of the Ising model on the composite network to the oscillating magnetic field is a weighted combination of responses of the spins on the two subnetworks, which can be maximum at different temperatures. Inputs to the SPA from the subnetworks can be identified both for weak and strong coupling between subnetworks. This explains the appearance of the multiple maxima of the SPA.

2. The model and methods of analysis

The Ising model with ferromagnetic coupling on a complex network with N nodes and with the degree distribution p_k consists of $i = 1, 2, \ldots N$ spins with two possible orientations $\sigma_i = \pm 1$ located in the nodes and subjected to thermal noise. The exchange integral between the spins σ_i, σ_j is $J_{ij} = J > 0$ if there is an edge between nodes i, j, and $J_{ij} = 0$, otherwise. In order to observe SR, the input periodic signal in the form of the oscillating magnetic field $h(t) = h_0 \sin \omega_0 t$ is applied to all spins. The Hamiltonian for the model is

$$H = -\frac{1}{\langle k \rangle} \sum_{i,j=1}^{N} J_{ij} \sigma_i \sigma_j - h_0 \sin \omega_0 t \sum_{i=1}^{N} \sigma_i \,, \tag{1}$$

where $\langle k \rangle$ is the average degree of nodes in the whole network. In this paper, the complex networks under study are composite networks, which consist of two subnetworks A, B with N_A, N_B nodes and the degree distributions $p_k^{(A)}$, $p_k^{(B)}$, respectively, so that $N = N_A + N_B$; then, the sum in Eq. (1) runs over all pairs of spins in both subnetworks. The model obeys the Glauber thermal-bath dynamics, with the transition rates between two spin configurations which differ by a single flip of one spin, *e.g.*, that in the node i, in the form

$$w_i(\sigma_i) = \frac{1}{2} \left[1 - \sigma_i \tanh \left(\frac{I_i(t)}{T} \right) \right] , \tag{2}$$

where

$$I_i(t) = \frac{J}{\langle k \rangle} \sum_{j \in \mathcal{K}_i} \sigma_j(t) + h_0 \sin \omega_0 t \tag{3}$$

is a local field acting on the spin i (with degree k_i) at time t, T is the temperature, and the sum in Eq. (3) runs over all neighbours of the node i in the whole (composite) network.

In order to observe SR, the output signal is assumed as the time-dependent order parameter $S(t)$ [21]

$$S(t) = (N \langle k \rangle)^{-1} \sum_{i=1}^{N} k_i \sigma_i(t) \,. \tag{4}$$

In the case of a composite network the time-dependent order parameters for each of the two subnetworks can be also analyzed as auxiliary output signals, *i.e.*

$$S^{(A)}(t) = (N_A \langle k \rangle_A)^{-1} \sum_{i=1}^{N_A} k_i \sigma_i^{(A)}(t) \,, \tag{5}$$

for the subnetwork A and similarly defined $S^{(B)}$ for the subnetwork B ($\langle k\rangle_A$ is the average degree of nodels in the subnetwork A). The SPA is evaluated from the output signal as

$$\mathrm{SPA} = |P_1|^2/h_0^2\,, \qquad P_1 = \lim_{\tau\to\infty}\frac{1}{\tau}\sum_{t=0}^{\tau-1} S(t)e^{-i\omega_0 t}\,, \tag{6}$$

and the dependence of the SPA on the temperature T is analyzed. Similarly, $\mathrm{SPA}_A(T)$, $\mathrm{SPA}_B(T)$ can be obtained for each subnetwork from $S^{(A)}(t)$, $S^{(B)}(t)$, respectively.

3. Mean field approximation

Let us consider the Ising model on a composite complex network consisting of two subnetworks. In the composite network there are both intra-network edges (connecting nodes belonging to the same subnetwork) and inter-network edges (connecting nodes belonging to different subnetworks). Let us assume that the probability that a node from the subnetwork A (B) is linked to a node in the subnetwork B (A) is $\pi^{(A)}$ ($\pi^{(B)}$), *i.e.*, it is constant for all nodes and does not depend on their degrees. Then the stationary value of the order parameter and the critical temperature for the possible order-disorder transition for the model in Sec. 2 can be evaluated in the mean-field (MF) approximation.

The Master equation for the probability that at time t the system is in the spin configuration $(\sigma_1, \sigma_2, \ldots \sigma_N)$ is

$$\begin{aligned} &\frac{d}{dt}P\left(\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots \sigma_N; t\right) = \\ &-\sum_{j=1}^{N} w_j\left(\sigma_j\right) P\left(\sigma_1, \sigma_2, \ldots, \sigma_j, \ldots \sigma_N; t\right) \\ &+\sum_{j=1}^{N} w_j\left(-\sigma_j\right) P\left(\sigma_1, \sigma_2, \ldots, -\sigma_j, \ldots \sigma_N; t\right)\,. \end{aligned} \tag{7}$$

Multiplying both sides of Eq. (7) by σ_i and performing an ensemble average, denoted by $\langle\ \rangle$, yields

$$\frac{d\langle\sigma_i\rangle}{dt} = -\langle\sigma_i\rangle + \left\langle \tanh\left(\frac{I_i(t)}{T}\right)\right\rangle\,. \tag{8}$$

The above N equations can be separated into two groups of N_A and N_B equations for the mean values of spins belonging to the sunbnetworks A, B, respectively,

$$\frac{d\left\langle\sigma_i^{(A)}\right\rangle}{dt} = -\left\langle\sigma_i^{(A)}\right\rangle + \left\langle\tanh\left(\frac{I_i^{(A)}(t)}{T}\right)\right\rangle \tag{9}$$

(and similar equations for $\sigma_i^{(B)}$), where, formally,

$$I_i^{(A)}(t) = I_i(t) = \frac{J}{\langle k\rangle}\left(\sum_{j\in\mathcal{K}_i^{(A)}}\sigma_j^{(A)}(t) + \sum_{l\in\mathcal{K}_i^{(B)}}\sigma_l^{(B)}(t)\right) + h_0\sin\omega_0 t\,. \tag{10}$$

Following the argument in Ref. [22], the nodes of the subnetworks can be divided into classes according to their degrees k, and the spins located in the nodes of the subnetwork A (B) belonging to the class with degree k are approximated by their average values $\langle\sigma_k^{(A)}\rangle$ ($\langle\sigma_k^{(B)}\rangle$). Replacing the sums over the nodes of the subnetworks with the sums over the classes of nodes, yields the following approximate MF values of the order parameters $S^{(A)}(t)$, $S^{(B)}(t)$ for the subnetworks

$$S^{(A)}(t) \approx \langle S^{(A)}(t)\rangle = \sum_{k=m}^{k_{\max}^{(A)}}\frac{kp_k^{(A)}}{\langle k\rangle_A}\left\langle\sigma_k^{(A)}(t)\right\rangle \tag{11}$$

(and similar equation for $S^{(B)}(t)$), where $k_{\max}^{(A)}$ ($k_{\max}^{(B)}$) are maximum degrees of nodes in the subnetwork A (B), respectively, and m is the minimum degree, for simplicity assumed equal for both subnetworks.

Similarly, taking into account that the probability that a link from a node i with degree k_i, belonging to the subnetwork A, points to a node with degree k, belonging to the subnetwork A, is $\left(1-\pi^{(A)}\right)kp_k^{(A)}/\langle k\rangle_A$, and to a node with degree k, belonging to the subnetwork B, is $\pi^{(A)}kp_k^{(B)}/\langle k\rangle_B$, the MF value of the local field in Eq. (10) becomes

$$\begin{aligned}\left\langle I_i^{(A)}(t)\right\rangle &= \frac{Jk_i}{\langle k\rangle}\left[\left(1-\pi^{(A)}\right)\sum_{k=m}^{k_{\max}^{(A)}}\frac{kp_k^{(A)}}{\langle k\rangle_A}\left\langle\sigma_k^{(A)}(t)\right\rangle\right.\\ &\left.+\,\pi^{(A)}\sum_{k=m}^{k_{\max}^{(B)}}\frac{kp_k^{(B)}}{\langle k\rangle_B}\left\langle\sigma_k^{(B)}(t)\right\rangle\right] + h_0\sin\omega_0 t\\ &= \frac{Jk_i}{\langle k\rangle}\left[\left(1-\pi^{(A)}\right)\left\langle S^{(A)}(t)\right\rangle + \pi^{(A)}\left\langle S^{(B)}(t)\right\rangle\right] + h_0\sin\omega_0 t \end{aligned}\tag{12}$$

(in a similar equation for $\langle I_i^{(B)}(t)\rangle$ superscripts (A), (B) should be interchanged).

Inserting this approximation on the right-hand side of Eq. (9), multiplying both sides by k_i, performing the sum over all nodes of the subnetwork A and replacing it with a sum over the classes of nodes, the equation for the continuous-time dynamics of $\langle S^{(A)}(t)\rangle$ is finally obtained (equation for the dynamics of $\langle S^{(B)}(t)\rangle$ can be obtained in a similar way),

$$\frac{d\left\langle S^{(A)}\right\rangle}{dt} = -\left\langle S^{(A)}\right\rangle + \sum_{k=m}^{k_{\max}^{(A)}} \frac{p_k^{(A)} k}{\langle k\rangle_A} \tanh\left(\frac{J_A k}{\langle k\rangle_A T}\left\langle \tilde{S}^{(A)}\right\rangle + \frac{h_0}{T}\sin\omega_0 t\right),$$
$$\frac{d\left\langle S^{(B)}\right\rangle}{dt} = -\left\langle S^{(B)}\right\rangle + \sum_{k=m}^{k_{\max}^{(B)}} \frac{p_k^{(B)} k}{\langle k\rangle_B} \tanh\left(\frac{J_B k}{\langle k\rangle_B T}\left\langle \tilde{S}^{(B)}\right\rangle + \frac{h_0}{T}\sin\omega_0 t\right), \quad (13)$$

where
$$\left\langle \tilde{S}^{(A)}\right\rangle = \left(1-\pi^{(A)}\right)\left\langle S^{(A)}\right\rangle + \pi^{(A)}\left\langle S^{(B)}\right\rangle \quad (14)$$

(in a similar equation for $\langle \tilde{S}^{(B)}\rangle$ superscripts (A), (B) should be interchanged), and $J_A = J\langle k\rangle_A/\langle k\rangle$, $J_B = J\langle k\rangle_B/\langle k\rangle$.

Henceforth let us assume that the two subnetworks are SF networks with the degree distributions $p_k^{(A)} = Ak^{-\gamma_A}$, $p_k^{(B)} = Bk^{-\gamma_B}$, $\gamma_A, \gamma_B > 2$, where A, B are normalization constants, and with the minimum degrees of nodes m. Provided that there are no artificial constraints on the maximum degrees of nodes $k_{\max}^{(A)}$, $k_{\max}^{(B)}$, for $N_A, N_B \to \infty$ nodes with arbitrarily large k are present in the subnetworks, and $A = (\gamma_A - 1)\, m^{\gamma_A - 1}$, $B = (\gamma_B - 1)\, m^{\gamma_B - 1}$. However, in practice each subnetwork, say A, has a finite number of nodes N_A and the distribution $p_k^{(A)}$ has a cutoff at a maximum value $k = k_{\max}^{(A)}$, which for $2 < \gamma_A \le 3$ can be estimated from the condition $\int_{k_{\max}^{(A)}}^{\infty} p_k^{(A)} dk < N_A^{-1}$ (since it is practically impossible to find a node with degree $k > k_{\max}^{(A)}$), which yields $k_{\max}^{(A)} = m N_A^{\frac{1}{\gamma_A - 1}}$, and for $\gamma_A > 3$ in practice scales as $k_{\max}^{(A)} \propto N_A^{1/2}$ (the same for $k_{\max}^{(B)}$) [23].

In the absence of the magnetic field, the system evolves towards a stable equilibrium with the corresponding values of the order parameters for the subnetworks $\langle S^{(A)}\rangle_0$, $\langle S^{(B)}\rangle_0$, which can be obtained from a stable fixed point of Eq. (13) with $h_0 = 0$,

$$\left\langle S^{(A)}\right\rangle_0 = \int_m^{k_{\max}^{(A)}} \frac{Ak^{-\gamma_A+1}}{\langle k\rangle_A} \tanh\left(\frac{J_A k}{\langle k\rangle_A T}\left\langle \tilde{S}^{(A)}\right\rangle_0\right) dk\,,$$
$$\left\langle S^{(B)}\right\rangle_0 = \int_m^{k_{\max}^{(B)}} \frac{Bk^{-\gamma_B+1}}{\langle k\rangle_B} \tanh\left(\frac{J_B k}{\langle k\rangle_B T}\left\langle \tilde{S}^{(B)}\right\rangle_0\right) dk\,, \quad (15)$$

($\langle \tilde{S}^{(A)} \rangle_0$, $\langle \tilde{S}^{(B)} \rangle_0$ are defined by Eq. (14) with $\langle S^{(A)} \rangle = \langle S^{(A)} \rangle_0$, $\langle S^{(B)} \rangle = \langle S^{(B)} \rangle_0$). The corresponding stationary values of the magnetization $\langle M^{(A)} \rangle_0$ of the subnetwork A is then

$$\left\langle M^{(A)} \right\rangle_0 = \int\limits_m^{k_{\max}^{(A)}} A k^{-\gamma_A} \tanh\left(\frac{J_A k}{\langle k \rangle_A T} \left\langle \tilde{S}^{(A)} \right\rangle_0 \right) dk \,, \qquad (16)$$

and $\langle M^{(B)} \rangle_0$ can be evaluated similarly (in Eq. (15), (16) summation was replaced by integration).

For $\pi^{(A)} = \pi^{(B)} = 0$, *i.e.*, two uncoupled subnetworks, the equations for $\langle S^{(A)} \rangle_0$, $\langle S^{(B)} \rangle_0$ in Eq. (15) become independent of each other. The equation for $\langle S^{(A)} \rangle_0$ has one stable fixed point $\langle S^{(A)} \rangle_0 = 0$ for $T > T_{\mathrm{c}}^{(A)}$ corresponding to the paramagnetic phase, and two stable symmetric fixed points $\pm\langle S^{(A)} \rangle_0$ with $\langle S^{(A)} \rangle_0 > 0$ for $T < T_{\mathrm{c}}^{(A)}$ corresponding to the ferromagnetic phase. The critical temperature $T_{\mathrm{c}}^{(A)} = J\langle k^2 \rangle_A / \langle k \rangle_A^2$, where $\langle k^2 \rangle_A$ is the second moment of the distribution $p_k^{(A)}$, depends on the scaling exponent γ_A and, possibly, on the number of nodes N_A. For $\gamma_A > 3$ the system undergoes a ferromagnetic phase transition at the critical temperature $T_{\mathrm{c}}^{(A)} = J\frac{\gamma_A - 2}{(\gamma_A - 1)(\gamma_A - 3)}$. For $2 < \gamma_A \leq 3$ the critical temperature diverges in the thermodynamic limit, however, for finite N_A there is a crossover (rather than critical) temperature $T_{\mathrm{c}}^{(A)} \propto \ln N_A$ for $\gamma_A = 3$ and $T_{\mathrm{c}}^{(A)} \propto N_A^{(3-\gamma_A)/(\gamma_A - 1)}$ for $2 < \gamma_A < 3$, separating the ordered and disordered phases [24, 25, 26, 27]. Of course, the same holds for the subnetwork B.

Critical properties of the Ising model on two coupled Barabási–Albert networks (SF networks with $\gamma = 3$) were studied in Refs. [28, 29] in the MF approximation and by means of Monte Carlo (MC) simulations. It was found that for a moderate and high fraction of the inter-network connections there is a critical temperature for the ferromagnetic transition for the composite network. Numerical solution of Eq. (15) as well as MC simulations of the model described in Sec. 5 show that this is also true in a more complex case of two coupled SF networks with possibly different numbers of nodes and scaling exponents. As the temperature is decreased, first the spins start ordering on the subnetwork which has higher critical temperature in the uncoupled case (*e.g.*, due to larger number of nodes or lower scaling exponent of the degree distribution), say B. This happens at the temperature T_{c} which for small $\pi^{(A)}$, $\pi^{(B)}$ is close to $T_{\mathrm{c}}^{(B)}$ and decreases with the increase of the fraction of inter-network edges so that for moderate and high $\pi^{(A)}$, $\pi^{(B)}$ it is between $T_{\mathrm{c}}^{(A)}$ and $T_{\mathrm{c}}^{(B)}$. Immediately, partial ordering of spins in the other subnetwork occurs, *i.e.*, there is no stable solution of Eq. (15) with

$\langle S^{(A)}\rangle_0 = 0$, $\langle S^{(B)}\rangle_0 \neq 0$. Hence, T_c can be treated as a sort of critical temperature for the Ising model on the composite network; its value cannot be, however, approximated analytically for a wide range of $\pi^{(A)}$, $\pi^{(B)}$.

4. Linear response theory

The response of the model to the weak oscillating magnetic field $h_0 \to 0$ for given T can be studied in the MF approximation in the framework of the linear response theory (LRT). It is assumed that the MF order parameters for the subnetworks $\langle S^{(A)}(t)\rangle$, $\langle S^{(B)}(t)\rangle$ oscillate around the stable stationary state, *i.e.*, $\langle S^{(A)}(t)\rangle = \langle S^{(A)}\rangle_0 + \xi_A(t)$, $\langle S^{(B)}(t)\rangle = \langle S^{(B)}\rangle_0 + \xi_B(t)$, where $\xi_A(t), \xi_B(t) \to 0$. Inserting this into Eq. (13), expanding the tanh function in the Taylor series up to linear terms and replacing the summation with integration yields

$$\begin{aligned}
\frac{d\xi_A}{dt} &= -\frac{\xi_A}{\tau_{AA}} - \frac{\xi_B}{\tau_{AB}} + \frac{h_0 Q_A}{T}\sin\omega_0 t\,, \\
\frac{d\xi_B}{dt} &= -\frac{\xi_B}{\tau_{BB}} - \frac{\xi_B}{\tau_{BA}} + \frac{h_0 Q_B}{T}\sin\omega_0 t\,,
\end{aligned} \tag{17}$$

$$\begin{aligned}
\tau_{AA} &= \left[1-\left(1-\pi^{(A)}\right)\frac{J_A}{\langle k\rangle_A^2 T}\sum_{k=m}^{k_{\max}^{(A)}} p_k^{(A)} k^2 \cosh^{-2}\left(\frac{J_A k\left\langle \tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right)\right]^{-1} \\
&= \begin{cases}
\left\{\frac{A\left(1-\pi^{(A)}\right)}{\langle k\rangle_A\left\langle\tilde{S}^{(A)}\right\rangle_0}\left[m^{-\gamma_A+2}\tanh\left(\frac{J_A m\left\langle\tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right)\right.\right. & \\
\left. -\left(k_{\max}^{(A)}\right)^{-\gamma_A+2}\tanh\left(\frac{J_A k_{\max}^{(A)}\left\langle\tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right)\right] & \\
\left. +1-\frac{\left\langle S^{(A)}\right\rangle_0}{\left\langle\tilde{S}^{(A)}\right\rangle_0}\left(1-\pi^{(A)}\right)(\gamma_A-2)\right\}^{-1} & \text{for } T\le T_c\,, \\
\left[1-\left(1-\pi^{(A)}\right)\frac{\langle k\rangle_A}{\langle k\rangle}\frac{T_c^{(A)}}{T}\right]^{-1} & \text{for } T>T_c\,,
\end{cases} \\
\tau_{AB} &= \left[\frac{\pi^{(A)}}{1-\pi^{(A)}}\left(\tau_{AA}^{-1}-1\right)\right]^{-1}, \\
Q_A &= \frac{1}{\langle k\rangle_A}\sum_{k=m}^{k_{\max}^{(A)}} p_k^{(A)} k \cosh^{-2}\left(\frac{J_A k\left\langle\tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right) \\
&= \begin{cases}
\frac{AT}{J_A\left\langle\tilde{S}^{(A)}\right\rangle_0}\left[\left(k_{\max}^{(A)}\right)^{-\gamma_A+1}\tanh\left(\frac{J_A k_{\max}^{(A)}\left\langle\tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right)\right. & \\
\left. -m^{-\gamma_A+1}\tanh\left(\frac{J_A m\left\langle\tilde{S}^{(A)}\right\rangle_0}{\langle k\rangle_A T}\right)\right]+(\gamma_A-1)\frac{\left\langle M^{(A)}\right\rangle_0}{J_A\left\langle\tilde{S}^{(A)}\right\rangle_0} & \text{for } T\le T_c\,, \\
1 \text{ for } T>T_c\,. &
\end{cases}
\end{aligned}$$

The quantities τ_{BA}, τ_{BB} and Q_B are defined analogously; to evaluate τ_{AA} and Q_A for $T \leq T_c$ the integration by parts was performed, and Eqs. (15), (16) were taken into account.

The asymptotic solution of Eq. (17) is

$$\begin{aligned} \xi_A(t) &= \alpha_1 \sin\omega_0 t + \alpha_2 \cos\omega_0 t\,, \\ \xi_B(t) &= \beta_1 \sin\omega_0 t + \beta_2 \cos\omega_0 t\,, \end{aligned} \tag{18}$$

where the coefficients can be obtained from the system of linear equations

$$\begin{bmatrix} \tau_{AA}^{-1} & -\omega_0 & \tau_{AB}^{-1} & 0 \\ \omega_0 & \tau_{AA}^{-1} & 0 & \tau_{AB}^{-1} \\ \tau_{BA}^{-1} & 0 & \tau_{BB}^{-1} & -\omega_0 \\ 0 & \tau_{BA}^{-1} & \omega_0 & \tau_{BB}^{-1} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \beta_1 \\ \beta_2 \end{bmatrix} = \begin{bmatrix} Q_A h_0 T^{-1} \\ 0 \\ Q_B h_0 T^{-1} \\ 0 \end{bmatrix}. \tag{19}$$

The SPA for the subnetworks are

$$\mathrm{SPA}_A(T) = \frac{\alpha_1^2 + \alpha_2^2}{4h_0^2}\,, \qquad \mathrm{SPA}_B(T) = \frac{\beta_1^2 + \beta_2^2}{4h_0^2}\,, \tag{20}$$

and for the composite network

$$\mathrm{SPA}(T) = \frac{N_A}{N}\mathrm{SPA}_A(T) + \frac{N_B}{N}\mathrm{SPA}_B(T)\,. \tag{21}$$

5. Model for numerical simulations

For the purpose of numerical simulations the Ising model on a composite network consisting of two SF subnetworks may be obtained as follows. The procedure starts with the construction of two independent SF networks A, B with required numbers of nodes N_A, N_B and scaling exponents γ_A, γ_B using the algorithm proposed in Ref. [19], which is an extension of the well-known Barabási–Albert preferential attachment algorithm [30]. First, for each network, a small number $m+1$ of fully connected nodes is fixed. Then, step by step, new nodes are added, and each new node is connected to existing nodes with m edges according to the following probabilistic rule: Probability of linking to a node i is $p_i = (k_i + \beta) / \sum_i (k_i + \beta)$, where k_i is the actual degree of the node i, $\sum_i k_i$ is the actual number of edges in the whole network, and β is a tunable parameter representing the initial attractiveness of each node, which can be different for each network. The growth process is continued until the desired number of nodes N_A or N_B is reached, when the network structure is frozen. For large number of nodes, this preferential attachment rule results in the network with the mean node degree $\langle k \rangle = 2m$ and the degree distribution $p_k \propto k^{-\gamma}$ for $k \gg \beta$, with

$\gamma = 3 + \beta/m$. However, it should be noted that in networks obtained in this way for small k the distribution p_k deviates from the power scaling law and is almost uniform.

After the two independent SF networks are constructed, cutting and rewiring of edges is performed in order to create links between nodes belonging to different networks. In this way, the two SF networks become subnetworks of a composite network. In each network A, B, pair of connected nodes is chosen at random, the two intra-network edges are cut, and, instead, two inter-network edges are created, each connecting a node from the subnetwork A to a node from the subnetwork B: the first pair of nodes to be connected is chosen at random, and the second inter-network edge connects the two remaining nodes. This cutting and rewiring procedure is repeated until a desired number of inter-network edges is created, and the fraction of the latter edges to the total number of edges in each network becomes $\pi^{(A)}$, $\pi^{(B)}$, respectively. The degree distributions of the two networks are not affected by this procedure since cutting and rewiring of edges does not change the degrees of individual nodes. Finally, after the required composite network is constructed, the Ising spins are located in its nodes.

In order to observe SR in the Ising model on the above-mentioned composite network MC simulations were performed using the Glauber heat-bath algorithm (2). The subnetworks had various numbers of nodes N_A, N_B and scaling exponents γ_A, γ_B; only the parameter $m = 5$ was fixed. The frequency of the slowly varying magnetic field was $\omega_0 = 2\pi/512$, and its amplitude $h_0 = 0.01$ since for smaller values prohibitively long simulation times were necessary to obtain reliable curves SPA *vs.* T due to intrinsic thermal fluctuations in the system. Typically, simulation time was 2^{15} steps of the MC algorithm (with one step corresponding to updating N spins), and the results were averaged over 10 random realizations of the network.

6. Results and discussion

6.1. Subnetworks with equal number of nodes $N_A = N_B$ and different scaling exponents $\gamma_A \neq \gamma_B$

Exemplary curves SPA *vs.* T for the Ising model on the composite network consisting of two subnetworks with $N_A = N_B = 5000$ and different scaling exponents $\gamma_A = 5$ and $\gamma_B = 2.5$ obtained from the LRT in the MF approximation (Sec. 4.2) for different fractions of inter-network edges $\pi^{(A)} = \pi^{(B)} = \nu$ are shown in Fig. 1 (a). For uncoupled subnetworks ($\nu = 0$) SPA exhibits two sharp maxima at temperatures close to the critical (for the subnetwork A) or crossover (for the subnetwork B) temperatures for the ferromagnetic transition. As the fraction of rewired edges increases the first maximum (at lower temperature) diminishes, but its location remains practically unchanged. In contrast, the second maximum (at higher temperature) grows and is shifted towards lower temperatures; it is located close to the critical temperature for the Ising model on the composite network. Hence, in a wide range of ν, double maxima of the SPA as a function of temperature occur, *i.e.*, SMR is observed. In Fig. 1 (b) exemplary curves SPA *vs.* T are shown, obtained from MC simulations of the model described in Sec. 5 with the same parameters as in Fig. 1 (a) ($\beta = 10$ for the subnetwork A and $\beta = -3$ for the subnetwork B). They resemble qualitatively the corresponding theoretical curves, although the rise and shift towards lower temperatures of the second maximum (at the higher temperature) of the SPA is less significant. Quantitative agreement between numerical and theoretical curves cannot be expected, mainly because the degree distributions of the subnetworks constructed using the algorithm of Ref. [19] deviate from the power-law behaviour for $k \to 0$, thus also the critical, or crossover, temperatures for the ferromagnetic transition are significantly different. An-

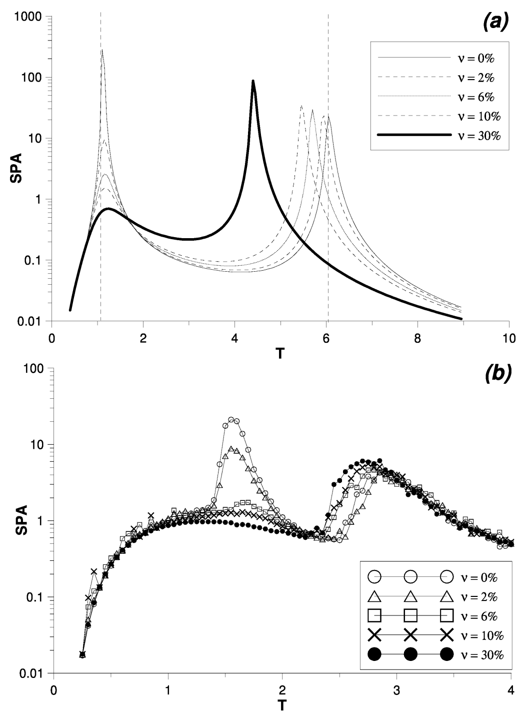

Fig. 1. SPA *vs.* T for the Ising model on the composite network consisting of two SF subnetworks with $N_A = N_B = 5000$ and $\gamma_A = 5$, $\gamma_B = 2.5$ for different fractions of inter-network edges ν (see legend); (a) theoretical results in the MF and LRT approximations, (b) results of MC simulations of the appropriate model of Sec. 5.

other source of discrepancy may be the presence of correlations between the degrees of nodes, which are unavoidable in the networks with $2 < \gamma < 3$ and without any artificial limit on the maximum degree [23]. The MF approximation neglects such correlations which leads to overestimation of the critical temperature. Thus, for example, in the case of the Ising model on a single SF network, the maxima of the SPA are shifted toward higher temperatures, even if the degree distribution obeys pure power scaling law [17].

In Fig. 2, the curves SPA_A, SPA_B *vs.* T are shown for individual subnetworks, for the system with the same parameters as in Fig. 1; again, the agreement between the theoretical (left column) and numerical (right column) results is only qualitative. As mentioned in Sec. 1 in the case of the Ising model on SF network with the tails of the degree distributions obeying the power scaling law with $2 < \gamma < 3$, and with no artificial constraints on the maximum degree of nodes, so-called structural SMR is observed [16, 17]:

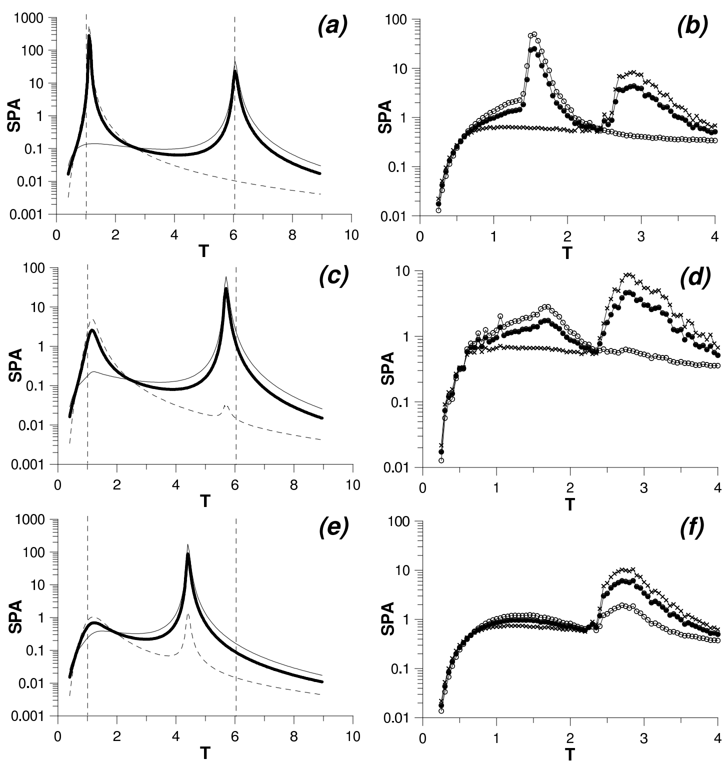

Fig. 2. SPA (thick solid lines and filled dots), SPA_A (dashed lines and empty dots), SPA_B (thin solid lines and crosses) *vs.* T for the same model as in Fig. 1, left column: theoretical results in the MF and LRT approximations, right column: results of MC simulations, (a), (b) $\nu = 0$, (c), (d) $\nu = 0.06$, (e), (f) $\nu = 0.3$.

the curve SPA *vs.* T exhibits two maxima, a sharp one close to the crossover temperature and a lower, broad one in the ordered phase. This is shown in Fig. 2 (a), (b), where the curve SPA_B *vs.* T with the above-mentioned properties can be seen for $\nu = 0$. On the other hand, in the case of the Ising model on SF networks with $\gamma > 3$ the curve SPA *vs.* T exhibits one maximum close to the critical temperature. This is the case of the curve SPA_A *vs.* T in Fig. 2 (a), (b). The location of the maximum of SPA_A coincides with that of the broad maximum of SPA_B in the ordered phase. As the fraction of inter-network edges ν increases, the sharp maximum of SPA_A (*i.e.*, of the subnetwork with $\gamma > 3$) quickly disappears, while the broad maximum of SPA_B (*i.e.*, of the subnetwork with $2 < \gamma < 3$) remains practically intact, and an additional maximum of SPA_A at the temperature close to the crossover temperature of the subnetwork B gradually appears.

From Eq. (21) it follows that SPA for the composite network is a weighted combination of SPA_A and SPA_B. Thus, the occurrence of the double maxima of the curve SPA *vs.* T in Fig. 1 is another example of structural SMR, and the individual maxima can be attributed to the response of the spins on the two subnetworks to the oscillating magnetic field. The shape of the dependence of the SPA on T for a network composed of densely connected subnetworks (larger ν) is mainly determined by the response of the subnetwork with higher critical, or crossover, temperature (with $2 < \gamma < 3$), even if the numbers of nodes in both networks are equal. The influence of the response of the subnetwork with lower critical temperature diminishes with the rise of ν. The critical behaviour of the Ising model on SF networks, and thus its susceptibility, is determined by "hubs", *i.e.* , spins located in nodes with a very high degree, which are with higher probability present in the networks with lower γ (thus, with higher critical temperature). As the number of inter-network edges increases the influence of these hubs spreads over the entire composite network, which yields the curves SPA *vs.* T similar to those of the SPA_B *vs.* T.

6.2. Subnetworks with different number of nodes $N_A \neq N_B$ and equal scaling exponents $\gamma_A = \gamma_B$

Exemplary curves SPA, SPA_A and SPA_B *vs.* T for the Ising model on the composite network consisting of two subnetworks with different numbers of nodes $N_A = 3000$, $N_B = 5000$ and equal scaling exponents $\gamma_A = \gamma_B = 2.5$ obtained from the LRT in the MF approximation are shown in Fig. 3 (a) for the case of small but not negligible fraction of inter-network connections $\pi^{(A)} = 0.02$, $\pi^{(B)} = 0.012$. Corresponding results of the MC simulations are shown in Fig. 3 (b). As in Fig. 1, the agreement between theoretical and numerical results is only qualitative. However, in both cases the curves

SPA *vs.* T exhibit three rather than two maxima: a broad maximum at low temperature, in the ordered phase, a sharp one close to the crossover temperature of the larger subnetwork with $N_B = 5000$ (which is also the critical temperature for the composite network), and a small one between the two above-mentioned ones, in the vicinity of the crossover temperature for the smaller subnetwork with $N_B = 3000$. The latter maximum seems negligible in Fig. 3 (b) (where it is marked by an arrow), however, it does not disappear as a result of averaging over many realizations of the network. For $\pi^{(A)} = \pi^{(B)} = 0$ this maximum is more distinct, and disappears gradually as the fraction of inter-network edges increases (not shown).

This is a more complex example of structural SMR. Since $2 < \gamma_A = \gamma_B < 3$, for $\pi^{(A)} = \pi^{(B)} = 0$ both curves SPA$_A$, SPA$_B$ *vs.* T exhibit double maxima [16, 17], one broad at similar temperatures, in the ordered phase,

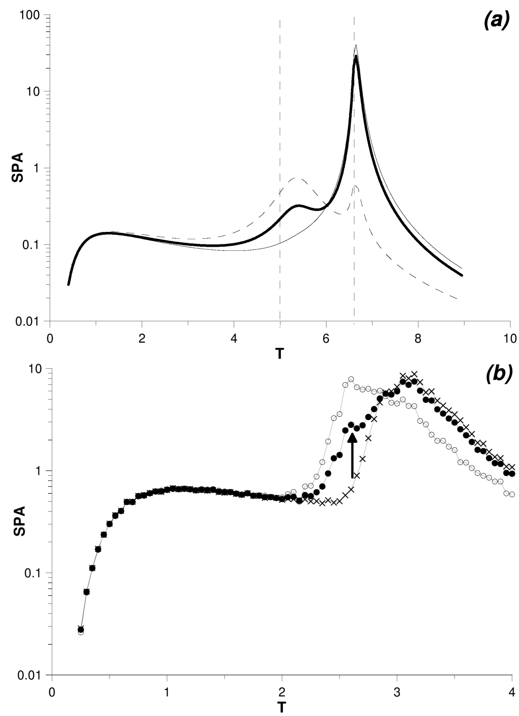

Fig. 3. SPA (thick solid lines and filled circles), SPA$_A$ (dashed lines and empty circles), SPA$_B$ (thin solid lines and crosses) *vs.* T for the Ising model on the composite network consisting of two SF networks with $N_A = 3000$, $N_B = 5000$ and $\gamma_A = \gamma_B = 2.5$ for $\pi^{(A)} = 0.02$, $\pi^{(B)} = 0.012$, (a) Theoretical results in the MF and LRT approximations, (b) results of MC simulations of the appropriate model of Sec. 5.

and one sharp, at crossover temperatures which are different due to different number of nodes in the two networks. As the fraction of inter-network edges increases the maximum of the SPA_A close to the crossover temperature for the smaller subnetwork A (lower than that for the larger subnetwork B) gradually diminishes. Thus, as in the case considered in Sec. 6.1, the shape of the dependence of the SPA on T for a network composed of densely connected subnetworks is mainly determined by the response of spins on the subnetwork with higher crossover temperature to the oscillating magnetic field.

7. Summary and conclusions

Structural SMR was observed in the Ising model on a complex composite network consisting of two coupled SF subnetworks. The subnetworks could differ by the number of nodes or the scaling exponent in the degree distribution, thus also by the critical temperature for the ferromagnetic transition. The curves SPA *vs.* T exhibited two or, possibly, three maxima, and their height and location depended on the fraction of inter-network edges. Their occurrence can be associated with the maxima of the response of spins in the two subnetworks to the weak oscillating magnetic field. For uncoupled subnetworks the Ising models on the subnetworks exhibited SR or even SMR (the respective curves SPA *vs.* T showed one or, possibly, two maxima). For non-zero fraction of inter-network edges the SPA from the model on the composite network was a weighted combination of inputs from the subnetworks: for sparsely connected subnetworks both components were comparable, while for more densely connected networks that from the network with higher critical temperature prevailed.

Although structural SMR was first observed in the Ising model on SF networks with no modular structure [16,17], the occurrence of multiple maxima of the SPA in the Ising model on a composite network can be understood more intuitively, since the sharp, high maxima appear at temperatures close to the critical (crossover) ones for the individual subnetworks, at least for small fraction of inter-network edges. This suggests that the curve SPA *vs.* T for the Ising model on a composite network consisting of more subnetworks, each with different critical (crossover) temperature for the ferromagnetic transition, can exhibit even more local maxima for small to moderate fraction of inter-network edges, and more complex picture of structural SMR can be obtained. This conjecture also opens way to search for structural SMR in systems other than the Ising model, with a topology of complex modular networks.

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