

CHAOS, SYNCHRONIZATION AND CONTROL
IN CELLULAR AUTOMATA*

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We investigate the mechanism of pinching synchronization (complete synchronization for a fraction of system) for cellular automata, considered as prototypes of discrete systems with unpredictable behaviour (finite-distance chaoticity). The pinching synchronization threshold is related to this chaoticity. Some control problems may be reformulated as *targeted* synchronization. In these problems one aims at discovering a protocol that keeps the distance between two replicas below a certain threshold with the minimum *effort*, given some constraints. We have chosen to investigate the behaviour of two control schemes based on the local number of non-zero first-order derivatives, taking as reference the “blind” pinching synchronization protocol. We have shown that, differently from usual chaotic systems, one can exploit *self-annihilation* of defects to obtain synchronization with a weaker control.

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1. Introduction

Control theory is a set of techniques for making a dynamical system behave in a desired way by exerting an external effort. In the case of a minimum effort one speaks of *optimal control*. It is obviously hard to reach the optimum limit, but many investigations are devoted to minimize the control for a desired behaviour. Let us suppose that we have a system following a discrete-time dynamical evolution $x(t+1) = f(x(t))$, that we write synthetically as $x' = f(x)$. We want to add a (small) control $u = u(t)$ such that, after a certain time interval, the trajectory $x(t)$, now evolving as $x' = f(x) + u$, follows a trajectory $y(t)$. In general, the problem of control of a dynamical system may be split into two parts. We have to bring the system x sufficiently near to some part of the trajectory $y(t)$, and then stabilize the trajectory $x(t)$ such that the distance $|x(t) - y(t)|$ is below a certain threshold.

Chaotic systems are ideal targets for control: their sensitivity to small changes may be exploited to drive them to the target area [1], after which chaos may be suppressed in order to make them follow, for instance, a desired periodic orbit [2].

In particular, if the trajectory $y(t)$ is a natural one for system x (i.e., it follows $y' = f(y)$), the control for the stabilization of the trajectory only corresponds to the suppression of the deviations, and can be null if at some time $x(t) = y(t)$. This is a condition that cannot exactly verify for continuous chaotic systems, but is reachable for discrete systems. We shall deal here with such systems, so we only face the problem of *synchronizing* a replica with a “drive” system. This type of synchronization can be called master-slave, identical or replica synchronization [3].

While in the usual studies about synchronization one exerts little attention to the optimisation of coupling, when formulated as a control problem this becomes a crucial issue.

There is a certain interest in modelling extended systems using discrete units, in many cases Boolean ones. Examples are genetic networks [4], some formalisation of neural networks [5], DNA replication and translation and VLSI digital circuits.

A discrete dynamical system is formed by units (nodes, sites) that take discrete values and evolve in discrete time steps. Cellular Automata (CA) [6] are the typical mathematical examples of such systems, even though one may be also interested in non-homogeneous contact networks and non-parallel dynamics [7].

Cellular automata are usually defined on a graph or a regular lattice, but may easily be extended to include mobile agents. The modelling of a system using cellular automata is conceptually much simpler than those using partial derivatives, and the evolution of such a system is easily performed by a digital computer, without rounding errors.

However, for such systems continuity and smoothness (differentiability) do not apply. It is therefore hard to extend the usual techniques used in control theory [8] and to define quantities like Lyapunov exponents and chaotic trajectories. It is still possible to define the derivatives of discrete systems [9], which prove useful in synchronization investigations [10].

In the case of replica synchronization, the “minimal strength” needed to synchronize a system is related to its chaoticity, defined by the largest Lyapunov exponent in low-dimensional systems. For extended systems, the correspondence between the minimal strength and Lyapunov exponents may break down [11].

In synchronization experiments, the “force” is generally applied blindly, without any relation with the dynamics. The corresponding synchronization effect is analogous to a directed percolation phase transition [12]. The two systems synchronize when their difference goes to zero. Their difference grows due to their “chaotic” dynamics, along the directions identified by the Jacobian matrix of the evolution rule. The synchronization “pressure” reduces the paths along which a difference can propagate. When this reduction overcomes the chaotic growth, the system synchronizes.

In control problems, one wants to exploit the knowledge about a system. It is, therefore, analogous to a synchronization problem of two different systems with a “targeted” force, that tries to “kill” the growing directions of the difference as soon as possible. We show how the concept of Boolean derivative and that of Boolean Jacobian matrix can be used to achieve this goal.

The result is however rather surprising: the control efforts should concentrate on the regions exhibiting less distance among replicas, while chaos can be exploited to “self-synchronize” the systems.

2. Synchronization and Lyapunov exponent

Let us start considering the following asymmetric coupling for a continuous one-dimensional map $f(x)$ [13]

$$\begin{aligned}x' &= f(x), \\y' &= (1-p)f(y) + pf(x),\end{aligned}\tag{1}$$

with $x = x(t)$, $x' = x(t+1)$ (idem for y) and $0 \leq p \leq 1$. Let assume that $f(x)$ is chaotic with Lyapunov exponent λ , and that $x(0) \neq y(0)$. Then, $x(t)$ is always different from $y(t)$ for $p = 0$, while for $p = 1$ x and y synchronize in one time step.

For a vanishing distance $h = |z|$, $z = x - y$, we can expand $f(y) = f(x) + \left(\frac{df}{dx}\right)_x (y - x)$ and obtain

$$h' = (1 - p) \left| \frac{df}{dx} \right|_x h. \quad (2)$$

By integrating Eq. (2), one obtains

$$h(t) = h(0) \exp(t(\log(1 - p) + \lambda(t))) ,$$

where

$$\lambda(t) = \frac{1}{t} \sum_{t'=0}^t \log \left(\frac{df}{dx} \right)_{x(t')}$$

is the finite-time Lyapunov exponent of the map f on the trajectory $x(t)$. In chaotic systems this exponent does not generally depend on the trajectory, except for very special initial conditions (like unstable fixed points, *etc.*).

In the limit $t \rightarrow \infty$, the synchronization threshold p_c for which the asymptotic distance $h(\infty) = 0$ loses its stability (the synchronization threshold) is therefore related to the Lyapunov exponent $\lambda = \lim_{t \rightarrow \infty} \lambda(t)$

$$p_c = 1 - \exp(-\lambda). \quad (3)$$

In what follows we shall try to develop similar relations for CA. We begin with a brief review of the definition of maximum Lyapunov exponent for CA based on a linear expansion of the evolution rule [14]. We then present a synchronization mechanism and show that the distance between two realizations goes to zero in a critical manner at p_c [10]. The numerical experiments show a relation between the synchronization threshold and the maximum Lyapunov exponent. We restrict our study to one dimensional, totalistic Boolean cellular automata with a limited number of inputs, since their number is reasonably manageable and their evolution can be efficiently implemented.

A Boolean CA F of range r is defined as a map on the set of configurations $\{\mathbf{x}\}$ with $\mathbf{x} = (x_0, \dots, x_{N-1})$, $x_i = 0, 1$, and $i = 0, \dots, N - 1$ such that

$$\mathbf{x}' = F(\mathbf{x}),$$

where $\mathbf{x} = \mathbf{x}(t)$, $\mathbf{x}' = \mathbf{x}(t + 1)$ and $t = 0, 1, \dots$. The map F is defined locally on every site i as

$$x'_i = f(\{x_i\}_r),$$

where $\{x_i\}_r = (x_i, \dots, x_{i+r-1})$ is the neighbourhood of range r of site i at time t , assuming periodic boundary conditions. For totalistic CA, the local function f is symmetric and depends only on s defined by

$$s(\{x_i\}_r) = \sum_{j=0}^{r-1} x_{i+j}.$$

That is $x'_i = f(s(\{x_i\}_r))$. It is useful to introduce the following operations between Boolean quantities: the sum modulo two (XOR), denoted by the symbol \oplus , and the AND operation, which is analogous to the usual multiplication and shares the same symbol. These operations can be performed between two configurations component by component. We introduce the difference, or damage, $\mathbf{z} = \mathbf{x} \oplus \mathbf{y}$, whose evolution is given by $\mathbf{z}' = F(\mathbf{x}) \oplus F(\mathbf{y})$ and we define the norm h of \mathbf{z} as $h = |\mathbf{z}| = (1/N) \sum_i x_i \oplus y_i$.

A function $f(x_i, \dots, x_j, \dots, x_{i+r})$ is sensitive to its j -th argument for a given neighbourhood $(\{x_i\}_r)$ if the Boolean derivative

$$\left. \frac{\partial f}{\partial x_j} \right|_{\{x_i\}_r} = f(x_i, \dots, x_j, \dots) \oplus f(x_i, \dots, x_j \oplus 1, \dots)$$

is 1. The Jacobian matrix J of F is an $N \times N$ matrix with components

$$J_{i,j}(\mathbf{x}) = \left. \frac{\partial f}{\partial x_j} \right|_{\{x_i\}_r}.$$

J is circulant matrix with zeroes everywhere except possibly on the main diagonal and the following $r - 1$ upper diagonals.

It is possible to ‘‘Taylor expand’’ a Boolean function around a given point using Boolean derivatives [14]. To first order in $|\mathbf{z}|$ we have

$$F(\mathbf{y}) = F(\mathbf{x}) \oplus J(\mathbf{x}) \odot \mathbf{z}, \quad (4)$$

where \odot denotes the Boolean multiplication of a matrix by a vector. Compared to algebraic multiplication of a matrix by a vector, the sum and multiplication of scalars are replaced by the XOR and the AND operations respectively. Using Eq. (4) we may approximate the evolution of the damage configuration \mathbf{z} by

$$\mathbf{z}' = J(\mathbf{x}) \odot \mathbf{z}.$$

However, $|\mathbf{z}|$ grows at most linearly with t since a damage cannot spread to more than r neighbours in one time step: a fixed site i at time $t + 1$ can be

damaged if at least one of its r neighbours at time t is damaged, but if more than one of the neighbours is damaged, the damage may cancel. Since

$$z'_i = \bigoplus_{j=i}^{i+r-1} J_{i,j}(\mathbf{x}) z_j ,$$

$z'_i = 1$ if $J_{i,j}(\mathbf{x}) z_j = 1$ on an odd number of sites. In order to account for all possible damage spreading, we choose to consider each damage independently. If, at time t , m damaged sites are present, we consider m replicas each one with a different damaged site. On each replica, the damage evolves for one time step, without interference effects and so on.

This procedure is equivalent to choosing a vector $\boldsymbol{\xi}(0) = \mathbf{z}(0)$ that evolves in time according to

$$\boldsymbol{\xi}' = J(\mathbf{x})\boldsymbol{\xi} , \quad (5)$$

where now the matrix multiplication is algebraic. The components ξ_i are positive integers that count the number of ways in which the initial damage can spread to site i at time t on the ensemble of replicas.

We define the maximum Lyapunov exponent λ of the cellular automaton F by

$$\lambda(\mathbf{x}^0) = \lim_{T \rightarrow \infty} \lim_{N \rightarrow \infty} \frac{1}{T} \log \left(\frac{|\boldsymbol{\xi}^T|}{|\boldsymbol{\xi}^0|} \right) ,$$

where $|\boldsymbol{\xi}|$ may be taken as the Euclidean norm or as the sum of its components.

We now discuss a synchronization mechanism for CA. Starting with two initial configurations chosen at random $\mathbf{x}(0)$ and $\mathbf{y}(0)$ we propose that

$$\begin{aligned} \mathbf{x}' &= \mathbf{F}(\mathbf{x}) , \\ \mathbf{y}' &= (\mathbf{1} \oplus \mathbf{S}(p))\mathbf{F}(\mathbf{y}) \oplus \mathbf{S}(p)\mathbf{F}(\mathbf{x}) , \end{aligned}$$

where $\mathbf{S}(p)$ is a Boolean random diagonal matrix with elements $s_i(p)$ that are one with probability p and zero with probability $1 - p$ (they change at each time step); $\mathbf{1}$ is the identity matrix and $\mathbf{1} \oplus \mathbf{S}(p)$ is the equivalent of $1 - p$ (bit by bit) of Eq. (1). On the average, y'_i will be set to the value of $x'_i = f(\{x_i\})$ on a fraction p of sites.

The evolution equation for the difference $\mathbf{z} = \mathbf{x} \oplus \mathbf{y}$ is

$$\mathbf{z}' = (\mathbf{1} \oplus \mathbf{S}(p)) [\mathbf{F}(\mathbf{x}) \oplus \mathbf{F}(\mathbf{y})] . \quad (6)$$

The control and order parameters are p and $h(p) = \lim_{t \rightarrow \infty} \lim_{N \rightarrow \infty} |\mathbf{z}(t)|$ respectively. We say that \mathbf{x} , the driver, and \mathbf{y} , the driven system, synchronize when $h(p) = 0$. For $p = 0$ both systems evolve independently, while for

$p = 1$ they synchronize in just one step; we expect then to find a synchronization threshold p_c . This behaviour is shared by all the CA with complex non-periodic space-time patterns. All others synchronize for $p \simeq 0$. This can be conversely expressed by saying that all CA that synchronize with a non-trivial p_c exhibit complex non-periodic space-time patterns.

In the limit of vanishing distance,

$$\mathbf{z}' \simeq (\mathbf{1} \oplus \mathbf{S}) \odot \mathbf{J}\mathbf{z}, \quad (7)$$

where the matrix product $\mathbf{J}\mathbf{u}$ is computed modulo two. The simplest mean-field approximation applied to this last equation leads to an equivalent of Eq. (2).

For totalistic linear rules, whose evolution rule is given by

$$f(\{x_i\}_r) = \bigoplus_{j=0}^{r-1} x_{i+j},$$

the synchronization equation (6) is equivalent to the dilution (with probability $1 - p$) of the rule. The presence of a single absorbing state and the absence of other conserved quantities (*i.e.* number of kinks) strongly suggests that the synchronization transition belongs to the directed percolation universality class [12].

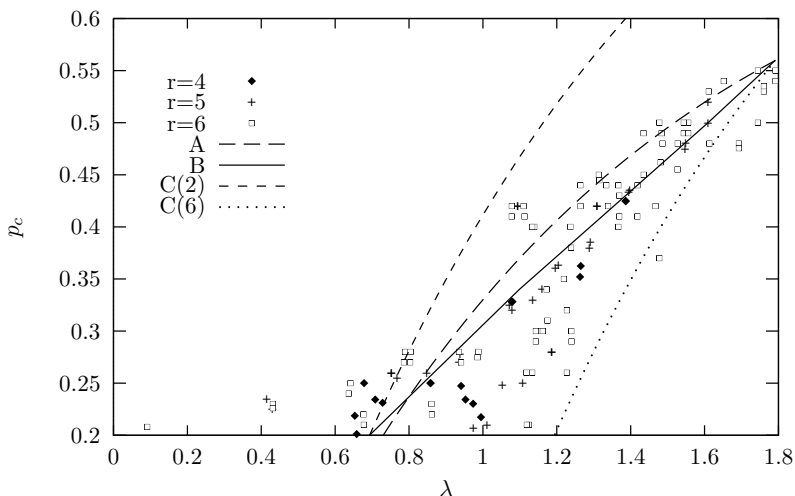


Fig. 1. Relationship between p_c and λ for all CA with range $r = 4, 5, 6$ (markers) and complex space-time patterns. The curves correspond to various analytic approximations, as specified Ref. [10] from which this figure is taken.

In order to illustrate that the relationship between the synchronization threshold and our definition of the Lyapunov exponent holds also for discrete systems, we report in Fig. 1 the values of the couple (p_c, λ) for all totalistic CA with $r = 4, 5, 6$ and a non-trivial value of p_c , from Ref. [10].

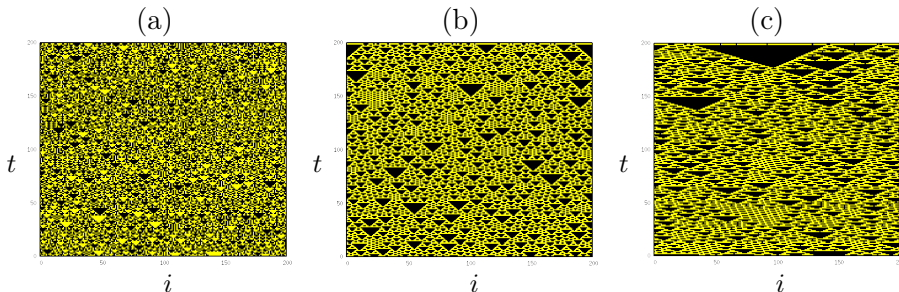


Fig. 2. Typical space-time patterns of “chaotic” rules. Time t runs from top to bottom, space i from left to right, sites in state one are marked in grey/yellow, sites in state 0 are marked in black. (a) $r = 3, R = 10$; (b) $r = 3, R = 6$; (c) $r = 6, R = 30$.

3. Control of cellular automata

In synchronization problems, synchronization is applied “blindly”. In control problems, the goal is that of exploiting available information in order to apply a smaller amount of control (or achieve a stronger synchronization).

We study here the application of synchronization to cellular automata (Fig. 2), *i.e.*, Eq. (6) where the effect of synchronization $s_i \in \{0, 1\}$ may depend on the position i (through x_i) [15].

As above, the efficacy of synchronization (order parameter) is the asymptotic distance $h = (\sum_i z_i)/N$, while the effort is the fraction of synchronized sites $k = k(p) = (\sum_i s_i)/N$.

It is possible in principle to find the absolute minimum of k by computing the effects of all possible choices of s_i , given an initial configuration $\mathbf{x}_0 = \mathbf{x}(0)$. This constitutes a great computational load. Since we are interested in possible real-time applications, we impose that the choice of $s_i = 1$ may only depend on *local* information: the neighbourhood configuration and a $t = 1$ time window.

We investigate three possible way of implementing a control $\mathbf{s}(p)$: (1) blindly with probability p (standard pinching synchronization); in this case $k = p$. (2) with a probability p proportional to the sum of the first-order derivatives and (3) with a probability p inversely proportional to the sum of first-order derivatives.

In order to keep the implementation simple, instead of fixing k and computing the probability p , we let p be a free parameter, and measure the actual fraction of synchronized sites k and the average asymptotic distance h . The previous schemes only require information about \mathbf{x} . If information about \mathbf{y} or about the damage distribution \mathbf{u} is available, the cost k is reduced by a factor h , since in this case we can apply the rule only when it is needed.

Simulation results are presented in Fig. 3. As expected, for linear rules there is no influence of the type of control, since all configurations have the same number of derivatives. For most nonlinear rules, the observed behaviour is the opposite of what is expected for continuous systems. Control 2, that minimizes the distance h for vanishing number of damages according to Eq. (7), gives worse results than the blind control 0. Control 1, inversely proportional to the sum of first-order derivatives, gives better results than the blind control 0. This result holds also for larger neighbourhoods, but not for all rules: there are CA rules for which control type 3 is not the best one (*e.g.* rules 5T45, 6T60 and 7T28), even if the differences are minimal. At present, we have no explanation for this behaviour. There are also non-chaotic rules, like rule 3T12 (the majority rule 232 in Wolfram notation) for which the above recipe fail: in the pattern ...001100110011... all sites has two non-zero derivatives and are therefore less affected by control 1, so by applying control 1 this state is actually favoured, and synchronization using control based on derivatives is worse than “blind” control. However, the large majority of chaotic rules follow the general pattern that control 1 is better than control 0 which is better than control 2.

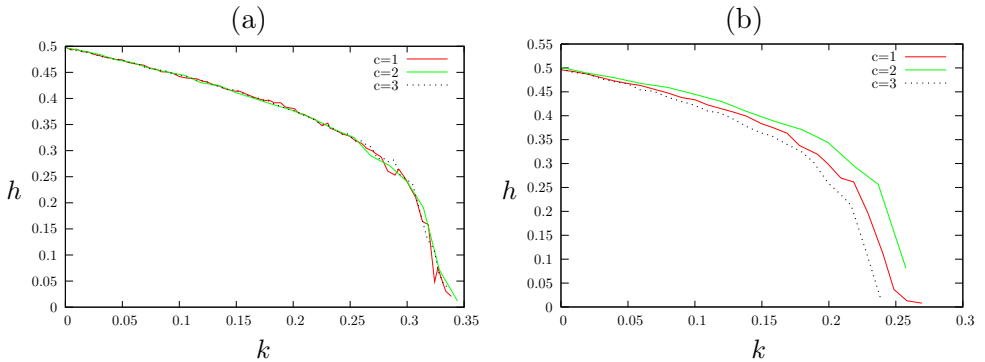


Fig. 3. (a) For linear rules, $r = 3$ and $R = 10$, no influence of different type of controls (all configurations have the same number of derivatives). (b) For a nonlinear CA, $r = 3$ and $R = 6$, control 2 is worse and control 3 is better than blind one. (c) The same result holds for larger neighbourhood, $r = 6$ and $R = 30$.

This surprising effect may be due to the fact that defects self-annihilate, as shown in Fig. 4. Our interpretation is the following, using a stochastic approximation of a chaotic dynamics: due to the fact that the state space is finite (only two values) and that the rule is chaotic, there is a finite probability that a patch of finite amplitude will spontaneously synchronize. Therefore, an effective strategy is that of applying the pinching synchronization mainly to limit the spreading of the defects. Sites with a small number of derivatives are “natural boundaries” of chaotic patches, so concentrating the application of control on such sites gives an optimised way of limiting damage spreading, while the self-annihilation of defects is responsible for the actual synchronization of patches.

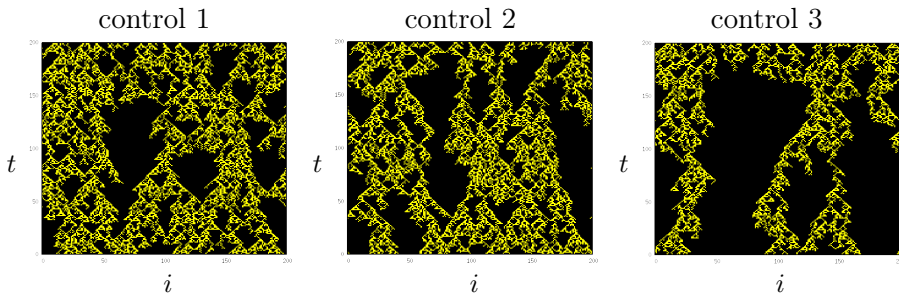


Fig. 4. Time evolution of defects for different types of control, Time t runs from top to bottom, space i from left to right, sites with a defects are marked in grey/yellow, synchronized sites are marked in black. Here $r = 3$ and $R = 6$, all cases starting from the same configuration. The effective probability p has been chosen so to have the same average control k in the three cases. One can notice that clusters of defects for control 3 are less dense than those for control 1 and 2.

In other words, we can exploit the characteristics of cellular automata (and other stable chaotic systems) in order to achieve a better control by exploiting the local contraction of the evolution rule.

4. Conclusions

Spatially extended stable systems (namely cellular automata) may exhibit unpredictable behaviour (finite-distance chaoticity). The pinching synchronization threshold is related to this chaoticity. On the other hand, Boolean derivatives and discrete Lyapunov exponents may be used to characterize this kind of chaos.

In the control problem, one aims at discovering a protocol that keeps the distance h below a certain threshold with the minimum “effort”, given some constraints. We have chosen to investigate the behaviour of two control schemes based on the local number of non-zero first-order derivatives, taking as reference the “blind” pinching synchronization protocol.

We have shown that, differently from usual chaotic systems, one can exploit *self-annihilation* of defects to obtain synchronization with a weaker control, corresponding to the case in which the control is inversely proportional to the number of non-zero derivatives.

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