THE PHASE DIAGRAM IN T–μ–N_c SPACE[*]

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We examine the phase diagram of hadronic matter when the number of colors, as well as temperature and density, are varied. We show that in this regime several phase transitions are possible, and we examine issues related to these transitions.

1. Introduction

The large-N_c limit [1, 2], while sharing several qualitative characteristics with QCD, exhibits significant simplifications. For this reason, it is a very popular technique for qualitative and semi-quantitative estimates of crucial QCD observables. For instance, the string-gauge correspondence is profoundly intertwined with the large N_c limit, since one of the requirements for a calculable dual holographic description is large N_c [3].

Yet, the existence of a smooth limit between $N_c = 3$ and $N_c = \infty$ is a conjecture which, for some observables, is *known* to be false. We know that, at zero chemical potential, $N_c = \infty$ and finite N_f deconfinement has to be a phase transition. At $N_c = 3, N_f = 1, 2$ lattice QCD seems to indicate it is a cross-over [4]. At finite chemical potential in the confining phase, nuclear matter is known to be a strongly bound crystal in the large N_c limit [5]. It seems to be a liquid at $N_c = 3$. This means that, before predictions at large N_c can be made qualitatively reliable, the phase diagram in T–μ–N_c space needs to be explored. The most current phenomenological application for these ideas is the existence of the so-called "quarkyonic matter" [6]. This paper explores these ideas, summarizing [7, 8, 9] and drawing connections between these works.

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2. The van der Waals gas and large N_c

Our first investigation of these issues [7] is to use the van der Waals (vdW) Ansatz to see under what conditions can we obtain both the familiar nuclear matter phase diagram at $N_c = 3$ and the commonly accepted large N_c phase diagram at $N_c \to \infty$ [6]. While the van der Waals gas is certainly a very rough description, it is part of a universality class, to which the nuclear liquid-gas phase transition seems to belong.

In the large N_c limit, the only N_c-invariant scale of the theory is Λ_{QCD}, the scale at which the 't Hooft coupling constant becomes $\lambda \sim \mathcal{O}(1)$. While a precise value of this scale depends on the scheme used to calculate it, it is roughly $\Lambda_{\text{QCD}} \sim N_c^0 \simeq 200\text{–}300$ MeV [10]. It is, therefore, natural to expect that any physical quantity is $\sim f(N_c)\Lambda_{\text{QCD}}^d$, a dimensionless function of N_c times a power of Λ_{QCD} set by the dimensionality d of the quantity. Henceforward, we shall adopt this assumption, and, for brevity, set Λ_{QCD} to unity in the equations. In this notation, the van der Waals parameters a, b and the curvature correction become dimensionless α, β, γ times the appropriate power of Λ_{QCD} (3 for α, 2 for β, 4 for γ), and the vdW equation becomes

$$\left(\rho^{-1} - \alpha\right)\left(P + \beta\rho^2 - \gamma\rho^3\right) = T. \tag{1}$$

We start by noting that, naturally, α can only go to $\Lambda_{\text{QCD}}^{-3}$ at $N_c \to \infty$, and is significantly larger than $\Lambda_{\text{QCD}}^{-3}$ at $N_c = 3$. Therefore,

$$\alpha \sim \mathcal{O}\left(\frac{N_N}{N_c}\right) + 1, \tag{2}$$

where N_N is a constant to be determined. The coefficients β, γ should, according to [2, 5] go as N_c. Recent work [11], however, has cast doubt on this assumption and proposed they go as $\sim N_c^0$ or $\sim \ln N_c$.

The chemical potential can be obtained by the textbook thermodynamic relation $\rho = (dP/d\mu)_T$. Inverting, and writing in terms of $\mu_q = \mu_B/N_c$ we have

$$\mu_q = 1 + \frac{1}{N_c}\left[\int_0^\rho f(\rho', T)d\rho' + F(T)\right], \tag{3}$$

where the first term is the nucleon mass and

$$f(\rho, T) = \left(\frac{dP}{d\rho}\right)_T \frac{1}{\rho} = \frac{T}{\rho(1 - \alpha\rho)^2} + 2\beta. \tag{4}$$

ρ and P are the density at the phase transition, which could be liquid ρ_l or gas ρ_g (if the calculation is performed correctly the same chemical potential

should come out). $\rho_{\mathrm{l,g}}$ are in turn the solutions to the equation (1) in the region, where this equation has two solutions (of which one is thermodynamically unstable). Obtaining all such solutions is trivial at the mathematical level through algebraically cumbersome. The reader can get the detailed results in [7].

As can be seen (Fig. 1), to interpolate between the currently accepted nuclear liquid phase diagram at $N_c = 3$ ($\rho_{\mathrm{c}} \ll \Lambda_{\mathrm{QCD}}^3, T_{\mathrm{c}} \ll \Lambda_{\mathrm{QCD}}$) and the "quarkyonic" phase diagram at $N_c \to \infty$ ($\rho_{\mathrm{c}} \sim \Lambda_{\mathrm{QCD}}^3, T_{\mathrm{c}} \sim \Lambda_{\mathrm{QCD}}$), $N_N \sim \mathcal{O}(10)$ must be used. How would we interpret physically such a value? Above all, what is the physical meaning of N_N? We conjecture that N_N tracks another dimensionless scale relevant at high density: The number of neighbors a nucleon has in a tightly packed nuclear material. N_N, of course, is a function not of N_c, but the (fixed) number of dimensions d and "packing scheme".
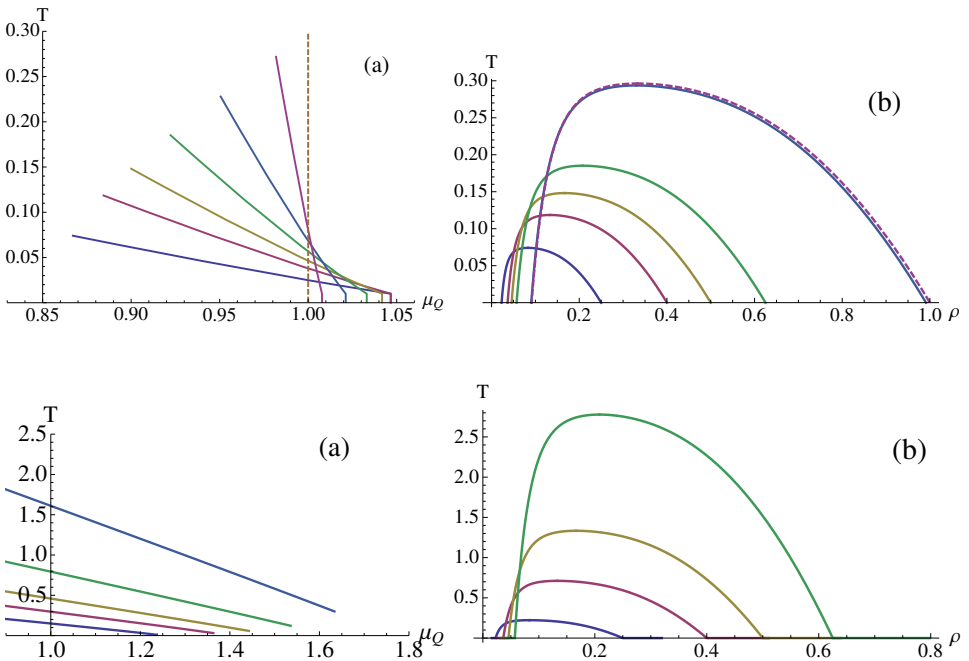

Fig. 1. (Color online) The phase diagram in T–μ space (left panel) and T–ρ space (right panel) as a function of N_c ($N_c = 3, 5, 8, 10, 30, 100$, with increasing color corresponding to a line with higher T, μ, ρ). Top panels assume nuclear interactions $\sim N_c^0$ or $\sim \ln N_c$, bottom panels as $\sim N_c$.

The sensitivity of the varying N_c equation of state to N_N is understandable due to the quantum nature of the problem and the fermionic nature of the quarks. The more neighbors, the more Pauli blocking of valence quarks must be important, and the more the presence of neighbors will disturb the configuration space part of the quark wavefunction inside the nucleons. Since, due to the uncertainty principle, any such disturbance of the nuclear wavefunction adds an energy of the order of the confinement scale $\sim \Lambda_{\mathrm{QCD}}$, the nuclear repulsive core will be larger than the inverse of the nuclear separation up to the deconfinement temperature. If the number of colors is larger than N_N, this problem will not exist since it will be possible to arrange the color part of the wavefunction so that the nearest quarks of neighboring baryons will be of different colors. In this limit, baryons can be tightly packed (interbaryonic separation $\sim \Lambda_{\mathrm{QCD}}$) without the configuration space part of the baryonic wavefunction being disturbed.

The number of neighbours scaling suggests that percolation phenomena might be relevant in the middle of the two regimes. In the next section, we investigate this possibility.

3. Percolation and dense matter at varying N_c

The key insight suggesting that interesting structures might be lurking in N_c is that 3D bond-percolation exhibits a phase transition at comparatively low critical link probability: for instance, $p_{\mathrm{c}} \sim 0.25$, 0.18, 0.12 for simple-cubic, body-centered-cubic and hexagonal-close-packed lattice, respectively [12]. Such values suggest that long-distance correlations on the quark level could occur even with a somewhat low percentage of quarks hopping between baryons, *i.e.* firmly in the confined phase. While below p_{c} the characteristic correlation distance ξ ($\sim$ cluster size) is $\sim \Lambda_{\mathrm{QCD}}^{-1}$, above the threshold this quantity explodes to the total system size in a comparable amount of time. We leave the meaning of "correlation" vague, as it can be either a quark hop or a gluon exchange; in our context, it implies exchange of colour degrees of freedom within a confined tightly packed medium. We encode the likelihood of exchange between neighbouring baryons in a link probability p, to be compared with the percolation threshold p_{c} in order to assess the formation of large-scale structures.

Two baryons will be correlated if *at least two* quarks are correlated. One has to sum over all possible multi-quark configurations, resulting in a strong N_c-dependence of p. We determine the latter by calculating the probability $q = 1 - p$ that *no* exchanges happen between neighbouring baryons. Assuming the quarks inside the nucleon are uncorrelated (Fermi motion dominates), this probability factorises into a geometric distribution $f_{A,B}(\boldsymbol{x})$ for quarks to be at a certain (vector) position $\boldsymbol{x}$, and a "squared propagator" transition

amplitude $F(d)$ for them

$$p = 1 - (q_{(1),ij})^{(N_c)^\zeta},$$
$$q_{(1),ij} = \int f_A(\boldsymbol{x}_i)d\boldsymbol{x}_i \int f_B(\boldsymbol{x}_j)d\boldsymbol{x}_j \left(1 - F(|\boldsymbol{x}_i - \boldsymbol{x}_j|)\right). \tag{5}$$

We assume a "hard-sphere" distribution for $f_{A,B}$ (since we keep μ_q fixed, the distance between centres of neighbouring baryons is always $2\Lambda_{\mathrm{QCD}}^{-1}$)

$$f_{A,B}(\boldsymbol{x}) \propto \Theta\left(1 - \Lambda_{\mathrm{QCD}}\left|\boldsymbol{x} - \boldsymbol{x}_{A,B}^{\mathrm{centre}}\right|\right) \tag{6}$$

and a probability of exchange $i \leftrightarrow j$ based on a range of "reasonable" propagators, compatible with confinement (fast fall-out in configuration space at distances greater than $r_T \sim 1$ in units of $\Lambda_{\mathrm{QCD}}^{-1}$) and with the large N_c limit of QCD, the interaction is $\sim g^2 \sim \lambda/N_c$ [1,2]. The propagators we use are the simple Θ-function in configuration space and the momentum-space Θ-function used in [13], all normalised so their area is $\lambda r_T/N_c$, where r_T is the range of propagation ($\sim \Lambda_{\mathrm{QCD}}^{-1}$). In configuration space the transition amplitudes are, respectively

$$F(y) = \frac{\lambda}{N_c} \begin{cases} \Theta(1 - \frac{y\Lambda_{\mathrm{QCD}}}{r_T}), \\ \frac{2r_T^2}{\pi y^2} \sin^2\left(\frac{y\Lambda_{\mathrm{QCD}}}{r_T}\right). \end{cases} \tag{7}$$

Other transition amplitudes, such as a Gaussian distribution in configuration space, were also tried with no significant modifications. This is unsurprising since, due to the fact that 3D percolation has a second-order phase transition at a certain $p_c < 1$, the results we obtain below have some degree of universality: as long as the qualitative features of confinement are observed (the transition amplitude $F(y)$ drops sharply above the scale r_T, and the hadron density profile $f_{A,B}(\boldsymbol{x})$ has a central plateau of radius $\sim \Lambda_{\mathrm{QCD}}^{-1}$ and a sharply decreasing tail outside), the results we show vary quantitatively but not qualitatively.

The crucial parameter left is ζ in Eq. (5). One can easily see that $\zeta = 1$ is in contradiction with the Skyrme crystal picture at large N_c: in this picture, $p(N_c)$ approaches a constant large-N_c value *from above*: low N_c nuclear matter would be more correlated (and hence more strongly bound) than *high* N_c nuclear matter. Comparing strongly coupled $N_c \to \infty$ nuclear matter [2] to the weakly bound nuclear liquid at $N_c = 3$ [7], this is obviously not right. We, therefore, assume $\zeta = 2$ henceforward, natural if the link is actually realised by a gluon exchange rather than a quark flip (it is obvious this does indeed dominate at large N_c).

In the large N_c limit for the case $\zeta = 2$, p asymptotically approaches unity. It is reasonable that this is the point, where the "dense baryonic matter as a Skyrme crystal", theorised in [2, 5, 6], is reached. If this is the case, however, one should remember that a percolation second-order phase transition occurs at a $p_c \ll 1$. Hence, keeping $\mu_q \sim \Lambda_{\mathrm{QCD}}$ fixed but varying N_c, the features of the Skyrme crystal should manifest not with a continuous approach, rather as a second order transition at a not too high N_c, whose order parameter can be though to be the "giant cluster" density. Below the critical N_c, there is little correlation between quarks of different baryons, while above this threshold they can correlate, with the distance boundary given only by causality. We reiterate that this is *not* deconfinement since $\mu \sim \Lambda_{\mathrm{QCD}} \ll N_c^{1/2} \Lambda_{\mathrm{QCD}}$ independently of the number of colours, and the *fraction* of correlated quarks from different hadrons is still ~ 0.1–$0.3 \ll 1$ at the percolation transition. Right above this transition, therefore, the baryonic wavefunction should *not* be too different from the large-N_c baryonic wavefunction described in [2]. The *correlation distance* of quarks will however be much larger than the baryon size. The features of this new phase are therefore similar to those of the quarkyonic matter [6].

Assuming the lowest 3D value $p_c = 0.12$, appropriate for a closely packed hexagonal lattice, the critical number of colours is shown in Fig. 2 as a function of λ and r_T. As can be seen, the critical number of colours is significantly larger than three for $r_T \sim \Lambda_{\mathrm{QCD}}^{-1}$, $\lambda \sim 1$. Considering Fig. 2 is a *lower limit* since p_c is at its minimum (p_c is significantly higher both in a Skyrme cubic crystal and in a disordered fluid), we can say that $N_c = 3$ is disfavoured, although it cannot be excluded. Changing temperature and μ_q
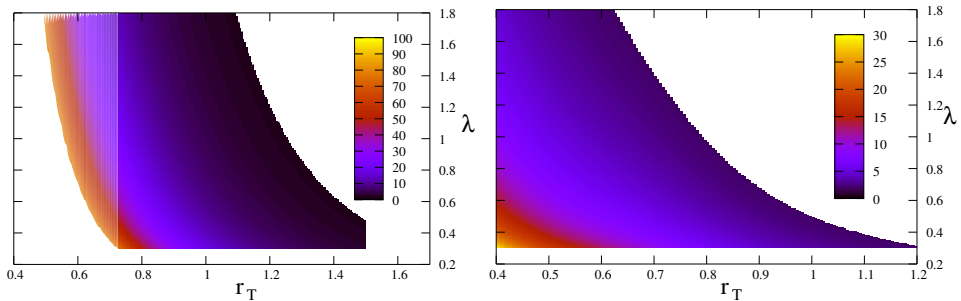

Fig. 2. (Color online) Contour plot of the critical N_c for the percolation transition in a hexagonally packed lattice as a function of the coupling λ and range r_T (in Λ_{QCD} units). The left panel assumes a Θ-function correlation probability in configuration space, the right panel assumes a correlation probability based on the propagator used in [13]. Diagram covers $2 \leq N_c \leq 80$. See [8] for details.

should further change the critical N_c. Exploring this parameter space, and seeing how it relates to the confinement phase transition, is the subject of a forthcoming work.

What are the phenomenological consequences of percolation? If by "correlation" we mean energy-momentum-exchange via quark tunneling between baryons, it is reasonable that pressure and entropy density $\sim N_c$ above the percolation threshold, while below it they stay $\sim N_c^0$. This is because above the threshold, where interbaryon tunnelling is significant, "typical" excitations of the Fermi surface will be superpositions across baryons of baryon-localized quark-hole excitations, similar of conduction band electrons in a metal; while the localized excitation energy $\geq \Lambda_{\mathrm{QCD}}$, the superposition makes its energy $\sim \Lambda_{\mathrm{QCD}}$ even if color degeneracy remains. Thus, the degrees of freedom of the system above percolation will be delocalized weakly interacting quarks in a lattice of confining potentials, a picture compatible with [6]. Below the threshold, where tunnelling is negligible, excitations are either color singlets or of energy $E \gg \Lambda_{\mathrm{QCD}}$, suppressed below deconfinement.

4. Confinement at finite N_c and quantum gravity

The transitions discussed here might also be visible with holographic techniques *beyond* the supergravity limit, since finite λN_c^{-1} corresponds, in gauge/string duality, to the string coupling constant g_s [3]. Thus, a transition at finite N_c will manifest itself as a transition between a classical supergravity regime and a quantum-gravitational regime. Since gravity to one loop still has not been solved for backgrounds dual to confining theories, we cannot rigorously say anything further.

However in [9], we have made a first step in that direction using the reasonable Ansatz, developed earlier [14, 15], that the effect of Quantum gravity corrections is to smoothen the black hole singularity to a Gaussian ℓ, the Plank length. The problem of the first order quantum black hole reduces, then, to the problem of the classical general relativity sourced by a Gaussian distribution of energy density ρ. The top energy-momentum tensor is then

$$T_0^0 = -\rho_\ell(r) = -\frac{M}{(4\pi\ell^2)^{3/2}} \exp\left(-\frac{r^2}{4\ell^2}\right) \tag{8}$$

with the other components of $T^{\mu\nu}$ being fixed by isotropy, the classical Einstein equation

$$R_{\mu n u} - \frac{1}{2}Rg_{\mu\nu} - \frac{3}{L_{\mathrm{AdS}}^2}g_{\mu\nu} = 8\pi G T_{\mu\nu} \tag{9}$$

and conservation laws ($T_{;\mu}^{\mu,\nu} = 0$). Thermodynamics can be obtained by the usual relationship between the entropy s and the horizon area A, $s = A/G$, as well as Maxwell's relations [16].

We rely on [9, 14] for details on solving this problem, and connecting it to the Hawking–Page phase transition [16], and just give the result

$$F(r_H) = F_{\text{Hawking}}(r_H) + \Delta F(r_H), \qquad (10)$$

where F_{Hawking} is the free-energy of a classical black hole in AdS space [16] and

$$\Delta F(r) = 2\pi T \int_{r_0}^{r_H} \frac{r\,dr}{G\frac{2}{\sqrt{\pi}}\gamma(3/2; r^2/4\ell^2)} \ .$$

Once we examine the minimum structure of this free energy, we find a critical point: For a "critical" $q = \ell/L_{\text{AdS}} \simeq 0.18$ there is a critical point after which the Hawking–Page phase transition becomes a *cross-over* (Fig. 3). The resulting phase diagram (in T *versus* ℓ) is shown to be in the universality class of the van der Waals gas.
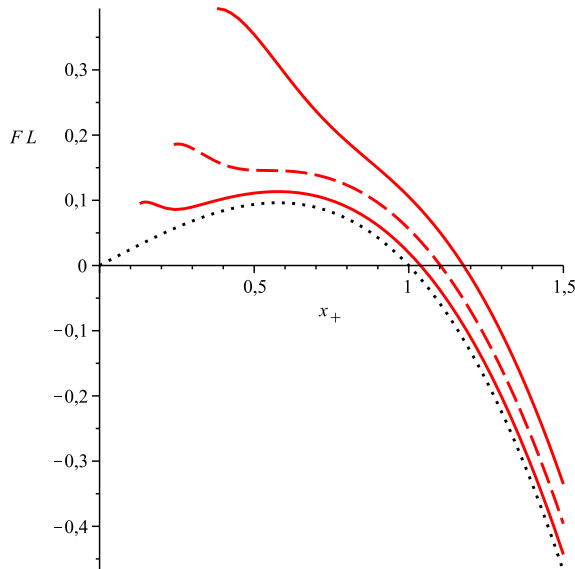

Fig. 3. The free energy F in units of the cosmological constant L for the system in equilibrium as a function of the Schwarzschild radius x_+ from top to bottom for, in terms of $q = \ell/L$, $q = 1/3$ and $q = 0.1$ (solid lines). The dashed lines represent the critical case $q = q^*$, while the dotted line is the classical Hawking–Page phase transition ($q = 0$). See [9] for details.

The relevance of this paper to the topic at hand becomes apparent given that the Hawking–Page transition is thought to be dual to the deconfinement transition [3]. We know that at finite N_f/N_c, deconfinement becomes a cross-over rather than a transition. Thus, this calculation can be thought of as an estimate of the critical N_f/N_c for the deconfinement critical point in two-dimensions (since the gravitatational setup is in 3D).

While the AdS_3 case presented here is considerably different from the $AdS_1 \times S_n$ or the $AdS_5 \times S_5$ background examined in [3], our results fall neatly into the van der Waals universality class. Because of that, and the structure of the calculations in the previous section, we expect that the general structure of this phase diagram will be maintained in $AdS_1 \times S_n$ or any other background where a deconfinement phase transition exists. The critical q^* will, of course, be different, but not the behavior around the critical point. This critical point will translate itself in the behaviour of N_c of the boundary theory, once again in a way that is universal around the critical point but sensitive to the exact nature of the background theory.

When the number of colors $N_c \gg$ the number of flavors N_f (the so-called t 'Hooft limit), confinement is rigorously known to be a phase transition in all dimensions greater than one, since the symmetries of the system are expected to change. The low temperature phase will have acquired an extra Z_N symmetry, signifying a zero expectation value of the Polyakov loop [17]. In the high-temperature phase the Polyakov loop will aquire a finite expectation value, and Z_N will be spontaneously broken. In the gravity picture, this can be thought of as a manifestation of the widely-believed cosmic censorship conjecture [18], the idea that any singularity in spacetime is surrounded by an event horizon (and hence a discontinuity in the energy density at the boundary theory). Since the existence of the black hole tracks the confining Polyakov loop behaviour in [3], making confinement into a cross-over would mean smoothening the black hole singularity. This can only be done, *e.g.* by our Ansatz, for $q > 0$.

At finite number of flavors N_f, the Z_N symmetry is no longer exact at a fundamental level. Hence, it is natural to expect that at some critical N_c^{crit} "a critical point" for deconfinement appears, where for higher N_c deconfinement is a phase transition and for lower N_c it is a crossover. This is indeed what emerges from lattice simulations [4] in three dimensions with a finite number of flavors. The free energy with respect to the Polyakov loop expectation value $\langle L \rangle$ and energy E is, therefore, of the form

$$F\left(\langle L \rangle, E\right) = F_{\text{gauge}}\left(\langle L \rangle, E\right) + \delta F\left(\frac{N_f}{N_c}, \langle L \rangle, E\right). \qquad (11)$$

F_{gauge} at the mixed phase is a function with two local minima, corresponding to the two phases. For high-enough N_f/N_c, however, δF could destroy one of the minima and make the phase transition into a crossover. The van der Waals form of the free energy Eq. (10) exactly parallels Eq. (11), with the q parameter playing the role of N_f/N_c in a Gauge theory in $2+1$ dimensions (since the bulk calculated here is $3+1$ dimensions).

5. Discussion and conclusions

The preceding three sections, at first sight, look like they deal with very different subjects. Indeed, the current status is that any correlation between them is conjectural.

When N_c is varied in the dense matter strongly coupled regime, deviations from the "classical" skyrme crystal arise due to the interplay between the number of colors and the number of neighbours N_N in the densely packed system. This "geometrical" picture is explored rigorously, in terms of an analytical model, in Sec. 3 and shown to give a percolation-type phase transition. A similar behaviour can be argued in terms of the Pauli exclusion principle, and if *it is assumed*, as shown in Sec. 2, leads to the transition between the loosely-bound nuclear liquid at $N_c = 3$ and the tightly bound nuclear crystal at $N_c \gg N_N$. Since the solid-liquid transition can be understood as a percolation of crystal bonds [19], it is, therefore, compelling to identify the percolation transition with the physics described in Sec. 2, with the critical N_c playing the role of "critical point" in T–μ–N_c space (the transition in Sec. 3 is necessarily second order).

The connection with the Hawking–Page transition described in Sec. 4 is, at the moment, not so clear. It is worth pointing out, however, that this transition is also in the van der Waals universality class. While Sec. 4 was at zero chemical potential (the black hole was not charged), the Hawking–Page phase diagram for classical ($\ell = 0$) charged black holes also falls into this class [20].

Thus, the phenomena examined at finite chemical potential in Secs. 3 and 2 might be analogous to the dynamics examined in Sec. 4 at zero chemical potential. The closeness of the critical q^* to the critical number of N_f/N_c required for percolation in 2D [8] (6 colors per flavor) is in this sense intriguing.

Physically, in AdS$_3$, the Hawking–Page transition coincides with the transition from a gas of black holes to one large black hole. It is plausible that the critical point to a Hawking–Page cross-over occurs because the black holes in the gas are actually linked by super-horizon quantum gravitational interactions, modeled in our approach by "smearing" the super-horizon black hole distance by a Planck-sized width. In this scenario, the connection

of percolation to the cross-over in deconfinement is straight-forward. Before this connection can be made more rigorous, however, a Gauge/string setup where percolation is relevant must be examined with a "smearing" analogous to Sec. 4. Work of this kind is in progress.

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