NEUTRINO OSCILLATIONS IN HAMILTONIAN DYNAMICS*

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Quantum mechanical description of neutrino oscillations can be developed in terms of the Gell-Mann–Goldberger formal theory of scattering [M. Gell-Mann, M.L. Goldberger, *Phys. Rev.* **91**, 398 (1953)] provided that the theory is slightly extended [S.D. Głazek, A.P. Trawiński, *Phys. Rev.* **D85**, 125001 (2012)]. The extension is needed because Gell-Mann and Goldberger considered only a very short period of time after a long incoming beam-preparation process ends and before a long outgoing transition-rate counting process starts, while in the case of neutrino oscillations the corresponding period of time is much longer than the beam-preparation and transition rate-counting processes. Besides the standard form of Hamiltonian dynamics, a slightly extended formal theory of scattering can also be defined in the so-called front form of Hamiltonian dynamics. The front form was distinguished by Dirac as particularly interesting in the context of particle physics [P.A.M. Dirac, *Rev. Mod. Phys.* **21**, 392 (1949); P.A.M. Dirac, *The Mathematical Foundations of Quantum Theory*, ed. A.R. Marlow, Academic Press, 1978, pp. 1–8]. We present here an example of a description of neutrino oscillations in the front-form version of the required scattering theory [S.D. Głazek, A.P. Trawiński, *Phys. Rev.* **D87**, 025002 (2013)].

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1. Introduction

In order to discuss neutrino oscillations, let us focus on the example of a typical long-baseline experiment called T2K [1]. In this experiment, the pion beam generated in Tokai is injected into a 100 m long tunnel. After a split-microsecond of free propagation (the pion half-life is $\tau_\pi \sim 1$ μs), pions decay into neutrinos and muons. The neutrino detector is located in

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Kamioka 295 km away. It takes at least 1 millisecond for light to travel this distance. A neutrino interacts with a neutron in Kamioka and thus a muon is created and subsequently detected in the Super Kamiokande detector, see Fig. 1 (left). The proton produced from the neutron is typically not observed. Fig. 1 (right) shows that the counting rate of muons with energies of about 600 MeV in Kamioka is much smaller than expected without account of the neutrino oscillations.

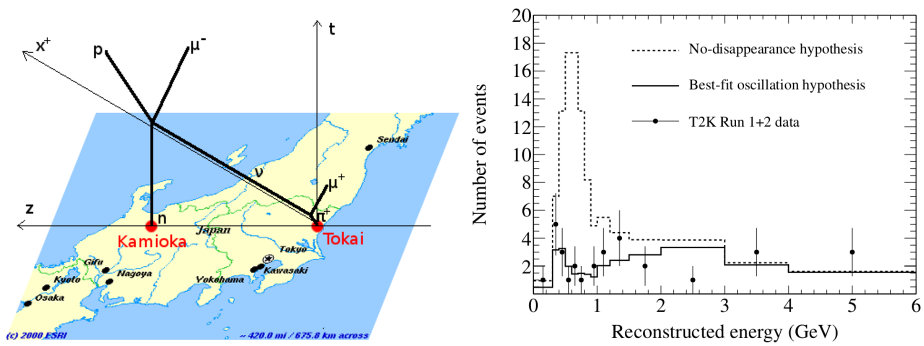


Fig. 1. Left: The thick lines represent word-lines of particles. The z-axis extends from Tokai to Kamioka. Neutrinos propagate over front-form "time" $x^+ \sim 2L/c \sim$ 2 ms, where $L = 295$ km is the distance between Tokai and Kamioka. Right: First Muon-Neutrino Disappearance Study with an Off-Axis Beam [1]. The "reconstructed energy" on the horizontal axis refers to intermediate neutrinos and the "number of events" refers to detection of muons in Super Kamiokande.

1.1. Standard interpretation of neutrino oscillation

The phrase *neutrino oscillations* was introduced by Pontecorvo in 1967 in a context of his study of the weak processes [2] that could become manifest when neutrinos propagate over long distances. Bilenky and Pontecorvo derived a formula for neutrino oscillations in 1977 [3]. Even if no weak processes interfere with neutrino propagation, the phase of a neutrino state changes with time t and in a linear combination of neutrino states with different energies the relative phase of their contributions changes with time. The neutrino oscillation occurs when the e-, μ- or τ-neutrino are composed of states with well defined masses m_i, $|\nu_\alpha\rangle = \sum_i U_{\alpha i}|\nu_i\rangle$, where $\alpha = e, \mu, \tau$. The probability that a μ-neutrino produced in Tokai will be detected in Kamioka

$$P_{\nu_\mu \to \nu_\mu} = \left| \sum_i |U_{\mu i}|^2 \, e^{i \frac{m_i^2}{2E_\nu} L} \right|^2 \approx 1 - \sin^2 (2\theta_{23}) \sin^2 \left(\frac{\Delta m_{23}^2}{4E_\nu} L \right) \qquad (1)$$

can be smaller than 1. It depends on the ratio of distance L to neutrino energy E_ν. For selected values of L/E_ν, the μ-neutrino detection probability decreases to nearly zero. This effect can explain the deficit of muon counts in Kamioka shown in Fig. 1 (right). The angle θ_{23} is related to coefficients $U_{\alpha i}$ and characterizes the electroweak interactions. The oscillation formula (1) can be fitted to the experimental results and such fits also yield most probable values of the differences of neutrino masses squared, which are basic items of information about the world of particles.

The relativistic quantum-mechanical basis of the formula (1) is provided by quantum field theory. For example, the theory helps in removing conceptual ambiguities related to the facts that *(i)* the detected particles are muons, rather than neutrinos, *(ii)* different neutrinos carry the same physical transfer of energy and momentum in a single event, and *(iii)* a single event includes the neutrino creation, propagation, and transformation into a final muon. Several theoretical issues related to these facts have been extensively discussed in literature [4–16]. Sources of relevant experimental results are available as Refs. [1, 17–20]. In view of its expected utility in particle physics [21–23], we explain below how the same facts can be accounted for in the front form (FF) of Hamiltonian dynamics.

2. The front-form neutrino oscillation formula

In the FF approach, the Hamiltonian $P^- = P_0^- + P_{\rm I}^-$ is used to build incoming state $|\Psi_{\rm i}\rangle$ of energy $p_{\rm i}^-$ over time $1/\epsilon^-$ from the corresponding eigenstate $|\phi_{\rm i}\rangle$ of P_0^- in such a way that the Schrödinger equation with full Hamiltonian P^- is satisfied [23]. Namely,

$$\left|\Psi_{\rm i}\left(x^+\right)\right\rangle = \frac{\epsilon^-}{2}\int_{-\infty}^{0} d\mathcal{X}^+\, e^{\epsilon^-\mathcal{X}^+/2} e^{-iP^-\left(x^+-\mathcal{X}^+\right)/2}\left|\Phi_{\rm i}\left(\mathcal{X}^+\right)\right\rangle \tag{2}$$

$$= e^{-iP^-x^+/2}\,\frac{i\,\epsilon^-}{p_{\rm i}^- - P^- + i\,\epsilon^-}|\phi_{\rm i}\rangle\,. \tag{3}$$

The transition rate of the evolving system to the final state $|\phi_{\rm f}\rangle$ is given by the derivative of the probability that the system is in state $|\phi_{\rm f}\rangle$ with respect to the laboratory time t

$$P_{\rm fi}\left(x^+\right) = \frac{d}{dt}\frac{|A\left(x^+\right)|^2}{||\Phi_{\rm f}||^2||\Psi_{\rm i}||^2}\,, \tag{4}$$

where $A(x^+)$ is the scattering amplitude

$$A\left(x^+\right) = \langle\phi_{\rm f}|e^{i\,p_{\rm f}^-x^+/2}\;e^{-i\,P^-x^+/2}\frac{i\epsilon^-}{p_{\rm i}^- - P^- + i\epsilon^-}|\phi_{\rm i}\rangle\,. \tag{5}$$

The norms of the states are constant. Using, the identity $\partial_{x^+} = \frac{\partial t}{\partial x^+}\partial_t + \frac{\partial z}{\partial x^+}\partial_z$, and constancy of z in a standard long-baseline experiment in the laboratory frame of reference, $\frac{\partial z}{\partial x^+} = 0$ and $\frac{\partial t}{\partial x^+} = 1/2$, one obtains

$$\frac{d}{dt}\left|A\left(x^+\right)\right|^2 = \frac{d}{dx^+/2}\left|A\left(x^+\right)\right|^2 = \frac{2\epsilon^-}{\left(p_{\mathrm{f}}^- - p_{\mathrm{i}}^-\right)^2 + \left(\epsilon^-\right)^2}\left|R_{\mathrm{fi}}^{\epsilon^-}\left(x^+\right)\right|^2 , \quad (6)$$

$$R_{\mathrm{fi}}^{\epsilon^-}\left(x^+\right) = \langle\phi_{\mathrm{f}}|P_{\mathrm{I}}^- e^{i\left(p_{\mathrm{f}}^- - P_0^-\right)x^+/2}\frac{i\epsilon^-}{p_{\mathrm{i}}^- - P^- + i\epsilon^-}|\phi_{\mathrm{i}}\rangle . \quad (7)$$

The parameter ϵ^-, which in the T2K experiments is $\sim 10^{-9}$ eV, smoothes out the transition rate as a function of energies of the initial and final states.

2.1. *Example of a scattering amplitude calculation*

Since in the T2K energy range neutrinos carry momenta on the order of 1 GeV, their interactions with hadrons can be approximately described without considering Hamiltonians for quarks and intermediates bosons. Instead, one can define $P^- = \int dx^- d^2x^\perp \mathcal{P}^-$ using the density $\mathcal{P}^-$ obtained from the effective Lagrangian density $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{\mathrm{I}}$, where [24, 25]

$$\mathcal{L}_0 = \sum_\psi \bar\psi(i\partial\!\!\!/ - m_\psi)\psi + \partial_\mu\pi^\dagger\partial^\mu\pi - m_\pi^2\pi^\dagger\pi , \quad (8)$$

$$\mathcal{L}_{\mathrm{I}} = g\bar\mu\gamma^\alpha(1-\gamma_5)\nu_\mu\,\bar p\gamma_\alpha(1-g_A\gamma_5)n - if\bar\nu_\mu\gamma^\alpha(1-\gamma_5)\mu\,\partial_\alpha\pi^\dagger + \mathrm{h.c.} \quad (9)$$

The coupling constants are $g = \frac{G_F}{\sqrt{2}}\cos\vartheta_C$ and $f = \frac{F_\pi}{\sqrt{2}}$. Solving the FF constraint equations in a series expansion in powers of g and f, one obtains the density $\mathcal{P}^- = \mathcal{P}_0^- + \mathcal{P}_1^- + \mathcal{P}_2^- + \mathcal{O}(g^2, f^2)$, where $\mathcal{P}_1^-$ denotes terms order g or f and $\mathcal{P}_2^-$ denotes terms order gf. Inserting the corresponding $P^- = P_0^- + P_1^- + P_2^-$ in Eq. (7), one arrives at the leading expression

$$R_{\mathrm{fi}}^{\epsilon^-}\left(x^+\right) = \langle\phi_{\mathrm{f}}|\left[P_1^-\frac{e^{i\left(p_{\mathrm{i}}^- - P_0^-\right)x^+/2}}{p_{\mathrm{i}}^- - P_0^- + i\epsilon^-}P_1^- + P_2^-\right]|\phi_{\mathrm{i}}\rangle . \quad (10)$$

The first term in the square bracket contains neutrinos or anti-neutrinos in the intermediate states. The second term is a FF instantaneous interaction that traditionally is called a seagull. In T2K, the physical momentum transfer from Tokai to Kamioka has a positive + component, $p_\nu^+ > 0$, and only neutrino intermediate states contribute. The seagull contribution is negligible. Thus,

$$R_{p\mu\bar\mu,n\pi^+}^{\epsilon^-}\left(x^+\right) \simeq \sum_i V_{\mu p,n\nu_i}|U_{\mu i}|^2\frac{1}{p_\nu^+}\frac{e^{i\,p_{\nu_i}^- L}}{p_\nu^- - p_{\nu_i}^- + i\epsilon^-}V_{\nu_i\bar\mu,\pi^+} , \quad (11)$$

where $V_{\mu p,n\nu_i}$ and $V_{\bar{\mu},\pi^+\nu_i}$ denote due vertex factors. Different neutrinos contribute with different $|U_{\mu i}|^2$ and phase factors in numerator and different denominators D_i, the vertex factors being negligibly different. The result is that for $x^+ = 2L$

$$\left|R^{\epsilon^-}_{p\mu\bar{\mu},n\pi^+}(2L)\right|^2 \propto \left|\sum_i |U_{\mu i}|^2 \frac{e^{i\,p^-_{\nu_i} L}}{D_i(p_\nu)}\right|^2 = \left|\sum_i |U_{\mu i}|^2 \frac{e^{i\,m^2_{\nu_i} L/p^+_\nu}}{p^2_\nu - m^2_{\nu_i} + i\epsilon^- p^+_\nu}\right|^2 . \tag{12}$$

2.2. The neutrino interference pattern

The origin of the standard neutrino oscillation formula is visible in Eq. (12). The three amplitudes appear in a sum and interfere with each other in the modulus squared that enters a cross section. This interference pattern is analogous to the interference pattern that is familiar from the elementary quantum slit experiment. The analogy is illustrated in Fig. 2. The role of a slit is played by the FF on-mass-shell "energy" $p^-_{\nu_i}$. The number of "energy slits" participating in the interference depends on the size of ϵ^-; it must be large enough for all potentially available neutrino intermediate states to contribute. This is explained in Fig. 3.

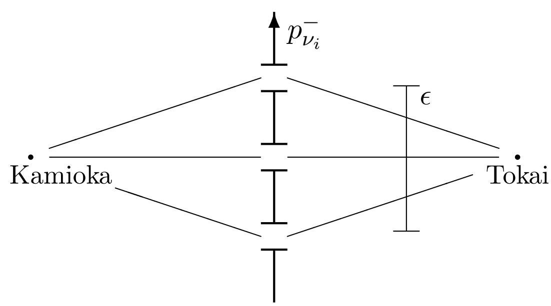
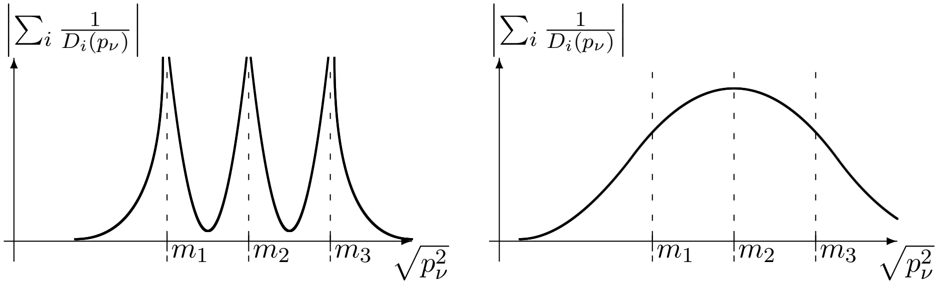


Fig. 2. Schematic representation of the interference of amplitudes mediated by virtual states with neutrinos of different FF free energies $p^-_{\nu_i}$ due to their different masses m_i assumed here to increase with the number i. The FF energy uncertainty ϵ^- must be large enough to create the interference pattern in total counting rate of muons in a far detector that is conventionally called "neutrino oscillation."



(a) for $\epsilon^- = 0$ (b) for $\epsilon^- \neq 0$

Fig. 3. Qualitative plots of the function $|\sum_{i=1}^{3} 1/D_i(p_\nu)|$ in arbitrary units in two cases: (a) $\epsilon^- = 0$ and (b) $\epsilon^- \neq 0$. The neutrino masses m_i are assumed to increase with i. The larger ϵ^- in comparison to $\Delta m_{ij}^2/p_\nu^+$, the more accurate the standard oscillation formula, because the wider the width $\epsilon^- p_\nu^+$ the more neutrino intermediate states uniformly contribute to the muon counting rate in the far detector.

3. Conclusion

The relativistic Hamiltonian interpretation of neutrino oscillation does not involve wave packets, does not refer to a concept of a neutrino state as a superposition of mass eigenstates, and does not involve the Feynman propagators for individual particles. Instead, the Hamiltonian interpretation says that the total probability amplitude for producing a muon in the far detector is a sum of amplitudes coming from virtual intermediate states with neutrinos of different masses. In the leading approximation, the detection rate of muons in the far detector is proportional to the standard formula, but only for sufficiently light neutrinos that can fit into the range of p^- allowed by ϵ^-. This means that heavy sterile neutrinos cannot contribute to oscillations in the low-energy experimental setups such as T2K.

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