NEW OPERATOR SOLUTION OF THE SCHWINGER MODEL IN A COVARIANT GAUGE AND AXIAL ANOMALY*

Ľubomír Martinovič

Bogoliubov Laboratory of Theoretical Physics
Joint Institute for Nuclear Research, Dubna, Russia
and
Institute of Physics, Slovak Academy of Sciences
Dúbravská cesta 9, 845 11 Bratislava, Slovakia

(Received January 15, 2013)

Massless QED (1+1) — the Schwinger model — is studied in a covariant gauge. The main new ingredient is an operator solution of the Dirac equation expressed directly in terms of the fields present in the Lagrangian. This allows us to study in detail residual symmetry of the covariant gauge. For comparison, we analyze first an analogous solution in the Thirring–Wess model and its implication for the axial anomaly arising from the necessity to correctly define products of fermion operators via point-splitting. In the Schwinger model, one has to define the currents in a gauge-invariant (GI) way. Certain problems with their usual derivation are identified, that obscure the origin of the massive gauge boson. We show how to define the truly GI interacting currents, reformulate the theory in a finite volume and clarify role of the gauge zero mode in the axial anomaly and in the Schwinger mechanism. A transformation to the Coulomb gauge representation is suggested along with ideas about how to correctly obtain other properties of the model.

DOI:10.5506/APhysPolBSupp.6.287
PACS numbers: 11.10.Ef, 11.15.Tk, 04.60.Kz

1. Introduction

The Schwinger model [1] is a prototype gauge model, studied in hundreds of papers using all kind of techniques. The natural question concerns, therefore, a necessity to perform another study of this subject. What can be added or improved in our understanding of the physics of the model? Surprisingly enough, no generally accepted picture of the physical content of the

* Presented at the Light Cone 2012 Conference, Kraków, Poland, July 8–13, 2012.

model is available and some controversies persist. This is nicely illustrated by comparing two representatives of the vast literature on the subject: the seminal work by Lowenstein and Swieca [2] and its mathematically rigorous reexamination [3]. Both start from the operator solution in Landau gauge in terms of "building block" fields, namely using Ansaetze for $A^\mu(x)$, $J^\mu(x)$ and $J^\mu_5(x)$ in terms of auxiliary fields. The second paper disagrees with the choice of dynamical variables (the issue of the correct "intrinsic algebra") and with the conclusions about the vacuum structure of the former work.

In this contribution, we will make an attempt to clarify the situation using a Hamiltonian approach that reformulates dynamics consistently in terms of true degrees of freedom, namely the free fields. In particular, we will focus on a few overlooked aspects related to truly gauge-invariant (GI) definitions of the interacting currents and the consequent issues of the axial anomaly and dynamical generation of the boson mass. We start with a brief discussion of the related Thirring–Wess (TW) model for comparison. The key element is the explicit solution of the Dirac equation in the covariant gauge in terms of the fields present in the starting Lagrangian, *i.e.* without using auxiliary fields that obscure some aspects of the problem. Interacting currents can then be calculated directly from the known solutions in a regularized form ("point-splitting") in both models. The difference is that one has to insert an exponential of the line integral of the gauge field to compensate violation of the local symmetry in the gauge model. The corresponding divergences of the quantum currents should, therefore, differ in the two models. However, this is not the case in the usual treatment! The explanation will be given and the key ideas and elements of the full solution of the model will be formulated.

2. The Thirring–Wess model

The model [4, 5] is defined by the classical Lagrangian

$$\mathcal{L} = \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \Psi - \frac{1}{4}\tilde{G}_{\mu\nu}\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}_\mu\tilde{B}^\mu - eJ_\mu\tilde{B}^\mu\,, \qquad \tilde{G}_{\mu\nu} = \partial_\mu\tilde{B}_\nu - \partial_\nu\tilde{B}_\mu\,. \tag{1}$$

The original solutions were either based on indirect methods using Ansaetze in terms of auxiliary fields or certain redundant definitions of "gauge-invariant" operators. No reliable solution of the model seems to have been obtained so far.

The above Lagrangian leads to the set of coupled field equations (Dirac+Proca)

$$i\gamma^\mu\partial_\mu\Psi(x) = e\gamma^\mu\tilde{B}_\mu(x)\Psi(x)\,, \qquad \partial_\mu\tilde{G}^{\mu\nu} + \mu_0^2\tilde{B}^\nu = eJ^\nu\,. \tag{2}$$

Taking ∂_ν of the Proca equation yields $\partial_\mu \tilde{B}^\mu = 0$. With this condition, the Dirac equation is solved in terms of $\tilde{B}^0(x)$ and the free massless fermion field $\psi(x), \gamma^\mu \partial_\mu \psi = 0$

$$\Psi(x) = \exp\left\{ -\frac{ie}{2}\gamma^5 \int_{-\infty}^{+\infty} dy^1 \epsilon\left(x^1 - y^1\right) \tilde{B}^0\left(y^1, t\right) \right\} \psi(x) . \tag{3}$$

Here $\epsilon(x) = \theta(x) - \theta(-x)$. Normal-ordering of the exponential is understood. With the notation $\hat{k} \cdot x \equiv E(k^1)t - k^1 x^1$, $E(k^1) = \sqrt{k_1^2 + \mu_0^2}$, $E(p^1) = |p^1|$, the quantum field expansions of the independent field variables of the model are

$$B^0(x) = \int_{-\infty}^{+\infty} \frac{dk^1}{\sqrt{4\pi E\left(k^1\right)}} \left[a\left(k^1\right) e^{-i\hat{k}\cdot x} + a^\dagger(k^1) e^{i\hat{k}\cdot x} \right] , \tag{4}$$

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dp^1 \left\{ b\left(p^1\right) u\left(p^1\right) e^{-i\hat{p}\cdot x} + d^\dagger\left(p^1\right) v\left(p^1\right) e^{i\hat{p}\cdot x} \right\} , \tag{5}$$

$$\left[a\left(p^1\right), a^\dagger\left(q^1\right) \right] = \left\{ b\left(p^1\right), b^\dagger\left(q^1\right) \right\} = \left\{ d\left(p^1\right), d^\dagger\left(q^1\right) \right\} = \delta\left(p^1 - q^1\right) .$$

The Fock vacuum is defined as $a(k^1)|0\rangle = b(k^1)|0\rangle = d(k^1)|0\rangle = 0$. The massless spinors are $u^\dagger(p^1) = (\theta(-p^1), \theta(p^1))$, $v^\dagger(p^1) = (-\theta(-p^1), \theta(p^1))$. The component $B^1(x)$ is determined from the operator relation $\partial_\mu B^\mu = 0$.

The product of two fermion operators is regularized by the point-splitting

$$\begin{aligned} J^\mu(x) &= \Psi^\dagger\left(x + \frac{\epsilon}{2}\right) \gamma^0 \gamma^\mu \Psi\left(x - \frac{\epsilon}{2}\right) , \\ J_5^\mu(x) &= \Psi^\dagger\left(x + \frac{\epsilon}{2}\right) \gamma^0 \gamma^\mu \gamma^5 \Psi\left(x - \frac{\epsilon}{2}\right) . \end{aligned} \tag{6}$$

Using $\psi^\dagger(x+\frac{\epsilon}{2})\gamma^0\gamma^\mu(\gamma^5)\psi(x-\frac{\epsilon}{2}) =: \psi(x)^\dagger\gamma^0\gamma^\mu(\gamma^5)\psi(x) : -\frac{i}{2\pi}\mathrm{Tr}\left(\frac{\gamma^\alpha\epsilon_\alpha\gamma^\mu(\gamma^5)}{\epsilon^2}\right)$ as well as the symmetric limit $s\ \lim_{\epsilon\to 0} \frac{\epsilon^\mu\epsilon^\nu}{\epsilon^2} = 1/2 g^{\mu\nu}$, we find

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}\tilde{B}^\mu(x) , \qquad J_5^\mu(x) = j_5^\mu(x) + \frac{e}{\pi}\epsilon^{\mu\nu}\tilde{B}_\nu(x) , \tag{7}$$

$j^\mu(x)$ and $j_5^\mu(x)$ are the (normal-ordered) free currents. The expression in the exponential contains a term of the order of $O(\epsilon)$ which cancels the singularity in the free-field contraction. In this way, a finite quantum correction is generated. The vector current is obviously conserved, the axial "anomaly" $a(x)$ is equal to

$$\partial_\mu J^\mu(x) = a(x) \equiv \frac{g}{2\pi}\epsilon^{\mu\nu}\tilde{G}_{\mu\nu}(x) . \tag{8}$$

It is remarkable that this is precisely the result known from the Schwinger model although no exponential of the integral over gauge field was inserted!

The Proca equations become, due to the relation $\partial_\mu \tilde{B}^\mu = 0$ and the form of the interacting current, also soluble. Defining retarded Green's function by $(\partial_\mu\partial^\mu + \mu^2)D_{\rm R}(x-y) = \delta^{(2)}(x-y)$, $(\partial_\mu\partial^\mu + \mu^2)B^\mu(x) = 0$, where $\mu^2 = \mu_0^2 - \frac{e^2}{\pi}$, the resultant equation $\partial_\mu\partial^\mu \tilde{B}^\nu(x) + \mu^2 B^\nu(x) = j^\nu(x)$ can indeed be inverted as

$$\tilde{B}^\nu(x) = B^\nu(x) + e\int_{-\infty}^{+\infty} d^2y D_{\rm R}(x-y) j^\nu(y)\,. \tag{9}$$

Then the Hamiltonian can be expressed in terms of the above independent fields. In the final-volume treatment, also the zero mode $b^1(t)$ will play a role. The questions to be studied are diagonalization of the Hamiltonian deriving thereby the true physical vacuum state of the model and a potential chiral symmetry breaking.

3. Schwinger model in the Landau gauge

We will start from the classical Lagrangian

$$\begin{aligned}
\mathcal{L} &= \frac{i}{2}\overline{\Psi}\gamma^\mu \overleftrightarrow{\partial_\mu} \Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eJ_\mu A^\mu - G(x)\partial_\mu A^\mu + \frac{1}{2}(1-\gamma)G^2(x)\,, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu,\ J^\mu(x) = \overline{\Psi}(x)\gamma^\mu\Psi(x)
\end{aligned} \tag{10}$$

that contains two additional terms with respect to the usual QED$(1+1)$. For arbitrary γ, these terms restrict the theory to an arbitrary Lorentz (covariant) gauge (replacing the usual term $-\frac{\lambda}{2}(\partial_\mu A^\mu(x))^2$) in which neither the condition $\partial_\mu A^\mu(x) = 0$ nor the Maxwell equations can be satisfied at the operator level

$$\partial_\mu F^{\mu\nu}(x) = eJ^\nu(x) - \partial^\nu G(x)\,, \qquad \partial_\mu A^\mu(x) = (1-\gamma)G(x)\,. \tag{11}$$

The auxiliary field $G(x)$ satisfies $\partial_\mu\partial^\mu G(x) = 0$. Choosing $\gamma = 1$, the gauge condition is satisfied at the operator level and the solution of the Dirac equation $i\gamma^\mu\partial_\mu\Psi(x) = e\gamma^\mu A_\mu(x)\Psi(x)$ is completely analogous to the TW model case

$$\Psi(x) = \exp\left\{-\frac{ie}{2}\gamma^5 \int_{-\infty}^{+\infty} dy^1 \epsilon\left(x^1 - y^1\right) A^0\left(y^1, t\right)\right\}\psi(x)\,, \quad \gamma^\mu\partial_\mu\psi = 0\,. \tag{12}$$

In order to guarantee that we are working with the original theory, the condition on physical states $G^{(+)}(x)|\text{phys}\rangle = 0$, generalizing the Gupta–Bleuler condition $\partial_\mu A^{(+)\mu}|\text{phys}\rangle = 0$, has to be used. Again, the vector and axial-vector currents have to be calculated via the point-splitting. It is important to keep in mind that the gauge freedom has been restricted only partially, the Lagrangian is still invariant with respect to gauge transformations parametrized by the gauge function obeying

$$\partial_\mu \partial^\mu \Lambda(x) = 0 \Rightarrow \partial_0^2 \Lambda = \partial_1^2 \Lambda \;\Rightarrow \frac{\partial_0}{\partial_1}\Lambda = \frac{\partial_1}{\partial_0}\Lambda \,. \tag{13}$$

The conclusion about appearance of a massive vector boson in the theory with gauge invariance crucially depends on the axial anomaly. For its derivation, one starts from the "gauge-invariant" definition of the axial current (see [6], *e.g.*), *i.e.* one inserts the gauge-field exponential to the point-split product of the fields

$$J^\mu_{(5)}(x) = \Psi^\dagger\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\left(\gamma^5\right)\exp\left\{-ie\int\limits_{x-\epsilon/2}^{x+\epsilon/2} dz_\mu A^\mu(z)\right\}\Psi\left(x-\frac{\epsilon}{2}\right) . \tag{14}$$

No gauge fixing has been done in (14). Both currents are formally GI under

$$\Psi(x) \to e^{ie\Lambda(x)}\Psi(x)\,, \qquad A^\mu(x) \to A^\mu(x) - \partial^\mu\Lambda(x)\,. \tag{15}$$

The vector current takes the form

$$J^\mu(x) = \left[:\Psi^\dagger(x)\gamma^0\gamma^\mu\Psi(x): + \underbrace{\Psi^\dagger\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\Psi\left(x-\frac{\epsilon}{2}\right)}\right]\Big[1 - ie\epsilon_\nu A^\nu(x)\Big] . \tag{16}$$

Note that in this derivation, the fermion and gauge fields are taken as independent and the free-field contraction has been used. The result is precisely

$$J^\mu(x) = j^\mu(x) + \frac{e}{\pi}A^\mu(x)\,, \qquad J^\mu_5(x) = j^\mu_5(x) + \frac{e}{\pi}\epsilon^{\mu\nu}A_\nu(x)\,, \tag{17}$$

i.e. gauge-NON-invariant expressions! This fact is hidden since one usually calculates directly the divergence which gives the "familiar" (gauge-invariant) anomaly (8). How should one understand the above contradiction? To answer this question, let us calculate the anomaly carefully using our Landau-gauge operator solution (12). We have to take into account that the general transformation law $A^\mu \to A^\mu - \partial^\mu\Lambda$ becomes $A^0(x) \to A^0(x) - \partial_0\Lambda(x)$, $\partial_\mu\partial^\mu\Lambda = 0$ in our gauge and *this completely determines the transformation law for the interacting fermion field* since the free fermion field $\psi(x)$ does not transform

$$\Psi(x) \to e^{\frac{ie}{2}\gamma^5\int_{-\infty}^{+\infty} dy^1\epsilon\left(x^1-y^1\right)\partial_0\Lambda\left(y^1,t\right)}\Psi(x) \equiv e^{\frac{ie}{2}\gamma^5\frac{\partial_0}{\partial_1}\Lambda}\Psi(x)\,. \tag{18}$$

The point, of course, is that $\Psi(x)$ and $A^\mu(x)$ are not independent. We have to modify the "gauge exponential" in such a way that the (split) currents are invariant under the specific transformations (18). The correct form of the current is

$$J^\mu_{(5)}(x) = \Psi^\dagger\Big(x+\frac{\epsilon}{2}\Big)\gamma^0\gamma^\mu\left(\gamma^5\right)\exp\left\{-ie\gamma^5\epsilon_{\mu\nu}A^\mu(x)\epsilon^\nu\right\}\Psi\Big(x-\frac{\epsilon}{2}\Big)\,, \qquad (19)$$

since the gauge variations in the exponential cancel. The interacting currents found in this way coincide with the free currents! The implication is no anomaly and, therefore, no Schwinger mechanism! This really looks like a very strange result.

To understand the situation better, let us analyze the residual gauge symmetry and interacting currents in an infrared-regularized framework by restricting $-L \le x^1 \le L$ and imposing (anti)periodic boundary conditions for the free fields

$$\begin{aligned} \psi(t,-L) &= -\psi(t,L)\,, \\ A^\mu(t,-L) &= A^\mu(t,L) \Rightarrow A^\mu(x) = A^\mu_N(x) + A^\mu_0(t)\,. \end{aligned} \qquad (20)$$

$A^\mu_0(t)$ is the gauge field zero mode (ZM). The Dirac equation and its solution is

$$i\gamma^0\partial_0\Psi + i\gamma^1\partial_1\Psi = e\left(\gamma^0 A^0_N - \gamma^1 A^1_N\right)\Psi - e\gamma^1 A^1_0(t)\Psi\,, \qquad (21)$$

$$\Psi(x) = \exp\left\{ie\gamma^5\left[\int_{t_0}^{t} d\tau A^1_0(\tau) - \int_{-L}^{+L} dy^1\epsilon_N\left(x^1-y^1\right)A^0_N\left(x^1-y^1\right)\right]\right\}\psi(x)\,. \qquad (22)$$

The gauge condition becomes $\partial_0 A^0_N(x) + \partial_1 A^1_N(x) = 0$ and $A^0_0(t) = 0$. The gauge transformations act also in the ZM sector

$$\begin{aligned} A^\mu_N(x) &\to A^\mu_N(x) - \partial^\mu\Lambda_N(x)\,, \\ A^0_0(t) &\to A^0_0(t) - \partial_0\Lambda_0(t)\,, \\ A^1_0(t) &\to A^1_0(t) + \partial_1\Lambda_0(t) = A^1_0(t)\,. \end{aligned} \qquad (23)$$

The GI currents have the form

$$J^\mu_{(5)}(x) = \exp\left\{-ie\gamma^5\epsilon^0 A^1_0(t)\right\}\psi\left(x+\frac{\epsilon}{2}\right)\gamma^0\gamma^\mu\left(\gamma^5\right)\psi\left(x-\frac{\epsilon}{2}\right)\,. \qquad (24)$$

Contraction in the discrete basis has the same singular structure as in the continuum and we obtain $J^\mu(x) = j^\mu(x) + \frac{e}{\pi}(0, A^1_0(t))$, $J^\mu_5(x) = j^\mu_5(x) + \frac{e}{\pi}(A^1_0(t), 0)$. Both currents are gauge invariant since $A^1_0(t)$ component is GI by itself. Then

$$\partial_\mu J^\mu(x) = \partial_\mu j^\mu(x) + \frac{e}{\pi}\left(0, \partial_x A_0^1(t)\right) = 0\,, \tag{25}$$

$$\partial_\mu J_5^\mu(x) = \partial_\mu j_5^\mu(x) + \frac{e}{\pi}\left(\partial_0 A_0^1(t), 0\right) = \frac{e}{\pi}\partial_0 A_0^1(t) \neq 0\,. \tag{26}$$

From the ZM part of the Maxwell equation one directly has

$$\partial_0^2 A_0^1(t) = -\frac{e^2}{\pi} A_0^1(t)\,. \tag{27}$$

We have thus found that the Schwinger mechanism works only in the zero-mode sector, where it gives rise to the massive Schwinger boson with $\mu^2 = \frac{e^2}{\pi}$.

Next steps in the analysis will involve an introduction of the indefinite-metric space, explicit solution of the Maxwell equations and a derivation of the Hamiltonian in terms of independent field variables along with a study of its invariances (chiral symmetry, large gauge transformations). For example, the (modified) Maxwell equations $\partial_\mu \partial^\mu \tilde{A}^\nu = e j^\nu - \partial^\nu G$ will be inverted as

$$\tilde{A}^\mu(x) = A^\mu(x) + e\int_{-\infty}^{+\infty} d^2y D_{\mathrm{R0}}(x-y) j^\mu(y) - \int_{-\infty}^{+\infty} d^2y D_{\mathrm{R0}}(x-y)\partial^\mu G(y)\,. \tag{28}$$

Presence of the unphysical fields $A^\mu(x)$ in (28) is related to the residual gauge freedom, which can be removed on the quantum level by means of a unitary transformation to the Coulomb gauge representation [7, 8]. It is also necessary to find a mechanism for the vacuum degeneracy in the present approach. Here the gauge zero mode and its residual (large) gauge symmetry may play a role (note that the covariant gauge admits transformations with the gauge function of the form cx^1). These topics are presently under study.

This work was supported by the grant VEGA 2/0070/2009.

REFERENCES

[1] J. Schwinger, *Phys. Rev.* **128**, 2425 (1962).

[2] J. Lowenstein, J. Swieca, *Ann. Phys.* **68**, 172 (1971).

[3] G. Morchio, D. Pierotti, F. Strocchi, *Ann. Phys.* **188**, 217 (1988).

[4] W. Thirring, J. Wess, *Ann. Phys.* **27**, 331 (1964).

[5] L.S. Brown, *Nuovo Cim.* **29**, 617 (1963).

[6] M. Peskin, D. Schroeder, *An Introduction to Quantum Field Theory*, Addison–Wesley, 1995.

[7] K. Haller, E. Lim-Lombridas, *Found. Phys.* **24**, 217 (1994).

[8] F. Lenz, H. Naus, K. Ohta, M. Thies, *Ann. Phys.* **233**, 17 (1994).