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ELEMENTARY EXAMPLE OF MASS MIXING DYNAMICS WITHOUT INVOLVEMENT OF THE VACUUM*

Stanisław D. Głazek

Institute of Theoretical Physics, Faculty of Physics
University of Warsaw, Warszawa, Poland

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Most relativistic quantum field theories of interest appear to involve the unsolved problem of constructing a ground state, called vacuum. An elementary example of the vacuum problem appears in a theory in which the entire interaction is reduced to a mass-mixing term. This example can be solved using a new renormalization group procedure for effective particles and the relativistic solution is obtained without any need to solve the vacuum problem. This contribution briefly reviews the vacuum problem and explains how the new procedure works around it in the example.

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1. The vacuum problem

It has been stressed by Dirac [1] that the concept of canonical quantum field theory (QFT) introduced by Heisenberg and Pauli [2, 3] involves a problem of defining a ground state in agreement with principles of quantum mechanics and special relativity. Quantum mechanics requires summation over all relevant basis states in a calculation of evolution of every state and special relativity requires that there are infinitely many basis states to sum over because field quanta must appear in infinitely many states of motion. The resulting divergences require cutoffs on momentum and such cutoffs violate Lorentz symmetry. In their presence, there is no ground state built from the basis states that could be invariant under Lorentz transformations. This difficulty led Dirac to question the possibility that QED exists in the Schrödinger picture. He developed an alternative formulation that he called the Heisenberg picture, in which he avoided the concept of a ground state as a state in a Hilbert space and used perturbation theory to calculate observables.

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The vacuum problem in QCD is more severe than in QED. One reason is that the coupling constant in QCD is considerably larger than in QED and the cutoffs on the space of relevant states have to be small in order to exclude large terms in perturbation theory. The approach that Dirac adopted for curing QED using cutoffs does not apply in QCD; small cutoffs on $|\vec{p}\,|$, where $\vec{p}$ is a momentum of a quantum of a quark or a gluon field, lead to effects that violate the Lorentz symmetry much stronger than in QED. Although asymptotic freedom [4, 5] allows one to carry out logical perturbative calculations using QCD, it does not solve the vacuum problem that is not perturbative. Namely, one needs to explain strong chiral symmetry breaking in QCD and a non-trivial structure of the vacuum can be invoked as its origin [6, 7]. One also needs to explain confinement of color [8]. Confinement can be associated with the concept of a complex ground state that contains the quark and gluon condensates [9, 10].

While the problem of constructing a ground state in QCD awaits a solution, the vacuum problem in QFT is not limited to the theory of strong interactions. It is also involved in the spontaneous symmetry breaking as a mechanism of mass generation in the standard model [11] and a host of theories that try to explain the standard model, *e.g.*, see Ref. [12]. Ultimately, the vacuum problem is also relevant to cosmology [13, 14].

In his search for a well-defined relativistic theory of particles, Dirac has identified the front form (FF) of Hamiltonian dynamics [15] as particularly promising [16]. The FF promises new avenues for studying dynamics of quantum fields because 7 out of 10 FF generators of the Poincaré group do not depend on interactions and are easy to construct. In contrast, in the standard form of dynamics, which Dirac called the instant form (IF), only 6 out of the 10 IF generators are easy to construct. So, in the IF there are 4 generators that depend on interactions and are difficult to construct and in the FF there are only 3. Such construction requires incorporation of Wilsonian understanding of renormalization of coupling constants in Hamiltonians of local QFT [17, 18]. The issue of constructing a FF Hamiltonian formulation of QCD using renormalization group (RG) concepts akin to Wilsonian was re-addressed only quite recently [19] on the time scale of progress in our understanding of the vacuum problem.

The FF dynamics does not involve the vacuum problem in the same sense as the IF does [20, 21]. The FF problem can be formulated as an RG problem for FF Hamiltonians [19]. Namely, the FF regularization cutoffs eliminate the field modes that could contribute to the vacuum structure. The only RG-admissible consequence of the cutoffs is new terms in the renormalized FF Hamiltonians. These FF terms must be responsible for the effects that are associated with the vacuum in the IF. An alternative approach is to deal directly with the modes that are cut off, called zero-modes [22]. As time attests [23], this is extraordinary difficult. Note also that the IF vacuum problem itself may be misconstrued by associating the phenomenological concepts of vacuum condensates [9, 10] with expectation values of quantum fields in the ultimate IF ground state rather than in the localizable excited states, such as hadrons in QCD [24, 25].

2. Example of solution to an elementary vacuum problem

Dirac expressed his worry about the ground state problem in QFT using an elementary model of IF Hamiltonians for fermions. Even simpler model is discussed here. It describes neutral scalar bosons. The FF solution of the problem is found using the renormalization group procedure for effective particles (RGPEP) whose details in the context of the example can be found in Ref. [26]. Recent descriptions of RGPEP can be found in Refs. [27, 28].

2.1. The IF vacuum problem due to mass mixing

Classical Lagrangian density $\mathcal{L}_f = \left[(\partial\phi)^2 - \mu^2\phi^2 + (\partial\chi)^2 - \nu^2\phi^2\right]/2$ for two free neutral scalar fields ϕ and χ implies a canonical IF Hamiltonian $H_f = \int d^3x\, \mathcal{H}_f$ with the Hamiltonian density $\mathcal{H}_f = [\pi_\phi^2 + (\vec\nabla\phi)^2 + \mu^2\phi^2 + \pi_\chi^2 + (\vec\nabla\chi)^2 + \mu^2\chi^2]/2$. Quantization amounts to writing for $t = 0$

$$\phi(\vec x\,) \;=\; \int [p]_\mu\, a_{\vec p}\, e^{-ipx} + \text{h.c.}\,, \qquad \chi(\vec x\,) = \int [p]_\nu\, b_{\vec p}\, e^{-ipx} + \text{h.c.}\,, \tag{1}$$

where $[p]_a = d^4p\,\delta(p^2 - a^2)\,\theta(p^0)(2\pi)^{-3}$, assuming that the IF time derivatives in $\pi_\phi = \dot\phi$ and $\pi_\chi = \dot\chi$ are determined by the energies $E_\phi(\vec p\,) = \sqrt{\mu^2 + \vec p^{\,2}}$ and $E_\chi(\vec p\,) = \sqrt{\nu^2 + \vec p^{\,2}}$, and imposing commutation relations

$$\left[\hat\phi(\vec x\,), \hat\pi_\phi\,(\vec y\,)\right] \;=\; [\hat\chi(\vec x\,), \hat\pi_\chi\,(\vec y\,)] = i\delta^3\,(\vec x - \vec y\,)\;, \tag{2}$$

through which the classical fields ϕ and χ are replaced by their quantum counterparts $\hat\phi$ and $\hat\chi$. The resulting normal-ordered quantum Hamiltonian,

$$H_f(a,b) \;=\; \int [p]_\mu\, E_\phi(\vec p\,)\, a^\dagger_{\vec p} a_{\vec p} + \int [p]_\nu\, E_\chi(\vec p\,)\, b^\dagger_{\vec p} b_{\vec p}\;, \tag{3}$$

contains no terms of the type $a^\dagger_{\vec p} a^\dagger_{-\vec p}$. They disappear because the combination of bilinear terms $\hat\pi_\phi^2 + (\vec\nabla\hat\phi)^2 + \mu^2\hat\phi^2$ yields them in the form

$$\int [p]_\mu \left(-E_\phi^2 + \vec p^{\,2} + \mu^2\right)\, a^\dagger_{\vec p} a^\dagger_{-\vec p} \tag{4}$$

and the right choice of energy E_ϕ in the time derivatives used to define $\hat{\pi}_\phi$ makes these terms vanish. Terms of the type $b^\dagger_{\vec{p}} b^\dagger_{-\vec{p}}$ vanish because of the right choice of E_χ in the time derivatives in $\hat{\pi}_\chi$.

Now let us add to the free Lagrangian density $\mathcal{L}_f$ the mass mixing term $\mathcal{L}_{\rm I} = -\, m^2 \, \phi \, \chi$ with m much smaller than μ, ν, and $|\mu - \nu|$, so that one might think it is a small perturbation. The corresponding interaction Hamiltonian is

$$H_{\rm I} \; = \; - \int d^3x \, \mathcal{L}_{\rm I} = \int [p]_\mu \, \frac{m^2}{2E_\chi} \, \left(a^\dagger_{\vec{p}} b^\dagger_{-\vec{p}} + a^\dagger_{\vec{p}} b_{\vec{p}} + b^\dagger_{\vec{p}} a_{\vec{p}} + a_{\vec{p}} b_{-\vec{p}} \right) \, . \quad (5)$$

One could develop a relativistic quantum theory using the Hamiltonian $H = H_f + H_{\rm I}$ if not the fact that the norm of the state

$$H_{\rm I} |0\rangle \; = \; \int [p]_\mu \, \frac{m^2}{2E_\chi} \, a^\dagger_{\vec{p}} b^\dagger_{-\vec{p}} |0\rangle \quad (6)$$

is infinite. Actually, every state that results from action of $H_{\rm I}$ on any state in the Fock space has infinite norm. The more powers of $H_{\rm I}$ one applies to any state, the more divergences are created. The evolution operator $U = \exp{(-iHt)}$ does not exist.

To make $H_{\rm I}$ finite, one may introduce a cutoff, such as $|\vec{p}| < \Lambda$. But such cutoff violates Lorentz symmetry. The ground state of the theory is not equal to the state $|0\rangle$ that is annihilated by a and b. It must involve pairs of particles of type a and b created by $H_{\rm I}$. Such state cannot satisfy the requirements of special relativity for finite cutoffs Λ. This is an elementary example of the Dirac vacuum problem in QFT.

The mass-mixing model is so simple that the problem can be discussed further (see below). However, when one considers $H_{\rm I}$ in any realistic theory, especially in gauge theories, similar divergences occur with three or four quanta being created from the bare Fock vacuum $|0\rangle$. Because of these divergences, Dirac suggested that we should abandon the Schrödinger concept of quantum mechanics in the case of fields.

2.2. IF recipe in the elementary example: re-quantization

The elementary model can be handled in a unique way further despite its divergent nature in the IF, because its Lagrangian is bilinear in fields. One can go back to the classical fields, diagonalize the mass-squared matrix in Lagrangian, and define new fields that are linear combinations of the initial ones

$$\xi \; = \; \cos c_\infty \; \phi - \sin c_\infty \; \chi \, , \quad (7)$$

$$\zeta \; = \; \sin c_\infty \; \phi + \cos c_\infty \; \chi \, , \quad (8)$$

where $c_\infty = -\arctan\sqrt{\frac{\epsilon-1}{\epsilon+1}}$ and $\epsilon = \sqrt{1+[2m^2/(\mu^2-\nu^2)]^2}$. The origin of the subscript ∞ will be explained on the next page. In terms of the fields ξ and ζ, the same classical Lagrangian takes the form

$$\mathcal{L} \;=\; \mathcal{L}_f + \mathcal{L}_\mathrm{I} = \left[(\partial\xi)^2 - m_1^2\xi^2 + (\partial\zeta)^2 - m_2^2\zeta^2\right]/2\,, \tag{9}$$

which is a free theory of scalar fields that can be quantized from scratch as before, except that the time derivatives in the construction of $\hat\pi_\xi$ and $\hat\pi_\zeta$ need to be $E_\xi(\vec p) = \sqrt{m_1^2+\vec p^{\,2}}$ and $E_\zeta(\vec p) = \sqrt{m_2^2+\vec p^{\,2}}$ in order to avoid the interactions that create particles from $|0\rangle$. As far as the author knows, there is no known extension of this IF recipe to realistic theories beyond perturbation theory.

2.3. FF RGPEP solution in the elementary example

Following the steps described in Refs. [29, 30], readers should be able to establish that the FF quantum Hamiltonian in the example is

$$\mathcal{P}^- = \int_{x^+=0} d^3x \;:\left\{\left[\left(\partial^\perp\hat\phi\right)^2 + \mu^2\hat\phi^2 + \left(\partial^\perp\hat\chi\right)^2 + \nu^2\hat\chi^2\right]/2 + m^2\hat\phi\,\hat\chi\right\}:\,. \tag{10}$$

No derivatives with respect to x^+ appear and it is sufficient to write

$$\hat\phi\left(x^{-\perp}\right) = \int [p]\, a_{\vec p}\, e^{-ipx} + \text{h.c.}\,, \qquad \hat\chi\left(x^{-\perp}\right) = \int [p]\; b_{\vec p}\, e^{-ipx} + \text{h.c.} \tag{11}$$

(The FF $\hat\pi_\phi$ and $\hat\pi_\chi$ are gradients of the fields with respect to x^-.) In comparison with Eq. (1) in the IF, one can also notice the absence of subscripts μ or ν in the FF integration measure over momentum $\vec p = (p^+, p^\perp)$; the measure does not depend on the mass parameter. In terms of the creation and annihilation operators

$$\mathcal{P}^- = \int [p]\left[\frac{p^{\perp\,2}+\mu^2}{p^+}\, a^\dagger_{\vec p} a_{\vec p} + \frac{p^{\perp\,2}+\nu^2}{p^+}\, b^\dagger_{\vec p} b_{\vec p} + \frac{m^2}{p^+}\left(a^\dagger_{\vec p} b_{\vec p} + b^\dagger_{\vec p} a_{\vec p}\right)\right]. \tag{12}$$

The mass-mixing interaction terms of the type $a^\dagger_{\vec p} b^\dagger_{-\vec p}$ are absent because $p^+ = \sqrt{a^2+p^{\perp\,2}+p^{z\,2}} + p^z > 0$ if $a^2 > 0$ and $|\vec p| \le \Lambda < \infty$. The presence of the cutoff Λ is not an obstacle in the FF dynamics in the example (see below). The state $|0\rangle$ is an eigenstate of $\mathcal{P}^-$ with eigenvalue 0 and plays the role of a ground state. But what is the spectrum of $\mathcal{P}^-$ due to the remaining mixing terms? The answer is found using RGPEP. There is no room or need to explain RGPEP here and the reader may consult Refs. [26–28].

The key idea is that one can transform the fields $\hat{\phi}$ and $\hat{\chi}$ according to the rule $\hat{\psi}_t = \mathcal{U}_t \, \hat{\psi}_0 \, \mathcal{U}_t^\dagger$, where t is the RGPEP scale-evolution parameter with interpretation of the size of effective particles. The canonical fields correspond to the pointlike, bare particles and $t = 0$. The $\mathcal{P}^-$ eigenstates built from physical particles are obtained using operators $\hat{\psi}_t$ with $t \to \infty$. This is the origin of the subscript ∞ in Eqs. (7) and (8). The operator $\mathcal{U}_t$ is found by solving the RGPEP equation for $\mathcal{P}_t^-$ (see the quoted literature),

$$\frac{d}{dt} \mathcal{P}_t^- \; = \; \left[\left[\mathcal{P}_f^- , \mathcal{P}_{Pt} \right] , \mathcal{P}_t^- \right] , \qquad \mathcal{P}_0^- = \mathcal{P}^- , \tag{13}$$

and integrating, $\mathcal{U}_t = T \exp\left(- \int_0^t d\tau \, [\mathcal{P}_f^- , \mathcal{P}_{P\tau}] \right)$, where T denotes ordering in the scale parameter. The operator $\mathcal{P}_{Pt}$ is obtained from $\mathcal{P}_t^-$ by multiplying its terms twice by P^+, where P^+ denotes the $+$-component of a total momentum carried by the particles that participate in the interaction. Thanks to the 7 kinematical symmetries of the FF dynamics, preserved in RGPEP, the non-perturbative operator Eq. (13) reduces to just one non-linear equation for a 2×2 mass-squared matrix, which is entirely independent of the cutoff Λ and thus respects the Poincaré symmetry irrespective of the value of the cutoff,

$$\frac{d}{dt} \left[\begin{array}{cc} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{array} \right] = \left[\left[\left[\begin{array}{cc} \mu^2 & 0 \\ 0 & \nu^2 \end{array} \right] , \left[\begin{array}{cc} 0 & m_t^2 \\ m_t^2 & 0 \end{array} \right] \right] , \left[\begin{array}{cc} \mu_t^2 & m_t^2 \\ m_t^2 & \nu_t^2 \end{array} \right] \right] \tag{14}$$

with initial conditions $\mu_0 = \mu$, $\nu_0 = \nu$, $m_0 = m$. This equation only slightly differs from Wegner's flow equation for a 2×2 Hamiltonian matrix [31–33]. The solution is expressible in terms of simple functions and for $t \to \infty$ one obtains $\mu_\infty^2 = m_1^2$, $\nu_\infty^2 = m_2^2$, $m_\infty^2 = 0$, *i.e.*, at the end of RGPEP the mass mixing interaction is eliminated and the spectrum of eigenstates of $\mathcal{P}^-$ is obtained in terms of the Fock space of free effective particles of masses m_1 or m_2, the same as in Sec. 2.2. However, the FF solution of the initial quantum theory did not involve any re-quantization, or guessing time derivatives to construct $\hat{\pi}$. In the entire RGPEP scale-evolution, $|0\rangle_t \equiv |0\rangle$.

3. Conclusion

It is not clear how to deal with the vacuum problem in the IF of dynamics in realistic theories. The elementary example discussed here is not sufficient as a guide when more than just two fields appear in a product. In contrast, the FF RGPEP, which was used here to solve the elementary example, can be also systematically attempted in realistic QFT, at least perturbatively (see the quoted literature). Since RGPEP is formulated non-perturbatively, it may in principle be attempted also beyond perturbation theory.

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