NAKANISHI REPRESENTATION ONTO THE NULL PLANE AND THE SOLUTION OF THE BETHE–SALPETER EQUATION*

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The Nakanishi perturbative integral representation of the 4D Bethe–Salpeter amplitude is used to solve the bound state problem in the Minkowski space. The main step to derive workable equations for Nakanishi weight function is provided by the projection onto the null-plane of the 4D Bethe–Salpeter Equation. We present a homogeneous equation for the Nakanishi weight-function for the ladder approximation of a bosonic model obtained by using uniqueness of the Nakanishi weight function. We provide numerical results and compare with another solution method.

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1. Introduction

The Bethe–Salpeter equation (BSE) is useful to treat the nonperturbative regime in two-body, three-body, *etc.*, in both nuclear and hadronic physics. The popular way to solve it for bound-states adopts a Wick rotation to the Euclidean space, and free propagators see (*e.g.* [1]), while other methods like the spectator approximation use a 3D reduction [2]. At the

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hadronic level, Dyson–Schwinger and BSE, with gluonic interactions and running coupling constant in Euclidean space, is a tool for dealing with the partonic structure of hadrons [3]. The Minkowski space methods of solving the BSE, were already developed exploiting the Light-Front (LF) projection and the expansion of the kernel according to the Fock-space content, *i.e.*, the reduction of the dynamics to the valence state with an effective interaction built, from a summation over terms, where higher Fock-states propagate between initial and final LF time. The kernel in the squared mass eigenvalue equation for the valence state, which embeds the full-covariant content of the BSE, can be systematically expanded by using the quasi-potential formalism [4, 5]. Then, the full BS amplitude is obtained through a suitable expansion. This approach resembles the "Iterated Resolvent Method" within Hamiltonian approach [6], where the full eigenvalue equation in Fock-space is reduced to the valence state giving rise to an effective squared mass operator. It is quite nice that a revival of such an approach is recovered within AdS/QCD models, where the mass squared eigenvalue equation for the string mode amplitude has a correspondence with the valence state and associated eigenvalue equation (see *e.g.* [7]).

The perturbation theory integral representation (PTIR) [8] of any multileg transition amplitude is based on a "parametric representation of *any Feynman diagram for interacting bosons*, with a denominator carrying the overall analytic behavior in the Minkowski space" [8]. The final expression of PTIR amplitude is given by a manyfold integral, where a function of real variables, the so-called Nakanishi weight-function, is present. A uniqueness theorem for the Nakanishi weight-function for bosonic systems is also demonstrated in [8]. Finally, once an integral equation determining the Nakanishi amplitude is introduced, it should be pointed out that PTIR can be extended to the nonperturbative regime, despite the perturbative framework where it has been originally devised. The solution of the bound-state BSE in ladder approximation for bosons interacting by exchanging a massive scalar, has been shown to be numerically tractable within PTIR approach [9]. Moreover, a substantial algebraic simplification can be achieved through the introduction of the LF projection of the BSE for the bound-state problem, as shown by Karmanov and Carbonell for the ladder approximation [10] and cross-ladder [11]. Notice that the numerical results are in agreement with both the ones found in [9] and by Euclidean approaches.

In this contribution, the Nakanishi PTIR of the 4D BS amplitude is used to treat bound states in the Minkowski space. We show some preliminary numerical results obtained by solving a homogeneous equation for the Nakanishi weight-function derived by applying uniqueness [12]. It is an alternative form of the bound state equation found in [10] for the ladder approximation of a bosonic model, close to Ref. [9], but within LF framework, which allowed for a much more simple formulation of the kernel. The same approach was extended to the inhomogeneous BSE within PTIR, in order to find the scattering amplitude [12].

We will briefly review the formal development of our approach for a bosonic system, composed by two massive scalars interacting through the exchange of a massive scalar. The explicit expression of the new integral equation will be shown, and, as simple applications of our formalism, some limiting cases, like the Wick–Cutkosky model will be discussed. We provide numerical results for the bound state for different intermediate boson masses, in order to compare the integral formulation by Karmanov and Carbonell and the new equation obtained by applying uniqueness.

2. Nakanishi weight function for a s-wave bound state

2.1. The Karmanov and Carbonell approach

In what follows, we will briefly present the method introduced in [10] to solve the BSE. The starting point is the projection on to the LF of the BS amplitude, determined by the BS integral equation. The LF projection is performed by integrating out the k^- dependence [4], as follows

$$\int \frac{dk^-}{2\pi}\Phi_b(k,p) \;=\; \int \frac{dk^-}{2\pi}G_0^{(12)}(k,p)\int \frac{d^4k'}{(2\pi)^4}i\mathcal{K}(k,k',p)\Phi_b(k',p)\,, \quad (1)$$

where $G_0^{(12)}(k,p) = (-i)^2\left[((\frac{p}{2}+k)^2 - m^2 + i\epsilon)((\frac{p}{2}-k)^2 - m^2 + i\epsilon)\right]^{-1}$. The function $\mathcal{K}(k,k',p)$ represents the sum of all two-body irreducible four-leg off-shell amplitudes. As shown in detail, for instance in Ref. [12], the left-hand side of Eq. (1) yields the valence component of the interacting two-body state under consideration.

The second step is given by the introduction of the Nakanishi PTIR form of the BS amplitude

$$\Phi_b(k,p) = -i\int\limits_{-1}^{1} dz' \int\limits_{0}^{\infty} d\gamma' \frac{g_b\left(\gamma',z';\kappa^2\right)}{\left[\gamma' + m^2 - \frac{1}{4}p^2 - k^2 - p\cdot k\; z' - i\epsilon\right]^{2+n}}\,, \quad (2)$$

where $p^2 = M^2$, $\kappa^2 = m^2 - \frac{M^2}{4}$, and g_b is the Nakanishi weight-function. For $n=1$, one gets

$$\int\limits_{0}^{\infty} d\gamma'\; \frac{g_b\left(\gamma',z;\kappa^2\right)}{[d(\gamma',\gamma,z)]^2} = \int\limits_{0}^{\infty} d\gamma' \int\limits_{-1}^{1} dz'\, V_b^{\rm LF}\left(\gamma,z;\gamma',z'\right) g_b\left(\gamma',z';\kappa^2\right)\,, \quad (3)$$

where $d(\gamma',\gamma,z) = \gamma' + \gamma + z^2m^2 + (1-z^2)\kappa^2$ and the kernel reads

$$V_b^{\rm LF}\left(\gamma,z;\gamma',z'\right) = \int_{-\infty}^{\infty} \frac{dk^-}{2\pi} \int \frac{d^4k'}{(2\pi)^4} \frac{p^+ \, G_0^{(12)}(k,p)\, \mathcal{K}(k,k',p)}{\left[k'^2 + p\cdot k' z' - \gamma' - \kappa^2 + i\epsilon\right]^3}\,, \quad (4)$$

where $\gamma = |\boldsymbol{k}_\perp|^2$ and $z = -2k^+/M$. In what follows, we specialize Eq. (4) by adopting the ladder approximation, that is $\mathcal{K}(k,k',p) = ((k-k')^2 - \mu^2 + i\epsilon)^{-1}$, with μ the exchanged boson mass.

2.2. Applying uniqueness

Assuming that the uniqueness theorem for g_b holds in a nonperturbative regime, the integral equation (3), in ladder approximation can be putted in a different form. In order to accomplish such a step, the expression of $V_b^{\rm LF}$ in ladder approximation, has to be rewritten factorizing an overall denominator equal to the one in the left-hand side of Eq. (3), *viz.*

$$V_b^{\rm (L)}\left(\gamma,z;\gamma',z'\right) = \frac{g^2}{2(4\pi)^2} \int_{-\infty}^{\infty} d\gamma'' \, \frac{\frac{(1+z)}{(1+z')}\theta(z'-z)\; h\left(\gamma'',z;\gamma',z';\mu^2\right)}{\left[\gamma + \gamma'' + z^2m^2 + \kappa^2(1-z^2) - i\epsilon\right]^2} + \left(z \to -z, z' \to -z'\right)\,, \quad (5)$$

where g is the coupling constant and the function h is given by

$$\begin{aligned} h\left(\gamma'',z;\gamma',z';\mu^2\right) = \theta\left[\gamma''\frac{(1+z')}{(1+z)} - \mu^2\right] \theta(\gamma'_- - \gamma')\;\theta(\gamma'_-)\Bigg[-\frac{\mathcal{B}_b}{\gamma''\,\mathcal{A}_b\;\Delta} \\ + \frac{(1+z')}{(1+z)} \int_{y_-}^{y_+} dy\; \frac{y^2}{\left[y^2\mathcal{A}_b + y(\mu^2+\gamma') + \mu^2\right]^2}\Bigg] \\ -\frac{(1+z')}{(1+z)} \int_0^{\infty} dy\; \frac{y^2}{\left[y^2\mathcal{A}_b + y(\mu^2+\gamma') + \mu^2\right]^2}\,, \end{aligned} \quad (6)$$

with

$$\begin{aligned} &\mathcal{A}_b = z'^2\frac{M^2}{4} + \kappa^2 + \gamma' \geq 0\,, \qquad \mathcal{B}_b\left(z,z',\gamma',\gamma'',\mu^2\right) = \mu^2 + \gamma' - \gamma''\frac{(1+z')}{(1+z)} \leq 0\,, \\ &\gamma'_- = \mu^2 + \gamma''\frac{(1+z')}{(1+z)} - 2\mu\sqrt{\gamma''\frac{(1+z')}{(1+z)} + z'^2\frac{M^2}{4} + \kappa^2} \geq 0\,, \\ &y_\pm = \frac{-\mathcal{B}_b \pm \Delta}{2\mathcal{A}_b}\,, \qquad \Delta^2 = \mathcal{B}_b^2 - 4\mu^2\mathcal{A}_b \geq 0\,. \end{aligned} \quad (7)$$

By applying the uniqueness of the Nakanishi weight-function, we get a simpler form of the integral equation for g_b by introducing Eq. (5) in (3), *viz.*

$$g_b\left(\gamma, z; \kappa^2\right) = \frac{g^2}{2(4\pi)^2} \int_0^\infty d\gamma' \int_{-1}^{1} dz' \left[\frac{(1+z)}{(1+z')}\, \theta(z'-z) h\left(\gamma, z; \gamma', z'; \mu^2\right) + \left(z \to -z, z' \to -z'\right)\right] g_b\left(\gamma', z'; \kappa^2\right) . \tag{8}$$

Equation (8) is a standard eigenvalue equation, where $1/g^2$ is the eigenvalue and g_b the eigenvector.

2.3. *Wick–Cutkosky model ($\mu = 0$)*

The integral equation for the Nakanishi weight-function (8) simplifies for $\mu = 0$ and it reads

$$g_b^{\mathrm{LW}}(\gamma, z) = \frac{g^2}{2(4\pi)^2}\theta(\gamma) \int_0^\infty \frac{d\gamma'}{\gamma'} \int_{-1}^{+1} dz' \frac{g_b^{\mathrm{LW}}(\gamma', z')}{\left[z'^2 \frac{M^2}{4} + \kappa^2 + \gamma'\right]} \times \left[\theta\left(z'-z\right)\theta\left(\gamma' - \frac{(1+z')}{(1+z)}\gamma\right) + \theta\left(z - z'\right)\ \theta\left(\gamma' - \frac{(1-z')}{(1-z)}\gamma\right)\right] . \tag{9}$$

Notice that if $\gamma \to \infty$ and $g_b^{\mathrm{LW}}(\gamma', z')$ is taken as a constant, then $g_b^{\mathrm{LW}}(\gamma, z) \to 1/\gamma$. By continuously iterating, one sees that $g_b^{\mathrm{LW}}(\gamma, z)$ decreases faster than any power of $1/\gamma$ and, therefore, it is reasonable to introduce a factorized form as: $g_b^{\mathrm{LW}}(\gamma', z') = f_b^{\mathrm{LW}}(z')\delta(\gamma' - \epsilon)$ $(\epsilon \geq 0)$ in (9), which simplifies to

$$f_b^{\mathrm{LW}}(z) = \frac{g^2}{2(4\pi)^2} \int_{-1}^{+1} dz' \frac{f_b^{\mathrm{LW}}(z')}{\left[z'^2 \frac{M^2}{4} + \kappa^2\right]} \left[\frac{(1+z)}{(1+z')}\theta\left(z'-z\right) + \frac{(1-z)}{(1-z')}\theta\left(z-z'\right)\right]$$

giving a well known expression (see *e.g.* [13]).

3. Results and perspectives

We have solved numerically both integral equations (3) and (8), the latter one found by applying uniqueness. The Nakanishi amplitude for the s-wave solutions of the BSE can be expanded in a bi-orthonormal basis, as follows

$$g(\gamma, z) = \sum_{i=1}^{N_g} \sum_{j=1}^{N_z} a_{ij} f_i(\gamma) h_j(z) \,, \tag{10}$$

where $f_i(\gamma) = \sqrt{\beta} L_{i-1}(\beta\gamma)\exp(-\beta\gamma/2)$ and $h_j(z) = \mathcal{N}_n(1-z^2)C_n^{(\frac{5}{2})}(z)$ with $n = 2(j-1)$ and $\mathcal{N}_n$ is a normalization factor. L_{i-1} (C_n) are Laguerre (Gegenbauer) polynomials, and β a parameter. In Table I, preliminary results for $\alpha = g^2/(4\pi\ m)^2$, at given $B/m = 2 - M/m$ and μ/m are shown. The results are numerically stable and the comparison between the different methods suggests that uniqueness holds nonperturbatively, at least at the level of the eigenvalues of the BSE. More detailed studies have still to be performed in order to investigate how the Nakanishi weight-functions, calculated with different methods, compare.

TABLE I

Comparison between $\alpha' s$ (see the text) from Ref. [10] and our solutions of (3) and (8).

B/m	μ/m	Ref. [10]	Eq. (3)	Eq. (8)	μ/m	Ref. [10]	Eq. (3)	Eq. (8)
1.	.5	6.712	6.7114	6.7113	.15	5.315	5.3137	5.3133
.5		4.901	4.9006	4.9005		3.611	3.6107	3.6090
.2		3.251	3.2511	3.2512		2.100	2.0993	2.0962
.1		2.498	2.4980	2.5001		1.437	1.4366	1.4373
.01		1.440	1.4401	1.4400		0.5716	0.57168	0.5736

In summary, within the light-front framework, we have reviewed the Nakanishi integral equation for bound states found by using uniqueness and provided preliminary numerical solutions, which show a successful comparison between different approaches. As a perspective, we would say that the Nakanishi PTIR is not constrained to 3+1 dimensions or to interacting bosons. Further extension to 2+1 dimensions (*e.g.*, useful to deal with the Dirac electrons in a graphene sample) and to fermionic systems (like NN) will be presented elsewhere.

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