BOUND STATE CALCULATIONS IN QED AND QCD USING BASIS LIGHT-FRONT QUANTIZATION*

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In order to describe self-bound systems, one needs a nonperturbative approach. We discuss the relativistic bound state equations of QED and QCD formulated in Basis Light-Front Quantization. In this approach, the light-front direction is discretized, and two-dimensional harmonic oscillator basis functions are used for the transverse direction. At present, the Fock-space in our calculations is limited to the minimal Fock sector, but the extension to an arbitrary number of (anti-)fermions and gauge bosons is in principle straightforward. We present initial results for the energies and distribution functions of two-body bound states obtained within this approach and discuss convergence issues.

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1. Nonrelativistic many-body bound state problem

Many-body bound state problems are recognized to be computationally hard problems. The challenge is how to deal with self-bound quantum systems, with many degrees of freedom, and strong interaction, while at the same time respecting all relevant symmetries. In recent years, we have made tremendous progress in describing the structure of atomic nuclei and their interactions with matter and radiation using realistic two-body and three-body potentials through state-of-the-art numerical methods on large-scale supercomputers [1, 2].

One of the commonly used methods in these nonrelativistic bound-state calculations is the Configuration Interaction (CI) approach for solving the many-body nuclear Hamiltonian in a (sufficiently large) basis space of Slater Determinants of single-particle states. In this approach, the wavefunction Ψ

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of a nucleus consisting of A nucleons is expanded in an A-body basis of Slater Determinants of single-particle states

$$\Psi\left(\vec{r}_1,\ldots,\vec{r}_A\right) \;=\; \sum c_k \Phi_k(\vec{r}_1,\ldots,\vec{r}_A)\,, \tag{1}$$

with $\Phi_k(\vec{r}_1,\ldots,\vec{r}_A) = \mathcal{A}[\phi_{n_1 l_1 j_1 m_1}(\vec{r}_1)\,\phi_{n_2 l_2 j_2 m_2}(\vec{r}_2)\ldots\phi_{n_A l_A j_A m_A}(\vec{r}_A)]$, and (typically) a harmonic oscillator (HO) basis for the single-particle states with quantum numbers n, l, j, and m. The nonrelativistic many-body Schrödinger equation then becomes an eigenvalue problem

$$H\,|\Psi_i\rangle \;=\; E_i\,|\Psi_i\rangle\,, \tag{2}$$

where the Hamiltonian $H = T + V$ consists of the kinetic energy operator T plus the nuclear potential V. For an A-body bound state and a two-body (plus three-body) potential, this becomes a large sparse matrix problem.

In a complete basis, this method would give exact results for a given input interaction V. However, practical calculations can only be done in a finite-dimensional truncation of a complete basis. The $N_{\max}$ truncation, in which the total number of HO quanta in a basis state is limited: $\sum N_k \leq N_0 + N_{\max}$, is an efficient truncation scheme for light nuclei. Here, $N_k = 2\,n_k + l_k$ is the number of quanta of each single-particle state in the many-body basis state, and $N_{\max}$ is the truncation parameter limiting the number of HO quanta above the minimum, N_0, for the system. This corresponds to a truncation on the system's total kinetic energy, and leads to an exact factorization of the Center-of-Mass (CM) wavefunction and the relative wavefunction.

2. Bound state problems in relativistic quantum field theory

There exist different methods to study bound state problems in QED and QCD: *e.g.* on the lattice, using the (covariant) Bethe–Salpeter equation, or with Hamiltonian methods, either in the equal-time formulation or in light-front formulation. One advantage of the latter is that it provides immediate access to light-front observables such as the distribution functions. Here we use Basis Light-Front Quantization (BLFQ) [3–5], in which the light-front direction is discretized, and two-dimensional HO basis functions are used for the transverse direction.

We use the usual Fock space expansion [6]. For each Fock sector we expand the many-body wavefunction in products of Slater Determinants of single-particle states for identical fermions (Permanents for identical bosons)

$$\Psi(x_i; p_{\perp,i}) \;=\; \sum c_j \Phi_j(x_i; p_{\perp,i})\,. \tag{3}$$

Since we use single-particle coordinates in the transverse direction, rather than relative (Jacobi) coordinates, the (anti-)symmetrization for identical particles is straightforward to implement. However, the obtained wavefunctions include the CM motion in the transverse direction. The main differences from the nonrelativistic case, see Eq. (1), are: (1) the LF wavefunctions depend on the longitudinal momentum fraction x_i and a two-dimensional transverse momentum $p_{\perp,i}$; (2) the mass operator is quadratic; and (3) the number of constituents (partons) is not conserved — different Fock space sectors are coupled to each other.

We discretize the momentum fraction in the longitudinal direction [6], with anti-periodic boundary conditions for fermions, and periodic boundary conditions for bosons, with the constraint $\sum x_i = 1$, that is, the total P^+ is kept fixed. We use N_x to indicate the number of intervals (of length $1/N_x$) in the longitudinal direction.

3. Basis Light-Front Quantization

For the transverse direction, we use a two-dimensional HO basis [3–5]. We expand the mass operator (or LF Hamiltonian) in this finite-dimensional basis, such that the bound state problem reduces to an eigenvalue problem

$$\mathcal{M}^2|\Psi_i\rangle \;=\; \lambda_i|\Psi_i\rangle\,, \tag{4}$$

where the discrete (positive definite) eigenvalues λ_i give the bound state spectrum, $M_i = \sqrt{\lambda_i}$. In BLFQ, the mass operator can be written as

$$\mathcal{M}^2 \;=\; P^+\,T_{\rm sp} - P_\perp^2 + P^+\,V_{\rm rel} \tag{5}$$

with $T_{\rm sp}$ the single-particle kinetic energy operator, and $V_{\rm rel}$ the interaction.

The total kinetic energy, including CM motion, is given by the sum of the single-particle kinetic energies

$$P^+\,T_{\rm sp} \;=\; \sum \frac{p_{\perp,i}^2 + m_f^2}{x_i}\,, \tag{6}$$

whereas the CM kinetic energy associated with the motion in the transverse direction is given by

$$P^+\,T_{\rm CM} \;=\; P_\perp^2 = \Big(\sum p_{\perp,i}\Big)^2\,. \tag{7}$$

In QED using light-front gauge we have two-body interactions, such as instantaneous one-photon and one-electron exchange, as well as interactions that couple different Fock space sectors: creation or annihilation of one

photon, electron–positron pair-creation, *etc.* In QCD, we have additional interaction terms coming from the gluon self-interactions. There are at most four (anti-)particles involved in each interaction term: the basis interactions can all be categorized as either $(2 \leftrightarrow 2)$, or $(1 \leftrightarrow 2)$, or $(1 \leftrightarrow 3)$ [6].

We use a transverse HO basis in the conjugate variables (q, s)

$$(q_i = p_{\perp,i}/\sqrt{x_i}\,, \qquad s_i = r_{\perp,i}\,\sqrt{x_i})\,, \tag{8}$$

defined by the eigenfunctions of the nonrelativistic HO equation in two dimensions $(q^2 + b^4\, s^2)\phi_i = e_i\phi_i$ with eigenvalues $e_i = (1+2n_i+|m_i|)2b^2$. The total transverse CM energy becomes

$$P^+\, H_{\rm CM} \;=\; \Big(\sum \sqrt{x_i}\; q_{\perp,i}\Big)^2 + b^4\,\Big(\sum \sqrt{x_i}\; s_{\perp,i}\Big)^2 \tag{9}$$

in this basis. In combination with a truncation on the total number of HO quanta in the system, $\sum(1 + 2n_i + |m_i|) \leq N_{\rm max}$, the transverse CM wavefunction and the transverse relative wavefunction factorize for any value of the basis parameters $N_{\rm max}$ and b. (Implicitly this also acts as a Fock space truncation: each additional parton adds at least one quantum.) Note that with a HO basis in the transverse momenta and coordinates $(p_\perp, r_\perp)$, as was used in Refs. [3–5], the CM wavefunction does not factorize in general.

We can now use the Lagrange multiplier method [7] in order to remove the CM excited states from the low-lying spectrum. That is, we replace the mass operator of Eq. (5) by

$$\mathcal{M}^2 \;=\; P^+\, T_{\rm sp} - P_\perp^2 + P^+\, V_{\rm rel} + \Lambda_{\rm CM}\,\big(P^+\, H_{\rm CM} - 2\,b^2\big) \tag{10}$$

with $\Lambda_{\rm CM} > 0$. This will increase the eigenvalues λ_i of CM excited states by at least $2b^2\Lambda_{\rm CM}$, but does not change the states with the lowest $(0s)$ CM motion. By choosing a sufficiently large $\Lambda_{\rm CM}$ we can eliminate all CM excitations from the spectrum up to the highest excitation energy of interest.

4. Numerical results for two-body bound states

In Fig. 1, we show results for a fermion–antifermion bound state in the minimal Fock sector, using the instantaneous interaction

$$\big(P^+ V_{\rm inst}\big)_{ij} \;=\; \frac{-4\,\alpha}{(x_i - x_j)^2 + \epsilon}\delta^2(r_i - r_j) \tag{11}$$

as well as a confining potential

$$\big(P^+ V_{\rm conf}\big)_{ij} \;=\; \kappa^4\; x_i\, x_j\, (r_i - r_j)^2 \tag{12}$$

motivated by soft-wall AdS/CFT [8]. Neither of these interactions flip any spins, so for simplicity we only consider states with anti-parallel spins. We use the parameters $\alpha = 0.3$, $\kappa = 0.2\,\mathrm{GeV}$, $m = 1.5\,\mathrm{GeV}$ and a regulator $\epsilon = 0.001$ for the instantaneous term. Without the instantaneous term, we would find a discrete HO spectrum due to the confining potential in the transverse direction, but with an infinite degeneracy in the limit $N_x \to \infty$ for each HO level. These discrete HO levels are shown as the green horizontal lines in the left panel of Fig. 1. The instantaneous interaction lifts this degeneracy, and we see converged discrete levels in the limit $N_x \to \infty$.

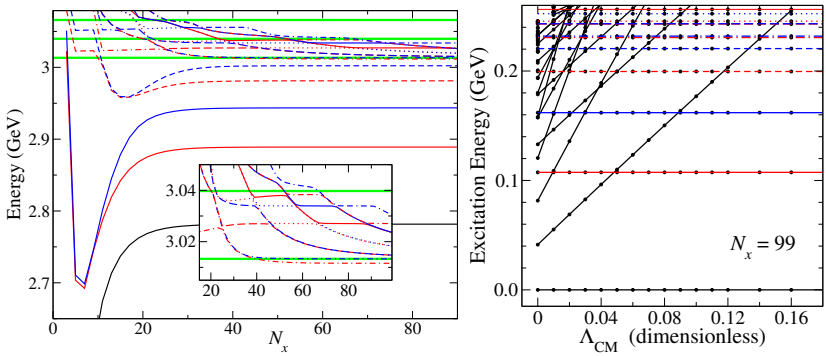

Fig. 1. Results for $N_{\max} = 12$: Lowest fifteen states as function of N_x (left). Solid light grey/green horizontal lines represent the spectrum of the unperturbed confining potential in the transverse direction. Low-lying spectrum as function of $\Lambda_{\mathrm{CM}} > 0$ (right).

Without the Lagrange multiplier term, however, it is not easy to interpret the spectrum. As expected, $\Lambda_{\mathrm{CM}} > 0$ increases the excitation energies of states with CM excitations, and by increasing Λ_{CM} we can remove CM excited states from the low-lying spectrum. Furthermore, we can clearly recognize states with different number of CM quanta in Fig. 1: the steepness of the lines is proportional to the number of CM excitations.

The lowest couple of excited states of Fig. 1 are excitations in the longitudinal direction, as is evident from Fig. 2. In addition to these longitudinal excitations (which become degenerate with the ground state in the absence of the instantaneous interaction), we can also recognize excited states corresponding to HO excited states in the absence of instantaneous interaction, despite all the level crossings in Fig. 1, *e.g.* at 3.027 GeV and at 3.034 GeV.

The extension of this approach to baryons is straightforward. We have performed initial calculations in finite model spaces for systems of three quarks, using the same interactions Eqs. (11) and (12), and find qualitatively similar results. In particular, we have confirmed the factorization of the transverse CM motion in this HO basis, independent of the quark masses.

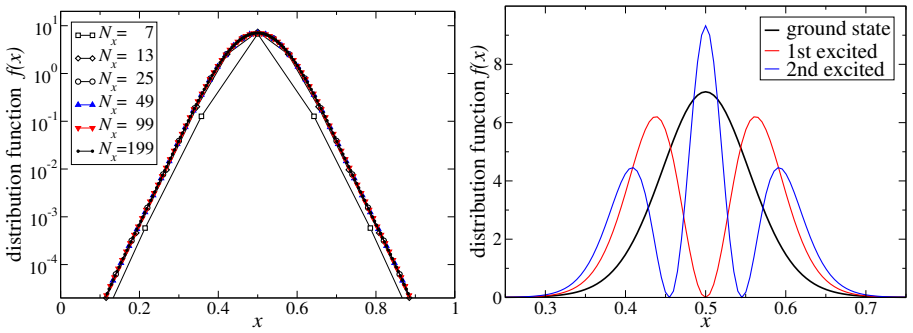

Fig. 2. Left: Convergence with N_x of the ground state distribution function $f(x)$ for $N_{\max} = 12$. Right: $f(x)$ of the lowest 3 states for $N_{\max} = 12$ and $N_x = 99$.

We also find exact factorization of the CM motion in small model space calculations including the next Fock space sectors, with one explicit photon or gluon. In order to judge convergence with N_x and $N_{\max}$, we have to implement consistent regularization and nonperturbative renormalization, following the sector-dependent renormalization procedures of Ref. [9].

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