# LORENTZ SYMMETRY FOR THE LIGHT-FRONT WIGHTMAN FUNCTIONS*

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New results for the Wightman function near the light-front hypersurface are presented. Mainly they disagree with the existing literature and suggest a substantial reformulation of the LF theory.

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## 1. Introduction

The 2-point Wightman function for a free scalar field is $\langle 0|\phi(x)\phi(y)|0\rangle = \varDelta_+(x-y)$, where the Lorentz invariant function $\varDelta_+(x)$ is defined by the covariant Fourier integral

$$\varDelta_+(x) := \int\limits_{\mathbb{R}^4} \frac{d^4k}{(2\pi)^3}\Theta(k_0)e^{-ik_\mu x^\mu}\,\delta\left(k^2-m^2\right)\,. \tag{1}$$

The integration over $k_0$ can be easily performed leading to the 3-dimensional equal-time (ET) Fourier integral

$$\varDelta_+(x) := \int\limits_{\mathbb{R}^3} \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{1}{2\omega}e^{-i\omega x^0}e^{+i\boldsymbol{k}\cdot\boldsymbol{x}}\,, \qquad \omega=\sqrt{m^2+\boldsymbol{k}^2}\,, \tag{2}$$

which then leads to the explicitly Lorentz invariant expression

$$\begin{aligned}\varDelta_+(x) \;=\; & -\frac{i}{4\pi}\mathrm{sgn}\left(x^0\right)\,\delta\left(x^2\right)+\Theta\left(-x^2\right)\frac{m}{4\pi^2\sqrt{-x^2}}K_1\left(m\sqrt{-x^2}\right) \\ & +\Theta\left(x^2\right)\frac{m}{8\pi\sqrt{x^2}}\left(N_1\left(m\sqrt{x^2}\right)+i\,\mathrm{sgn}\left(x^0\right)\,J_1\left(m\sqrt{x^2}\right)\right)\,. \end{aligned} \tag{3}$$

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This well known result can be studied for different limits. For the invariant interval $x^2 = (x^0)^2 - \boldsymbol{x}^2 \sim 0$, one celebrates the light-cone singularity

$$\Delta_+(x) \sim -\frac{i}{4\pi}\mathrm{sgn}\left(x^0\right)\,\delta\left(x^2\right) + \frac{1}{4\pi^2}\mathcal{P}\frac{1}{x^2}\,. \tag{4}$$

Also the ET limit $x^0 \to 0$ is well defined

$$\lim_{x^0\to 0} \Delta_+\left(x^0, \boldsymbol{x}\right) = \Delta_+\left(0, \boldsymbol{x}\right) = \frac{m}{4\pi^2 r}K_1(mr)\,, \qquad r = |\boldsymbol{x}|\,. \tag{5}$$

Taking the light-front (LF) limit naively, one finds

$$\lim_{x^+\to 0} \Delta_+\left(x^+, x^-, \boldsymbol{x}_\perp\right) = \Delta_+\left(0, x^-, \boldsymbol{x}_\perp\right) = -\frac{i}{8}\mathrm{sgn}\left(x^-\right)\delta^2(\boldsymbol{x}_\perp) + \frac{m\,K_1\left(mx_\perp\right)}{4\pi^2 x_\perp}\,. \tag{6}$$

Comparing the mass dependent parts of (5) and (6) one finds for small arguments $r \to 0$ and $x_\perp \to 0$, respectively

$$\frac{m}{4\pi^2 r}K_1(mr) \sim \frac{1}{4\pi^2}\frac{1}{r^2}\,, \qquad \frac{m}{4\pi^2 x_\perp}K_1(mx_\perp) \sim \frac{1}{4\pi^2}\frac{1}{x_\perp^2}\,. \tag{7}$$

This pole singularity is integrable in 3 dimensions, but is not in 2 dimensions, thus in terms of distributions $\Delta_+(0, x^-, \boldsymbol{x}_\perp) \notin \mathcal{S}'(\mathbb{R}^2)$, while $\Delta_+(0, \boldsymbol{x}) \in \mathcal{S}'(\mathbb{R}^3)$. This means that the naive LF limit is at least inconsistent. Actually, the careful analysis leads for $x^+x^- \sim 0$ to the expression

$$\begin{aligned}\Delta_+(x) \;=\; & -\frac{1}{4\pi}\left[\ln\frac{m^2\left|x^+x^-\right|}{2} + 2\gamma_E + i\frac{\pi}{2}\left[\mathrm{sgn}\left(x^+\right) + \mathrm{sgn}(x^-)\right]\right]\delta^2\left(\boldsymbol{x}_\perp\right) \\ & -\frac{1}{4\pi^2}D_i\left[\frac{x^i}{x_\perp^2}K_0\left(mx_\perp\right)\right] + 0\left(x^+x^-\right)\,,\end{aligned} \tag{8}$$

where $D_i$ is a distributional partial derivative. Accordingly, $\Delta_+(x)$ is singular for points lying along a light-like direction at the LF hypersurface $x^+ = 0$. Appearance of such singularity disagrees with the existing literature [1, 2] and one may ask if this singularity is relevant to physical problems.

One may implement these results for the commutator function — the Jordan–Pauli function — defined as

$$\langle 0|\left[\phi(x),\,\phi(y)\right]|0\rangle = \Delta_+(x-y) - \Delta_+(y-x) =: i\Delta(x-y)\,. \tag{9}$$

Near $x^+x^- \sim 0$, one finds

$$\Delta(x) \;=\; -\tfrac{1}{4}\left[\mathrm{sgn}\left(x^+\right) + \mathrm{sgn}\left(x^-\right)\right]\delta^2\left(\boldsymbol{x}_\perp\right) + 0\left(x^+x^-\right)\,, \tag{10}$$

thus evidently, the LF limit exists

$$\lim_{x^+ \to 0} \varDelta(x) = \varDelta(0, \bar{x}) = -\tfrac{1}{4}\mathrm{sgn}\left(x^-\right)\, \delta^2\left(\boldsymbol{x}_\perp\right)\, , \tag{11}$$

where we denote hereafter $\bar{x} = (x^-, \boldsymbol{x}_\perp)$. Starting from

$$\varDelta(x) \;\sim\; -\tfrac{1}{4}\left[\mathrm{sgn}\left(x^+\right) + \mathrm{sgn}\left(x^-\right)\right] \delta^2\left(\boldsymbol{x}_\perp\right)\, , \tag{12}$$

one finds

$$\partial_+ \varDelta(x) \sim -\tfrac{1}{2}\delta\left(x^+\right) \delta^2\left(\boldsymbol{x}_\perp\right)\, , \tag{13}$$

which has no LF limit. Accordingly, the Jordan–Pauli function $\varDelta(x)$ is a smooth but non-analytic function of $x^+$ at $x^+ = 0$, *i.e.* at the LF hypersurface. Therefore, the temporal evolution in $x^+$ variable from $\varDelta(0, \bar{x})$ to $\varDelta(x^+, \bar{x})$, with arbitrary $x^+ \neq 0$, cannot be reduced to the Taylor expansion series. This means that the LF Cauchy problem for $\varDelta(x)$ must be reformulated without using analyticity at $x^+ = 0$.

## 2. General properties of the LF Wightman function

Let us assume the translational invariance, the Lorentz invariance of the vacuum state $|0\rangle = U(\varLambda)|0\rangle$ and the Lorentz covariance of $\phi(x)$

$$U(\varLambda)\, \phi(x)\, U^{-1}(\varLambda) = \phi\left(x'\right)\, , \qquad x'^\mu = \varLambda^\mu{}_\nu x^\nu\, . \tag{14}$$

These assumptions lead us to the relations for the Wightman function

$$\left(x_\nu \partial_\mu - x_\mu \partial_\nu\right) \langle 0|\phi\left(x^+, \bar{x}\right)\, \phi(0)|0\rangle = 0\, . \tag{15}$$

Then, adding the LF canonical commutator

$$\left[\phi\left(x^+, \bar{x}\right), \phi\left(x^+, \bar{y}\right)\right] = -\frac{i}{4}\, \mathrm{sgn}\left(x^- - y^-\right)\, \delta^2\left(\boldsymbol{x}_\perp - \boldsymbol{y}_\perp\right) \tag{16}$$

we find that the Lorentz symmetry is consistent with

$$\lim_{x^+ \to 0} x^\pm \partial_\pm \langle 0|\phi\left(x^+, \bar{x}\right)\, \phi(0)|0\rangle = -\frac{1}{4\pi}\delta^2\left(\boldsymbol{x}_\perp\right)\, , \tag{17}$$

which means that the logarithmic terms in $x^+$ and $x^-$ will appear near $x^+ = 0$. This conclusion is valid for general scalar fields. Then, restriction to a free field case allows us to take Klein–Gordon equation for a scalar field

$$\left(2\partial_+\partial_- - \varDelta_\perp + m^2\right) \phi(x) = 0\, , \tag{18}$$

which leads to the equation of motion for Wightman function $W_{(2)}(x^+,\bar{x}) = \langle 0|\phi(x^+,\bar{x})\,\phi(0)|0\rangle$

$$\left(\partial_+\partial_- - \mathscr{D}\right) W_{(2)}\left(x^+,\bar{x}\right) = 0\,, \qquad \mathscr{D} = \tfrac{1}{2}\left(\Delta_\perp - m^2\right)\,. \tag{19}$$

Then, we find the integral equation for the Wightman function

$$\begin{aligned} W_{(2)}\left(x^+,\bar{x}\right) &= -\frac{\delta^2\left(\boldsymbol{x}_\perp\right)}{4\pi}\left[\log\left(m\,x^+\right)+\log\left(m\,x^-\right)\right] - \frac{1}{4\pi^2}D_iF^i\left(\boldsymbol{x}_\perp\right) \\ &\quad + \mathscr{D}\int\limits_0^{x^+} d\tau \int\limits_0^{x^-} d\xi\, W_{(2)}\left(\tau,\xi,\boldsymbol{x}_\perp\right)\,, \end{aligned} \tag{20}$$

where $\log(z) = \ln|z| + i(\pi/2)\,\mathrm{sgn}(z)$ and

$$F^i(\boldsymbol{x}_\perp) = \frac{x^i}{x_\perp^2}K_0(mx_\perp) = \frac{i}{4\pi}\int\limits_{\mathbb{R}^2} d^2\boldsymbol{k}_\perp\, e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\frac{k^i}{k_\perp^2}\ln\frac{m^2}{m^2+k_\perp^2}\,. \tag{21}$$

Then, from (20), we obtain the integral equation for the Pauli–Jordan function $\Delta(x) = -i\left[W_{(2)}(x^+,\bar{x}) - W_{(2)}(-x^+,-\bar{x})\right]$

$$\Delta(x) = -\tfrac{1}{4}\delta^2\left(\boldsymbol{x}_\perp\right)\left[\mathrm{sgn}\left(x^+\right)+\mathrm{sgn}\left(x^-\right)\right] + \mathscr{D}\int\limits_0^{x^+} d\tau\int\limits_0^{x^-} d\xi\,\Delta\left(\tau,\xi,\boldsymbol{x}_\perp\right) \tag{22}$$

and its partial derivative

$$\partial_+\Delta(x) = -\tfrac{1}{2}\delta^2\left(\boldsymbol{x}_\perp\right)\delta\left(x^+\right) + \mathscr{D}\int\limits_0^{x^-} d\xi\,\Delta\left(x^+,\xi,\boldsymbol{x}_\perp\right)\,. \tag{23}$$

Evidently, $\partial_+\Delta(x)$ is singular at $x^+ = 0$ but we may subtract the singular term as follows,

$$\begin{aligned} \lim_{x^+\to 0}\left[\partial_+\Delta(x) + \tfrac{1}{2}\delta^2\left(\boldsymbol{x}_\perp\right)\delta\left(x^+\right)\right] &= \lim_{x^+\to 0}\mathscr{D}\int\limits_0^{x^-} d\xi\,\Delta\left(x^+,\xi,\boldsymbol{x}_\perp\right) \\ &= -\tfrac{1}{4}\mathscr{D}\left|x^-\right|\delta^2\left(\boldsymbol{x}_\perp\right)\,, \end{aligned} \tag{24}$$

where the final expression agrees with the literature, for example [3]. This suggests that the subtraction of singular terms may be a reasonable procedure, thus we also may define a subtracted Wightman function as

$$S\left(x^+,\bar{x}\right) := W_{(2)}(x^+,\bar{x}) + \frac{\delta^2\left(\boldsymbol{x}_\perp\right)}{4\pi}\log\left(m_0\,x^+\right)\,. \tag{25}$$

From (20), we easily find the integral equation for $S(x^+,\bar{x})$ as follows

$$\begin{aligned} S\left(x^+,\bar{x}\right) &= -\frac{\delta^2\left(\boldsymbol{x}_\perp\right)}{4\pi}\log\left(m_0\,x^-\right) - \mathscr{D}\delta^2\left(\boldsymbol{x}_\perp\right)\frac{x^-x^+}{4\pi}\left[\log\left(m_0\,x^+\right)-1\right] \\ &\quad + \mathscr{D}\int\limits_0^{x^+} d\tau\int\limits_0^{x^-} d\xi\, S\left(\tau,\xi,\boldsymbol{x}_\perp\right) - \frac{1}{4\pi^2}D_iF^i\left(\boldsymbol{x}_\perp\right)\,. \end{aligned} \tag{26}$$

Thus, $\partial_+S(x^+,\bar{x})$ is singular at $x^+=0$, but the LF limit for $S(x^+,\bar{x})$ exists and defines regular distributions in the transverse coordinates

$$\lim_{x^+\to 0} S\left(x^+,\bar{x}\right) = S\left(0,\bar{x}\right) = -\frac{\delta^2\left(\boldsymbol{x}_\perp\right)}{4\pi}\log\left(m_0\,x^-\right) - \frac{1}{4\pi^2}D_iF^i\left(\boldsymbol{x}_\perp\right) \in \mathcal{S}'\left(\mathbb{R}^2\right). \tag{27}$$

In summary, the Jordan–Pauli function $\Delta(x)$ and the subtracted Wightman function $S(x^+,\bar{x})$ are smooth but non-analytic functions at $x^+=0$.

## 3. Momentum representation for $\Delta_+$

In the covariant formula,

$$\Delta_+(x) = \int\limits_{\mathbb{R}^4}\frac{d^4k}{(2\pi)^3}\Theta(k^++k^-)e^{-ik_\mu x^\mu}\,\delta\left(2k^+k^- - k_\perp^2 - m^2\right)\,, \tag{28}$$

usually one integrates over $k^-$

$$\Delta_+(x) = \int\limits_{\mathbb{R}^2}\frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3}e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\int\limits_0^\infty\frac{dk^+}{2k^+}e^{-ik^+x^-}e^{-i\left(\mu^2/k^+\right)x^+}\,, \tag{29}$$

where $\mu^2=(m^2+k_\perp^2)/2$. Thus at the LF $x^+=0$, the mass dependence disappears but the integral over $k^+$ becomes divergent. When the cut-off $\delta$ is introduced $k^+>\delta$, then the Lorentz symmetry is definitely lost. But since there are two identities in the sense of distributions

$$\delta\left(2k^+k^- - k_\perp^2 - m^2\right) = \frac{1}{2|k^+|}\,\delta\left(k^- - \frac{\mu^2}{k^+}\right) = \frac{1}{2|k^-|}\,\delta\left(k^+ - \frac{\mu^2}{k^-}\right), \tag{30}$$

thus the integration over both $k^+$ and $k^-$ gives

$$\begin{aligned} \Delta_+(x) &= \int\limits_{\mathbb{R}^2}\frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3}e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp}\left[\int\limits_{K^+}^\infty\frac{dk^+}{k^+}e^{-ik^+x^-}e^{-i\left(\mu^2/k^+\right)x^+}\right. \\ &\quad \left. + \int\limits_{K^-}^\infty\frac{dk^-}{k^-}e^{-i\left(\mu^2/k^-\right)x^-}e^{-ik^-x^+}\right] \end{aligned} \tag{31}$$

with constraint $K^+K^- = \mu^2 = (m^2 + k_\perp^2)/2$. At LF $x^+ = 0$, we need to cut-off high momentum $k^-$

$$\Delta_+^\Lambda(0,\bar{x}) = \int\limits_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3} e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \left[ \int\limits_{K^+}^{\infty} \frac{dk^+}{k^+} e^{-ik^+x^-} + \int\limits_{K^-}^{\Lambda} \frac{dk^-}{k^-} e^{-i\left(\mu^2/k^-\right)x^-} \right] \tag{32}$$

and one could implement the Pauli–Villars regularization for removing cut-off $\Lambda$, but this would kill all mass independent terms. Therefore, we propose to subtract terms which become singular for $\Lambda \to \infty$. We may take such term as

$$\Delta_+^{\mathrm{sing}}(\bar{x}) := \int\limits_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3} e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \int\limits_{\mathcal{M}}^{\Lambda} \frac{dk^-}{k^-} = \frac{1}{2\pi}\delta^2(\boldsymbol{x}_\perp)\ln(\Lambda/\mathcal{M}), \tag{33}$$

where $\mathcal{M} > 0$ is an arbitrary parameter. This leads to the subtracted function

$$\begin{aligned} S(\bar{x}) \;:=\; \lim_{\Lambda\to\infty}\left[\Delta_+^\Lambda(0,\bar{x}) - \Delta_+^{\mathrm{sing}}(\bar{x})\right] &= \int\limits_{\mathbb{R}^2} \frac{d^2\boldsymbol{k}_\perp}{(2\pi)^3} e^{i\,\boldsymbol{k}_\perp\cdot\boldsymbol{x}_\perp} \\ \times \left[ \int\limits_{\mu^2/\mathcal{M}}^{\infty} \frac{dk^+}{k^+} e^{-ik^+x^-} + \int\limits_{0}^{\mu^2/\mathcal{M}} \frac{dk^+}{k^+}\left(e^{-ik^+x^-} - 1\right) \right] \end{aligned} \tag{34}$$

which agrees with the formerly defined function $S(0,\bar{x})$. The integrals over $k^+$ are very instructive. In the first part, the lower limit $(m^2 + k_\perp^2)/(2\mathcal{M})$ is commonly interpreted as a mass dependent infra-red cut-off. However, we argue that without the second part one definitely looses the Lorentz symmetry, violates unitarity and obtains incorrect dependence on mass of the Wightman function.

## 4. Conclusions and prospects

We have shown that the Lorentz symmetry induces singularities at $x^+{=}0$ for points lying along a light-like direction. For a free scalar field, Wightman functions can be uniquely determined without using the Fock representation. For the momentum representation of $\Delta_+(x)$, we need to use both longitudinal momenta $k^+$ and $k^-$. Then at $x^+ = 0$, we have the mass dependent terms and the large momentum $k^-$ needs to be regularized. As we have found that the Jordan–Pauli function $\Delta(x)$ is smooth but non-analytic function of $x^+$ at $x^+ = 0$, then nontrivial problems may appear within the canonical LF quantization procedure for fields with higher spin like Dirac fermions, massive vector fields, gauge vector fields, *etc.* Also the Cauchy problem needs to be reexamined near $x^+ = 0$.

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