SOLUTION OF BETHE–SALPETER EQUATION IN THE MINKOWSKI SPACE FOR THE SCATTERING STATES*

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The Bethe–Salpeter equation for the scattering states in the Minkowski space is solved for spinless particles and ladder kernel. The off-mass-shell scattering amplitude is computed for the first time.

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1. Introduction

The inhomogeneous Bethe–Salpeter (BS) equation in the Minkowski space [1] provides a covariant four-dimensional description of two-body scattering states. For scalar particles it reads

$$
\begin{aligned}
&F(p,p_{\rm s};P)\\
&= K(p,p_{\rm s};P) - i\int \frac{d^4p'}{(2\pi)^4}\frac{K(p,p';P)F(p',p_{\rm s};P)}{\left[\left(\frac{P}{2}+p'\right)^2-m^2+i\epsilon\right]\left[\left(\frac{P}{2}-p'\right)^2-m^2+i\epsilon\right]}\,.
\end{aligned}
\tag{1}
$$

The one-boson exchange kernel K has the form

$$
K\left(p,p';P\right) = -\frac{g^2}{(p-p')^2-\mu^2+i\epsilon}\,. \tag{2}
$$

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We introduce the coupling constant α defined by $g^2 = 16\pi m^2\alpha$ and make the partial wave expansion of the amplitude F according to [2]

$$F(\theta) = 16\pi \sum_{l=0}^{\infty}(2l+1)F_l P_l(\cos\theta)\,.$$

In the center of mass frame, one has $\vec{P}=0$, $P_0=\sqrt{s}=2\varepsilon_{p_\mathrm{s}}=2\sqrt{m^2+p_\mathrm{s}^2}$ and for a given incident momentum p_s, the partial wave off-mass-shell amplitude F_l depends on two scalar variables p_0 and $|\vec{p}|$. It will be hereafter denoted by $F_l(p_0,p;p_\mathrm{s})$ setting $p=|\vec{p}|$, $p_\mathrm{s}=|\vec{p_\mathrm{s}}|$. For the S-wave, equation (1) obtains the form

$$F_0(p_0,p;p_\mathrm{s}) = K_0(p_0,p;p_\mathrm{s}) - \frac{4i}{\pi^2}\int_0^{\infty} p'^2 dp' \int_{-\infty}^{\infty} dp'_0 \times \frac{K_0\left(p_0,p;p'_0,p'\right)F_0\left(p'_0,p';p_\mathrm{s}\right)}{\left(p'^2_0+2p'_0\varepsilon_{p_\mathrm{s}}+{p_\mathrm{s}}^2-p'^2+i\epsilon\right)\left(p'^2_0-2p_0\varepsilon_{p_\mathrm{s}}+{p_\mathrm{s}}^2-p'^2+i\epsilon\right)}\,, \quad (3)$$

where

$$\begin{aligned} K_0(p_0,p;p'_0,p') &= -\frac{1}{32\pi}\int_{-1}^{1} dz \frac{g^2}{(p_0-p'_0)^2-\left(p^2-2pp'z+p'^2\right)-\mu^2+i\epsilon} \\ &= -\frac{\alpha m^2}{4pp'}\log\frac{|\eta+1|}{|\eta-1|}+\frac{i\alpha\pi m^2}{4pp'}U(\eta)\,, \end{aligned} \quad (4)$$

and

$$U(\eta) = \begin{cases} 1, & \text{if } |\eta|\le 1 \\ 0, & \text{if } |\eta|>1 \end{cases}\,, \qquad \eta = \frac{(p_0-p'_0)^2-p^2-p'^2-\mu^2}{2pp'}\,.$$

The on-shell amplitude $F_l^{\mathrm{on}} = F_l(p_0=0, p=p_\mathrm{s}; p_\mathrm{s})$ determines the phase shift according to

$$\delta_l = \frac{1}{2i}\log\left(1+\frac{2ip_\mathrm{s}}{\varepsilon_{p_\mathrm{s}}}F_l^{\mathrm{on}}\right)\,. \quad (5)$$

The knowledge of this function in the entire domain of its arguments — *i.e.* the off-shell amplitude — is mandatory for some interesting physical applications, like for instance computing the transition e.m. form factor $\gamma^* d \to np$ or solving the BS–Faddeev equations. This quantity has not been obtained until now.

The numerical solution of the BS equation in the Minkowski space is complicated by the existence of singularities in the amplitude as well as in the integrand of (1). These singularities are integrable in the mathematical sense, due to $i\epsilon$ in the denominators of propagators, but their integration is a quite delicate task and requires the use of appropriate analytical as well as numerical methods.

To avoid these singularities, the BS equation was first solved in the Euclidean space. These solutions provided on-shell quantities like binding energies and phase shifts [3]. However, we have shown [4] that the Euclidean BS amplitude cannot be used to calculate electromagnetic form factors, since the corresponding integral does not allow the Wick rotation. One, therefore, needs the BS amplitude in the Minkowski space.

This amplitude has been computed for a separable kernel (see [5] and references therein). For a kernel given by a Feynman graph — ladder and cross ladder — the Minkowski BS amplitude was first obtained in our preceding works [6, 7] in the case of bound state problem. We developed to this aim an original method based on the Nakanishi integral representation of the BS amplitude. A similar method for the scattering states has been proposed in [8] although the numerical solutions are not yet available.

We present in this contribution a new method providing a direct solution of the original BS equation. It is based on an accurate treatment of the singularities and allows us to compute the corresponding off-shell scattering amplitude in the Minkowski space. We will give the low energy parameters in the case of spinless particles and ladder kernel. First results have been published in [9].

2. The direct method

There are four sources of singularities in the r.h. side of the BS equation (1), which are detailed below.

(i) The constituent propagators in (3) *vs.* p'_0 have two poles, each of them represented as

$$\frac{1}{p'_0 - a - i\epsilon} = \mathrm{PV}\frac{1}{p'_0 - a} + i\pi\delta\left(p'_0 - a\right) ,$$

where PV means the principal value. In the product of four pole terms, the only non vanishing contributions come from the product of four PVs without delta-functions, from the terms with three PVs and one delta-function and from the term with two PVs and two deltas. After partial wave decomposition the 4D integral BS equation (1) is reduced to a 2D one, Eq. (3). Integrating in (3) over p'_0, we obtain in addition to the 2D part, a 1D integral over p' and a non-integrated term. The singularities due to the PVs are eliminated by subtractions according to the identity

$$\mathrm{PV}\int_0^\infty \frac{f\left(p_0'\right) dp_0'}{{p_0'}^2 - a^2} = \int_0^\infty \left(\frac{f\left(p_0'\right)}{{p_0'}^2 - a^2} - \frac{f(a)}{{p_0'}^2 - a^2}\right) dp_0' \,.$$

The integrand in r.h.-side is not singular.

(ii) The propagator of the exchanged particle has the pole singularities which, after partial wave decomposition, turn into logarithmic ones, Eq. (4). Their positions are found analytically and the numerical integration over p_0' variable is split into intervals between two consecutive singularities, namely

$$\int_0^\infty [\ldots]\, dp_0' = \int_0^{\mathrm{sing}_1} [\ldots]\, dp_0' + \int_{\mathrm{sing}_1}^{\mathrm{sing}_2} [\ldots]\, dp_0' + \ldots$$

Each of these integrals is made regular with an appropriate change of variable. We proceed in a similar way for the p' integration.

(iii) The inhomogeneous (Born) term is given by the ladder kernel and is also singular in both variables. The positions of these singularities are analytically known.

(iv) The amplitude F_0 itself has many singularities, among which the strongest one results from the Born term $K_0(p_0, p; p_\mathrm{s})$. This makes difficult its representation on a basis of regular functions as well as its numerical integration in (3). To circumvent this difficulties we made the replacement $F_0(p_0, p; p_\mathrm{s}) = K_0(p_0, p; p_\mathrm{s}) f_0(p_0, p; p_\mathrm{s})$, where f_0 is a smooth function. After that, the singularities of the inhomogeneous term are canceled. We obtain in this way a non-singular equation for f_0 which we solve by standard methods. Then we restore the BS off-mass-shell amplitude F_0 in the Minkowski space.

3. Numerical results

We first applied this method to solve the bound state problem by dropping the inhomogeneous term in (1). The binding energies coincide, within four-digit accuracy, with the ones calculated in our previous work [6] and with the Euclidean space results.

Solving Eq. (3), the S-wave off-shell scattering amplitude F_0 is calculated and the phase shifts are extracted by means of Eq. (5). Above the first inelastic threshold $p_s^*(\mu) = \sqrt{m\mu + \mu^2/4}$, the phase shifts have an imaginary part which has been also found. By performing a Wick rotation in (3) — and taking into account the contributions of singularities which, in contrast to the bound state case, crossed the rotated contour — we derived an Euclidean space equation similar to one obtained in [3]. The phase shifts found by these two methods — *i.e.*, solving Eq. (3) and the Euclidean space equation — coincide with each other within 3–4 digits. Furthermore, the imaginary part of the phase shifts vanishes with high accuracy below threshold. The unitarity condition is not automatically fulfilled in our approach, but appears as a consequence of handling the correct solution. It thus provides a stringent test of the numerical method. Our results reproduce the phase shifts given in [3] within the accuracy allowed by extracting numerical values from published figures.

Figure 1, left panel, shows the phase shifts calculated via BS equation (solid curve) and via the Schrödinger one with the Yukawa potential (dashed curve) for the constituent mass $m = 1$, exchange mass $\mu = 0.5$ and coupling constant $\alpha = 1.2$. For this value of α there exists a bound state. Therefore, according to the Levinson theorem, the phase shift starts at 180 degrees. One can see that the relative difference between relativistic and non-relativistic results is considerable large specially for small incident momentum. This difference increases with α.

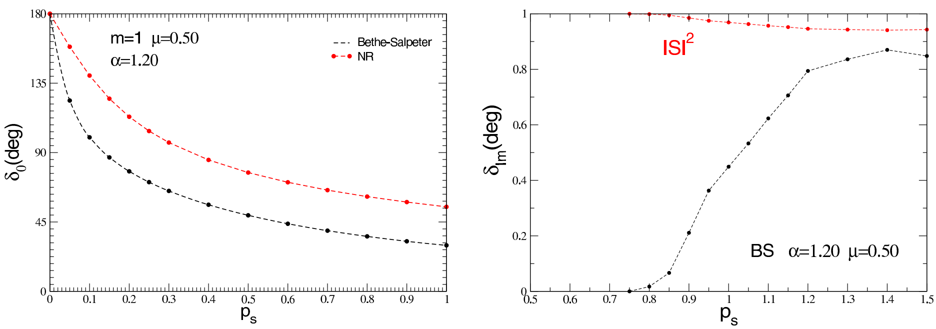

Fig. 1. Left: phase shift calculated via BS equation (solid curve) are compared to the non-relativistic results (dashed curve) for $\mu = 0.5$. Right: solid curve — imaginary part of the phase shift; dashed curve — modulas of the two-body S-matrix.

Right panel in Fig. 1 shows the imaginary part of the phase shift which automatically appears when the incident momentum exceeds the threshold value for the creation of one exchange meson. For $m = 1$ and $\mu = 0.5$ this value is $p_s^* = 0.75$. Simultaneously, the modulus of the two-body S-matrix differs from 1. For $p_\mathrm{s} = 1.118$ the second inelastic threshold, corresponding to the creation of two mesons, is open. It also contributes to this curve.

We have displayed in Fig. 2 the real (left panel) and imaginary (right panel) parts of the off-shell scattering amplitude $F_0(p_0, p; p_\mathrm{s})$ *vs.* p_0 and p calculated for $p_\mathrm{s} = \mu = 0.5$. Its real part shows a non-trivial structure with a ridge and a gap resulting from the singularities of the inhomogeneous term. Its on-shell value $F_0^{\mathrm{on}} = F_0(p_0 = 0, p = p_\mathrm{s}; p_\mathrm{s})$, determining the phase shift calculated previously, corresponds to a single point $p_0 = 0$, $p = p_{\rm s}$ on theses surfaces. Our calculation, shown in Fig. 2, provides the full amplitude $F_0(p_0, p; p_{\rm s})$ in a two-dimensional domain. It cannot be found from the Euclidean equation. Computing this quantity is the main result of this work.

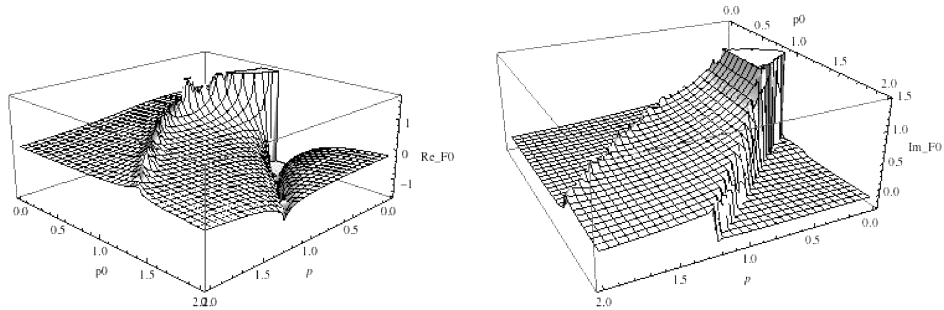

Fig. 2. Left: real part of the off-shell amplitude $F_0(p_0, p; p_{\rm s})$ for $p_{\rm s} = 0.5$, $\mu = 0.5$. Right: imaginary part of $F_0(p_0, p; p_{\rm s})$.

4. Conclusion

We solved the BS equation for the scattering states in the Minkowski space for the ladder kernel. The off-mass-shell amplitude is found for the first time. Coming on mass shell, we obtain the phase shifts which coincide with ones calculated by other methods. They considerably differ, even at low energy, from the non-relativistic phase shifts calculated by the Schrödinger equation. Above the meson creation threshold, the inelasticity appears which is also calculated. The off-mass-shell amplitude can be used to calculate the transition form factor and as an input in the three-body BS–Faddeev equations.

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