LORENTZ SYMMETRY AND GAUGE DEPENDENCE IN THE ZWANZIGER MODEL OF TWO-POTENTIAL QED*

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Zwanziger model of quantum electrodynamics (QED) introduces two independent vector gauge fields: A_μ and B_μ, which allows for the local form of interaction and the electromagnetic duality transformation. This formulation is based on a fixed space-like 4-vector n^ν, which appears in the definition of the electromagnetic field strength tensor $F_{\mu\nu}$. One finds a gauge invariant differential condition on the Wightman function $\langle 0|A_\mu(x)B_\nu(y)|0\rangle$. For a a free field, due to Peierls's formula, this condition has no dependence on n^μ. One proves that this condition is inconsistent with the Lorentz covariance for vector fields, thus there is no Lorentz covariant Wightman function $\langle 0|A_\mu(x)B_\nu(y)|0\rangle$ in any gauge. Therefore, one may freely take different choices for the equal-time and the light-front formulations.

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1. Introduction

The Wightman functions $\langle 0\,|\,V_\mu(x)\,W_\nu(y)|\,0\rangle$ for vector fields provide us with a practical tool for studying different properties of quantum field theories. Assuming the translational invariance

$$\langle 0\,|\,V_\mu(x)\,W_\nu(y)|\,0\rangle = \langle 0\,|\,V_\mu(x-y)\,W_\nu(0)|\,0\rangle = \langle 0\,|\,V_\mu(0)\,W_\nu(y-x)|\,0\rangle\,, \quad (1)$$

we may consider $\langle 0\,|\,V_\mu(x)\,W_\nu(0)|\,0\rangle$ as a generic case. Taking the Lorentz transformation for a vector field, say $V_\mu(x)$ field,

$$U_\Lambda^{-1}\,V_\mu(x)\;U_\Lambda \;=\; \Lambda^\nu{}_\mu V_\nu(\Lambda x)\,, \quad (2)$$

where for an infinitesimal case $(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu = x^\mu + \omega^{\mu\nu}x_\nu$ with $\omega^{\mu\nu} = -\omega^{\nu\mu}$ and assuming the Lorentz symmetric vacuum state, one finds the

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condition for the Lorentz covariant Wightman function for two arbitrary vector fields $V_\mu(x)$ and $W_\nu(x)$

$$\begin{aligned}&(x_\lambda\partial_\rho - x_\rho\partial_\lambda)\,\langle 0\,|\,V_\mu(x)\,W_\nu(0)|\,0\rangle\\&= g_{\rho\mu}\,\langle 0\,|\,V_\lambda(x)\,W_\nu(0)|\,0\rangle - g_{\lambda\mu}\,\langle 0\,|\,V_\rho(x)\,W_\nu(0)|\,0\rangle\\&+ g_{\rho\nu}\,\langle 0\,|\,V_\mu(x)\,W_\lambda(0)|\,0\rangle - g_{\lambda\nu}\,\langle 0\,|\,V_\mu(x)\,W_\rho(0)|\,0\rangle\ .\end{aligned} \tag{3}$$

When the vector fields $V_\mu(x)$ and $W_\mu(x)$ are Hermitian operators and the system is symmetric under the PCT transformation, then the Wightman functions respect symmetry under the inversion of their ordering

$$\langle 0\,|V_\mu(x)W_\nu(0)|\,0\rangle \;=\; \langle 0\,|W_\nu(x)V_\mu(0)|\,0\rangle\ . \tag{4}$$

For a single vector field $W_\mu = V_\nu = A_\mu$, one may easily find the Wightman function, which satisfies both the Lorentz covariance equation (3) and the discrete symmetry (4) is

$$\langle 0\,|A_\mu(x)\,A_\nu(y)|\,0\rangle = g_{\mu\nu}F_+(x)\,, \tag{5}$$

where $F_+(x)$ is a Lorentz invariant (generalized) function. This analysis can be compared with the canonical quantization procedure. In the standard formulation of QED, when A_μ is a single vector gauge potential, one needs to introduce some gauge fixing condition for a consistent canonical quantization. For different gauges, one finds that the Wightman function for a free field case has the general structure

$$\langle 0\,|A_\mu(x)\,A_\nu(y)|\,0\rangle = -g_{\mu\nu}D_+(x) + \partial_\mu\Phi_\nu(x) + \partial_\nu\Phi_\mu(x)\,, \tag{6}$$

where $D_+(x)$ is the massless invariant singular function, defined by the Fourier integral as

$$D_+(x) = \int\limits_{-\infty}^{+\infty}\frac{d^3k}{(2\pi)^3}\,\frac{e^{-ik\cdot x}}{2\left|\vec{k}\right|}, \tag{7}$$

while $\Phi_\mu(x)$ depends on the choice of gauge condition. Evidently, this general Wightman function agrees with the discrete symmetry condition (4), while its common gauge independent part $g_{\mu\nu}D_+(x)$ agrees with the Lorentz covariance (3) and (5). In the standard formulation of QED, one expresses the electromagnetic field tensor $F_{\mu\nu}(x)$ in terms of a single gauge vector potential $A_\mu(x)$ as $F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$. Thus from (6), one finds Peierls's formula [1] for the Wightman function of the electromagnetic field tensor

$$\langle 0|F_{\mu\nu}(x)\,F_{\lambda\rho}(0)|0\rangle = [g_{\mu\lambda}\,\partial_\nu\partial_\rho - g_{\nu\lambda}\,\partial_\mu\partial_\rho + g_{\nu\rho}\,\partial_\mu\partial_\lambda - g_{\mu\rho}\,\partial_\nu\partial_\lambda]D_+(x)\,, \tag{8}$$

where all dependence on a gauge fixing condition, which is contained in $\Phi_\mu(x)$, disappears as expected. Thus in the standard formulation of QED, the gauge invariant information is located in the Lorentz covariant part of the Wightman function (6).

2. Two potential formulation of QED

Two potential formulation of QED starts with Maxwell's equations with the electric and magnetic external sources J^μ and K^μ

$$(\partial\cdot F)^\mu=\partial_\mu F^{\mu\nu}=J^\nu\,,\qquad \left(\partial\cdot\widetilde{F}\right)^\mu=\partial_\mu\widetilde{F}^{\mu\nu}=K^\nu\,,\tag{9}$$

where $\widetilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\lambda\sigma}F_{\lambda\sigma}$ is the dual electromagnetic tensor. These equations are symmetric under the electromagnetic duality transformation

$$F_{\mu\nu}\mapsto\widetilde{F}_{\mu\nu}\mapsto\widetilde{\widetilde{F}}_{\mu\nu}=-F_{\mu\nu}\,,\qquad J^\mu\mapsto K^\mu\mapsto -J^\mu\,.\tag{10}$$

Since now, one cannot express $F_{\mu\nu}$ by a single gauge field potential, then one may try the explicitly covariant definition by Cabbibo–Ferrari [2]

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu-\epsilon_{\mu\nu\lambda\rho}\partial^\lambda B^\rho\tag{11}$$

with two gauge field potentials A_μ and B_μ. Now, the electromagnetic duality (10) for $F_{\mu\nu}$ takes the equivalent form

$$A_\mu\mapsto B_\mu\mapsto -A_\mu\,.\tag{12}$$

Unfortunately, this formulation has no action principle, thus one cannot use (11) in a consistent canonical quantization. For a consistent formulation of two potential QED, we will follow Zwanziger [3], where one starts with the algebraic identities for tensors $F^{\mu\nu}$ and $\widetilde{F}^{\mu\nu}$

$$F^{\mu\nu}=\frac{1}{n^2}\left[n^\mu(n\cdot F)^\nu-n^\nu(n\cdot F)^\mu\right]-\frac{1}{n^2}\epsilon^{\mu\nu\lambda\rho}n_\lambda\left(n\cdot\widetilde{F}\right)_\rho\,,\tag{13}$$

$$\widetilde{F}^{\mu\nu}=\frac{1}{n^2}\left[n^\mu(n\cdot\widetilde{F})^\nu-n^\nu(n\cdot\widetilde{F})^\mu\right]+\frac{1}{n^2}\epsilon^{\mu\nu\lambda\rho}n_\lambda\left(n\cdot F\right)_\rho\,,\tag{14}$$

where n^μ is a fixed 4-vector, preferably a space-like one $n^2<0$. Zwanziger defines two vector gauge potentials A_μ and B_μ implicitly by

$$(n\cdot F)^\nu=(n\cdot\partial)A^\nu-\partial^\nu(n\cdot A)\,,\qquad \left(n\cdot\widetilde{F}\right)^\nu=(n\cdot\partial)B^\nu-\partial^\nu(n\cdot B)\,.\tag{15}$$

These gauge potentials A_μ and B_μ are independent, with two gauge transformations

$$B_\mu(x)\to B_\mu(x)+\partial_\mu\chi_g(x)\,,\qquad A_\mu(x)\to A_\mu(x)+\partial_\mu\chi_e(x)\,,\tag{16}$$

and the electromagnetic duality transformation coincides with (12). Further for the mixed Wightman functions (MWF), one may extend (4) to

$$\langle 0\,|A_\mu(x)B_\nu(0)|\,0\rangle=\langle 0\,|B_\nu(x)A_\mu(0)|\,0\rangle=-\,\langle 0\,|A_\nu(x)B_\mu(0)|\,0\rangle\;,\tag{17}$$

thus MWF are antisymmetric in their tensor indexes. Starting from the definitions (15), we find a new identity for MWF

$$(n\cdot\partial)^2\epsilon^{\lambda\alpha\beta\rho}\partial_\lambda\langle 0|A_\alpha(x)\,B_\beta(0)|0\rangle = (n\cdot\partial)\langle 0|(n\cdot F)_\beta(x)\,F^{\rho\beta}(0)|0\rangle + n^\rho\langle 0|(n\cdot F)_\beta(x)\,\partial_\lambda F^{\lambda\beta}(0)|0\rangle \tag{18}$$

which is explicitly invariant under gauge transformations (16).

3. MFW for free electromagnetic fields

For free fields, one may use Peierls's formula (8), which changes (18) to

$$(n\cdot\partial)^2\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\langle 0|A_\mu(x)\,B_\nu(0)|0\rangle = 2(n\cdot\partial)^2\partial^\rho D_+(x)\,. \tag{19}$$

Then, for $n^2<0$, we can integrate out the differential operator $(n\cdot\partial)^2$ by assuming vanishing of gauge potentials at the boundary at spatial infinity and we arrive at

$$\epsilon^{\mu\nu\lambda\rho}\partial_\lambda\langle 0|A_\mu(x)\,B_\nu(0)|0\rangle = 2\partial^\rho D_+(x)\,, \tag{20}$$

which has the explicitly Lorentz covariant form. We may introduce a simplifying notation

$$\langle 0|A_i(x)\,B_j(0)|0\rangle = \epsilon_{ijk}\mathscr{P}_k(x)\,, \tag{21}$$

$$\langle 0|A_0(x)\,B_i(0)|0\rangle = \mathscr{N}_i(x) = -\langle 0|A_i(x)\,B_0(0)|0\rangle\,, \tag{22}$$

where $\epsilon_{ijk}=\epsilon^{0ijk}$, $\epsilon^{0123}=1$, so we can rewrite equation (20) as

$$\partial_0\mathscr{P}_k(x)-\epsilon_{kij}\partial_i\mathscr{N}_j(x)=\partial_k D_+(x)\,,\qquad \partial_k\mathscr{P}_k(x)=\partial_0 D_+(x)\,. \tag{23}$$

Evidently, these equations allow for many solutions, which are connected with different gauge fixing conditions for gauge potentials. One may look for the Lorentz covariant solution, thus one imposes additionally equation (3). For the Lorentz boosts one imposes

$$(x_l\partial_0-x_0\partial_l)\mathscr{P}_j=-\epsilon_{ljk}\mathscr{N}_k\,,\qquad (x_l\partial_0-x_0\partial_l)\mathscr{N}_j=\epsilon_{ljk}\mathscr{P}_k\,, \tag{24}$$

while for the spatial rotations one imposes

$$(x_k\partial_l-x_l\partial_k)\mathscr{P}_i(x) = -\delta_{li}\mathscr{P}_k(x)+\delta_{ki}\mathscr{P}_l(x)\,, \tag{25}$$

$$(x_k\partial_l-x_l\partial_k)\mathscr{N}_i(x) = -\delta_{li}\mathscr{N}_k(x)+\delta_{ki}\mathscr{N}_l(x)\,. \tag{26}$$

The consistency condition for all these differential equations becomes extremely simple

$$x_0\partial_0 D_+(x) = x_k\partial_k D_+(x)\,, \tag{27}$$

which disagrees with the consequence of the scaling invariance of $D_+(x)$: $x_0\partial_0 D_+(x)=(x_k\partial_k-2)D_+(x)$. Accordingly, there is no Lorentz covariant form of MWF in the Zwanziger formulation of QED.

4. Special noncovariant solutions for MWF

In the equal-time (ET) quantization the spatial rotations leave the quantization hypersurface $x^0 = 0$ invariant, thus one may look for a spherically symmetric solution for Wightman functions. Thus we take from (23)

$$\mathscr{P}_i(x) = \partial_i \partial_0 \Delta^{-1} \star D_+(x)\,, \qquad \mathscr{N}_i = 0\,, \tag{28}$$

where

$$\partial_0 \Delta^{-1} \star D_+(x) = i \int\limits_{-\infty}^{+\infty} \frac{d^3k}{(2\pi)^3} \frac{e^{-ik\cdot x}}{2\left|\vec{k}\right|^2} = \int\limits_0^{x^0} d\tau\, D_+(\tau, \vec{x}\,) + i\frac{1}{8\pi}\frac{1}{|\vec{x}|}\,. \tag{29}$$

Accordingly, the non-vanishing MWF are

$$\langle 0| A_i(x)\, B_j(0) |0\rangle \;=\; \epsilon_{ijk} \partial_k \partial_0 \Delta^{-1} \star D_+(x)\,, \tag{30}$$

and one may write in a compact notation as

$$\langle 0| A_\mu(x)\, B_\nu(0) |0\rangle \;=\; -\epsilon_{\mu\nu\alpha\beta} \bar{\partial}^\alpha \partial^\beta \Delta^{-1} \star D_+(x)\,, \tag{31}$$

where $\bar{\partial}^\mu = \partial^\mu - t^\mu \partial_0$ with $t^\mu = (1,0,0,0)$. Evidently, this solution can be interpreted as the Coulomb gauge condition for both gauge potentials $\bar{\partial}^\mu A_\mu = \partial_i A_i = 0$ and $\bar{\partial}^\mu B_\mu = \partial_i B_i = 0$. Thus in the ET formulation we expect the general form of MFW

$$\langle 0| A_\mu(x)\, B_\nu(0) |0\rangle = -\epsilon_{\mu\nu\alpha\beta} \bar{\partial}^\alpha \partial^\beta \Delta^{-1} \star D_+(x) + \partial_\mu \Psi_\nu(x) - \partial_\nu \Psi_\mu(x)\,, \tag{32}$$

where $\Psi_\mu(x)$ depends on the gauge fixing conditions.

In the light-front (LF) formulation, we introduce the LC coordinates: $x^\pm = (x^0 \pm x^1)/\sqrt{2}, x_i = (x_2, x_3)$. Now the equation (20) becomes

$$\epsilon_{ij}\left[\partial_i \mathcal{A} + \partial_- \mathcal{B}_i - \partial_+ \mathcal{C}_i\right] \;=\; -\partial_j D_+(x)\,, \tag{33}$$

$$\epsilon_{ij}\left[-2\partial_j \mathcal{B}_i + \epsilon_{ij} \partial_+ \mathcal{D}\right] \;=\; 2\partial_+ D_+(x)\,, \tag{34}$$

$$\epsilon_{ij}\left[2\partial_j \mathcal{C}_i - \epsilon_{ij} \partial_- \mathcal{D}\right] \;=\; 2\partial_- D_+(x)\,, \tag{35}$$

where $\epsilon_{ij} = \epsilon^{+-ij}$, $i,j = 2,3$, and we denote

$$\langle 0| A_+(x)\, B_-(0) |0\rangle \;=\; \mathcal{A}(x)\,, \qquad \langle 0| A_i(x)\, B_+(0) |0\rangle = \mathcal{B}_i(x)\,, \tag{36}$$

$$\langle 0| A_i(x)\, B_-(0) |0\rangle \;=\; \mathcal{C}_i(x)\,, \qquad \langle 0| A_i(x)\, B_j(0) |0\rangle = \epsilon_{ij} \mathcal{D}(x)\,. \tag{37}$$

We have found the simplest solution

$$\mathcal{B}_i(x) = -\epsilon_{ik} \partial_k \partial_+ \Delta_\perp^{-1} \star D_+(x)\,, \qquad \mathcal{C}_i(x) = \epsilon_{ik} \partial_k \partial_- \Delta_\perp^{-1} \star D_+(x)\,, \tag{38}$$

$$\mathcal{D}(x) = \mathcal{A}(x) = 0\,, \tag{39}$$

where

$$\partial_i \Delta_\perp^{-1} \star \partial_+ D_+(x) = -\frac{x^i}{8\pi^2 x_\perp^2} \frac{1}{x^+ - i0} + \frac{1}{2}\partial_i \int_0^{x^-} d\xi D_+\left(x^+, \xi, x_\perp\right) , \quad (40)$$

$$\partial_i \Delta_\perp^{-1} \star \partial_- D_+(x) = -\frac{x^i}{8\pi^2 x_\perp^2} \frac{1}{x^- - i0} + \frac{1}{2}\partial_i \int_0^{x^+} d\tau D_+\left(\tau, x^-, x_\perp\right) . \quad (41)$$

This solution can be expressed compactly as

$$\langle 0|A_\mu(x)\, B_\nu(0)|0\rangle = -\epsilon_{\mu\nu\alpha+}\partial^\alpha \partial_- \Delta_\perp^{-1} \star D_+(x) - \epsilon_{\mu\nu\alpha-}\partial^\alpha \partial_+ \Delta_\perp^{-1} \star D_+(x) . \quad (42)$$

Thus one may claim that the general form of MFW in the LF formulation can be expressed as

$$\begin{aligned}\langle 0|A_\mu(x)\, B_\nu(0)|0\rangle &= -\epsilon_{\mu\nu\alpha+}\partial^\alpha \partial_- \Delta_\perp^{-1} \star D_+(x) - \epsilon_{\mu\nu\alpha-}\partial^\alpha \partial_+ \Delta_\perp^{-1} \star D_+(x) \\ &\quad + \partial_\mu \Psi_\nu^{\mathrm{LF}}(x) - \partial_\nu \Psi_\mu^{\mathrm{LF}}(x) , \end{aligned} \quad (43)$$

where $\Psi_\mu^{\mathrm{LF}}(x)$ depends on the gauge fixing conditions.

5. Summary

We have shown that the Lorentz symmetry is broken for the mixed Wightman functions within the Zwanziger model of QED. Accordingly, only noncovariant solutions of equation (20) exist. In the ET and LF formulations, different solutions (32) and (43) are preferable. None of them depends explicitly on the fixed 4-vector n^μ, which is used in the Zwanziger formulation. Thus only the gauge terms Ψ_μ and Ψ_μ^{LF} may contain some dependence on n^μ.

This investigation is the first step in the perturbative formulation of the local QED with electric and magnetic currents, where the independence on n^μ becomes explicit for the gauge invariant quantities. Such independence on n^μ has been proved in [4], when electric and magnetic currents are given by the closed paths of classical point particles. In our approach, the charged particle currents are the local quantum field operators.

REFERENCES

[1] R.E. Peierls, *Proc. R. Soc. Lond.* **A214**, 124 (1952).

[2] N. Cabibbo, E. Ferrari, *Nuovo Cim.* **23**, 1147 (1966).

[3] D. Zwanziger, *Phys. Rev.* **D3**, 880 (1971).

[4] R.A. Brandt, F. Neri, D. Zwanziger, *Phys. Rev.* **D19**, 1153 (1979).