ON ANALYTIC SOLUTIONS OF MULTI-PARTON 't HOOFT EQUATIONS*

Jacek Wosiek

The Marian Smoluchowski Institute of Physics, Jagiellonian University
Reymonta 4, 30-059 Kraków, Poland

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Recent progress in understanding the structure of QCD_2 with adjoint matter is reported. In particular, the analytic solutions of the most singular (in infrared) part of the QCD equations are constructed in arbitrary Fock sector of the theory.

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After 40 years of research, there are basically two ways of attacking gauge theories in their non-perturbative regime. The lattice approach has developed into a standard and precise method allowing to compute the low-energy spectrum and plethora of important hadronic matrix elements from first principles [1], thereby confirming that QCD is *the* theory of strong interactions. On the other hand, the Light Cone formulation provides an intuitive and practical method to study higher energy phenomena like scattering, structure functions, excited states, *etc.* [2, 3].

The spectrum and structure of hadrons in the LC approach is determined *ab initio* by the infinite hierarchy of linear (in the large N_c limit), integral equations which couple subsequent Fock components, ψ_n, of hadronic wave functions

$$M^2\psi_n(x_1\dots x_n) = A\otimes\psi_n + B\otimes\psi_{n-2} + C\otimes\psi_{n+2}\,. \tag{1}$$

They constitute the eigenequation of the LC Hamiltonian, $H_{\rm C}$, (which is proportional to the mass squared operator, A, B, and C are amplitudes for elementary processes induced by $H_{\rm C}$). At finite number of colours, these equations become non-linear. Needless to say that complete solution of (1) is still not available.

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In this paper, I will review a proposal to reorganize Eqs. (1) according to the degree of infrared divergence of various terms in the kernel. Consequently, one may conceive the complete solution being constructed in an iterative process, where the most singular equations are solved first and then the less singular terms included subsequently in the controlled way. This Infrared Dominance Approximation (IRDA) will be also referred to as the Coulomb Approximation, since the most singular terms correspond to the straightforward Coulomb scatterings between partons [4][1]. Moreover, in the IRDA the hierarchy of equations decouple, since the Coulomb process obviously does not change a parton number. To further simplify the matter, we consider only the two-dimensional case. Nevertheless, we include full complexity of a field theory by allowing arbitrary parton multiplicities n. Formally, we consider the dimensional reduction of the full four-dimensional supersymmetric Yang–Mills theory to two dimensions. Then the IRDA equations (1) become diagonal in n and describe two species of partons (transverse gluons (*i.e.* bosons) and gluinos (fermions)) with additional two spin degrees of freedom each. The choice of the supersymmetric model has two-fold motivation. First, supersymmetry requires the matter fields to be in the adjoint representations of the gauge group, which in turns allows for the non-trivial physics in the higher parton sectors at large N. With matter fields in the fundamental representation, only two parton sector has non-vanishing contributions. Second, supersymmetry implies many relations, *e.g.* degeneracies, between the solutions, which provide additional tests of the whole procedure. In fact, even in the Coulomb Approximation, which breaks SUSY, we see traces of degenerate supermultiplets [4].

The paper is organized as follows: first, we briefly summarize numerical solutions of (1) in the first three ($n = 2, 3, 4$) Fock sectors. Armed with these, we then solve the problem analytically in the semiclassical approximation. Generalization for arbitrary parton numbers will be summarized in the last part together with determination of the entropy and the Hagedorn temperature of our solutions.

In Fig. 1, we show solutions of the eigenequation (1) in the LC configuration space, *i.e.* in x^- variable. To be more specific, we used the Light Cone Discretized Quantization of Brodsky *et al.* [2] upon which the total momentum P, together with parton momenta $p_i > 0$, were replaced by integers K and $k_i > 0$ with $\Sigma_i k_i = K$. Given the cutoff K, the Fock space with fixed parton number n is spanned by states defined by partitions of K. In this basis, the matrix elements of our Coulomb Hamiltonian can be readily calculated and eigenenergies, together with corresponding eigenstates, obtained by numerical diagonalization. What is shown, in the left part of

[1] And the self energy insertions which regularize Coulomb singularities.

Fig. 1, are the profiles of the Fourier transforms of the eigenstates from the LC momenta to the relative LC distance *i.e.* $d_{12} = d_1^- - d_2^-$. All distances are in the units of the LC momentum $a = 2\pi/P$.

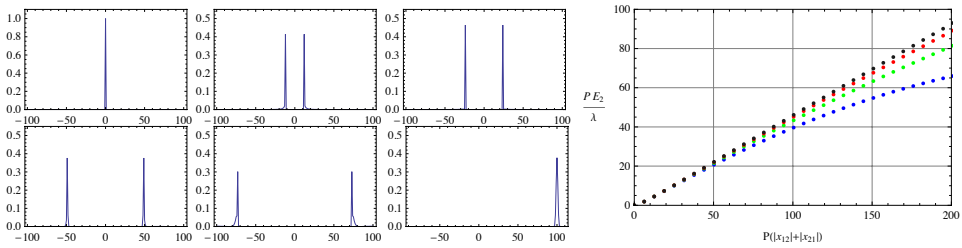

Fig. 1. Density profiles, in $x_{12} \equiv \Delta x^-$, of the six eigenstates with two partons (left) and eigenenergies in the two-parton sector, as the function of the relative LC distance, for four values of the cutoff K (right).

The first, astonishing at first glance, observation is that partons are very well localized, in the relative distance, in the x^- space. The quantum fluctuations are of the order of the ultraviolet cutoff $2\pi/P$ and vanish at infinite P. This property of 't Hooft equation is rarely, if at all, emphasized. As a consequence, the eigenenergies of corresponding eigenstates are well defined function of d_{12} which is, in fact, linear and grows with the appropriate, for two-dimensional theory, string tension [4, 5].

The same phenomenon occurs in the sector with three partons, *cf.* Fig. 2, where the contour plots of profiles of eigenstates as a function of three relative distances are shown. The situation is somewhat less trivial in this case. In addition to the localization in x^-, a new variable which controls

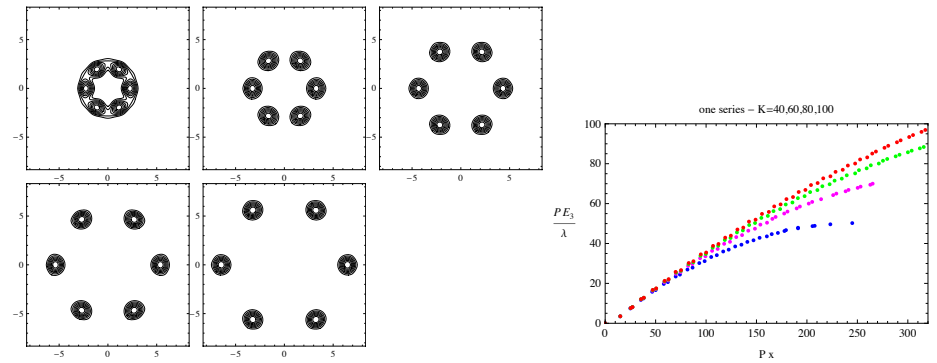

Fig. 2. As above but for three partons. Left: Profiles on the Dalitz plot in the three relative distances. Right: Eigenenergies *versus* the combined string length $x = |x_{12}| + |x_{23}| + |x_{31}|$.

the eigenenergies of three partons emerges. It is the combined string length $x = |x_{12}| + |x_{23}| + |x_{31}|$. This nicely confirms the string interpretation of two-dimensional gauge theories. Again, the eigenenergies are linear in x and the string tension extracted from Fig. 2 (right) is consistent with the one obtained from the two parton analysis above.

The situation is similar, and yet less trivial, in the four parton sector. Eigenstates are well localized, the eigenenergies grow linearly with the combined string length and consistent value of the string tension is obtained.

At present, numerical simulations beyond $p = 4$ become very time consuming. However, by then, enough regularities were observed to allow for construction of the analytic solutions which work for arbitrary parton numbers.

The IRDA form of master equations (1) reads, for n massless (because of supersymmetry) partons

$$\frac{\lambda}{2\pi}\int_0^{p_1+p_2} dk \frac{\psi_n(p_1,p_2,p_3\ldots p_n)-\psi_n(k,p_1+p_2-k,p_3\ldots p_n)}{(p_1-k)^2}$$
$$\pm \text{ cyclic permutations of } (p_1\ldots p_n) = E_{\rm C}\psi_n(p_1\ldots p_n)\,. \qquad (2)$$

In spite of many efforts, exact solutions, even of the simplest ($n = 2$, *i.e.* 't Hooft) equation, are not available [6]. Instead, we shall construct the complete set of approximate (WKB-like) solutions but for arbitrary parton number. For two partons, such solutions were already considered by 't Hooft [7], however, the generalizations to arbitrary n were unknown[2].

For large eigenenergies, the singularity of the kernel dominates, and one can extend the boundaries of integrations to $\pm\infty$. In this way, one obtains the *asymptotic* equations whose exact, generic solution reads

$$\psi_n(k_1,\ldots,k_n) \;=\; \exp\left(ik_1r_1 + ik_2r_2 + \ldots + ik_nr_n\right) \qquad (3)$$

and is labelled by positions of n partons in x^- space. This explains the localization we observed numerically. The exact eigenvalue of (3) is

$$E_{\rm C} = \frac{\lambda}{2}\Sigma_{i=1}^{n}|\Delta_{i,i+1}| \equiv \frac{\lambda}{2}La\,, \qquad \Delta_{i,j} = r_i - r_j\,, \qquad n+1=1 \qquad (4)$$

with unrestricted $n-1$ relative distances $\Delta_{i,j} = r_i - r_j$. Again, this is precisely what was observed numerically above — L being the combined length of strings in LC units, $a = \frac{2\pi}{P}$. To turn (3) into a complete basis of independent solutions, one has to impose appropriate boundary conditions at the edges of the physical region $k_1+\ldots+k_n = P$, $k_i > 0$. Their origin can

[2] See [8] for particular solutions with not suitable boundary conditions.

be simply seen in the two parton case, where (3) assumes the well known 't Hooft form (for the massive case)

$$M^2 f(x) = m^2 \left(\frac{1}{x} + \frac{1}{1-x} \right) f(x) + \frac{\lambda}{\pi} PV \int_0^1 dy \frac{f(x) - f(y)}{(y-x)^2} \tag{5}$$

with $f(x) = f(k_1/P) = \psi_2(k_1, k_2)$. For massive partons (5) implies the Dirichlet boundary conditions: $f(0) = f(1) = 0$. In the massless case, however, the leading singularities at the boundaries are governed by the Coulomb singularity of the kernel. Namely, if $x = 0, 1$ the principal value prescription is not effective, and the numerator of the integrand has to vanish in the first order as well, which implies the Neumann boundary conditions: $f'(0) = f'(1) = 0$. Until now, these consistent conditions were known and classified only for the $n = 2$ case [7, 9]. For arbitrary number of "supersymmetric" partons, the correct boundary conditions are of the Neumann type and follow from the generalization of the principal value argument above. Namely, one has to require the cancellation of all of the IR singularities, at the boundaries, to the first order in the soft momentum. The final result reads[3]

$$\begin{aligned} k_1 = 0 \ &: \ (\partial_2 - 2\partial_1)\psi_n = 0\,, \\ k_i = 0 \ &: \ (\partial_{i+1} - 2\partial_i + \partial_{i-1})\psi_n = 0\,, \qquad 2 \le i \le n-2\,, \\ k_{n-1} = 0 \ &: \ (\partial_{n-2} - 2\partial_{n-1})\psi_n = 0\,, \\ k_n = 0 \ &: \ (\partial_1 + \partial_{n-1})\psi_n = 0\,, \end{aligned} \tag{6}$$

where ψ_n is the function of the first $n-1$ independent momenta.

With the above BCs the problem is well posed, however, to satisfy (6) in practice, and for general n, is a non-trivial task. We have shown that it can be done, and have found simple combinatorial rules how to achieve that for the arbitrary Fock sector [10]. Let us illustrate the procedure in a few cases.

For $n = 2$ the BCs are satisfied by a linear combination of two generic solutions (3) — the second component being the Z_2 image of the first one. Moreover, the conditions quantize the allowed relative distance and fix the final combinations to read

$$\psi_2^{(m)}(k_1) = \cos\left(\Delta_{12}^{(m)} k_1 \right), \qquad \Delta_{12}^{(m)} = \frac{m}{2} a \tag{7}$$

which is nothing but the massless case of the WKB basis used by 't Hooft. The distances (and, consequently energies/masses) are quantized in terms

[3] It was derived during the discussion with G. Veneziano, and independently confirmed by Z. Ambroziński.

of one integer m. For even (odd) m, eigenfunctions are symmetric (antisymmetric) under the Z_2 symmetry and corresponding relative distance Δ_{12} is integer (half integer) in LC units a.

Things get more interesting for three partons. BCs are satisfied by well defined combinations of six generic solutions with the same L. Again, individual components can be obtained by the three Z_3 shifts of (3), together with inversions[4] along the closed string $(1,2,3)$. As for $n=2$, the relative distances are quantized

$$\begin{aligned} \Delta_{13} &= r/2\,, \qquad \Delta_{23} = s/2\,, \qquad r,s \text{ even} - \mathrm{Z}_3 \text{ invariant},\ \nu = 0\,, \\ \Delta_{13} &= r/2+\nu/2\,,\ \Delta_{23} = s/2-\nu/2\,,\ r,s \text{ odd} - \mathrm{Z}_3 \text{ covariant},\ \nu = \pm 1/3\,, \end{aligned} \tag{8}$$

and solutions with appropriate charge ν under Z_3 can be readily constructed. The two integers labelling states are not independent — the total degeneracy of a given energy level is known and proportional to L.

Instead of quoting the explicit solutions, which are a bit lengthy, we compare in Fig. 3 (left), the lowest spectra of Z_3 invariant states, calculated numerically with the analytic results following from (8) and (4). At first glance, there is very little correspondence between the two. Exact (numerical) eigenenergies seem pretty irregular, while analytic results predict equally spaced spectrum, with growing degeneracy. However, the comparison gets more interesting if one checks the overlaps between actual states. This is coded by colours of the lines. It turns out that only states corresponding to the same colour on the left and right panel have substantial overlaps. Moreover, degeneracies of the analytic spectrum closely match numbers of

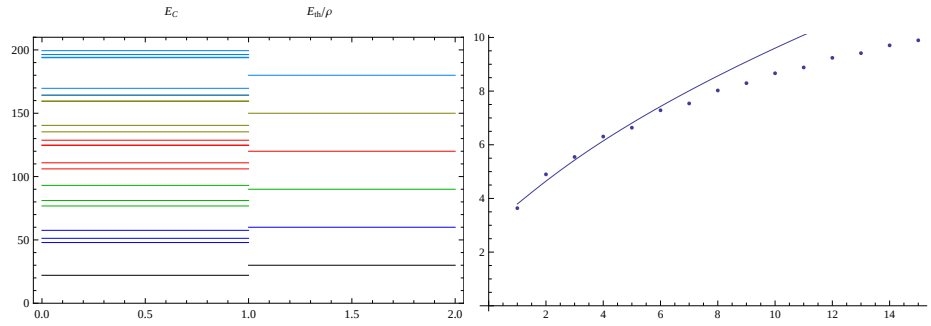

Fig. 3. Left: Numeric — DLCQ (left) — and analytic -WKB (right) — spectra of the infrared dominated Hamiltonian for three partons. The levels in the right panel are g-fold degenerate: $g = 1, 3, 3, 5, 5$ and 7, respectively. Right: The entropy of solutions (*versus* M^2/λ) from the first six multiplicity sectors.

[4] And complex conjugation.

exact levels (left) with significant overlap. As a consequence, the complicated and seemingly irregular spectrum of exact solutions can be thought of as resulting from splitting of the regular analytic solutions by some perturbation consisting of taking properly into account the Light Cone kinematics of an ensemble of partons.

Notice, however, that there exists an overall discrepancy of $\sim 30\%$ between actual values of the energies obtained by two methods. We think that this is the effect of the WKB approximation. Indeed, it was checked that the "disagreement" factor ρ decreases towards 1 in the higher part of the spectrum [10].

Situation becomes yet more complex and interesting with four partons. There, the linear combinations of Z_4 shifts and inversions are not sufficient to satisfy BCs (6). However, with more partons additional degeneracies appear. Namely, there are states with different configurations of relative distances, which nevertheless have the same combined length of strings, thereby the same energy. Consequently, the degenerate spaces are rich enough to admit combinations that satisfy (6). Constructing all such combinations was a challenging exercise, nevertheless, we have developed the algorithm which can generate a complete basis of solutions with arbitrary L. As an example, in Fig. 4, we compare one profile (there are three independent relative distances now) of one of lower states, obtained numerically by DLCQ, with the analytic expression just discussed. The agreement is quite satisfactory — even the complicated structure of the state is quite well reproduced. Checking the entropy of states, obtained in both ways, we have also confirmed that the disagreement factor ρ tends to 1 at higher energies, similarly to the three parton case. The number of states with given L grows now as L^2.

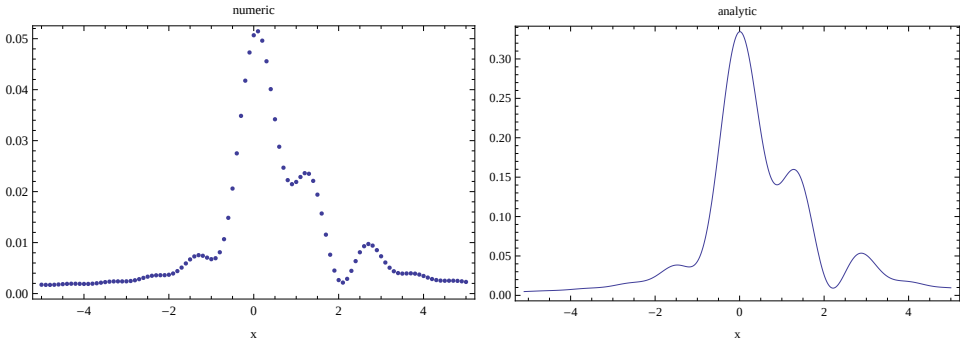

Fig. 4. d_{14} profile of one four-parton eigenstate: numeric (left) and analytic (right), $d_{24} = d_{34} = 1.3$.

The above construction can be generalized to higher (arbitrary in fact) number of partons. A number of regularities was found in $n = 3, 4$ sectors which can be turned into the exact, and easy to implement, rules how to identify combinations obeying (6). As a result, the complete bases of WKB solutions are available analytically for arbitrary parton number [10].

As the last illustration, we show in Fig. 3 (right) the total number of states in the first six sectors as a function of L, or equivalently the mass squared of a bound state. The onset of a famous Ramanujan square root dependence of the entropy of partitions of an integer is clearly seen. In physical terms: the entropy of our solutions has the Hagedorn behaviour with the Hagedorn temperature consistent with he one, $(1.3–1.4)\sqrt{\lambda/\pi}$, seen by Bhanot *et al.* [11] in DLCQ studies of similar systems.

To conclude, we have constructed analytically the WKB solutions of the most singular part of, master equations of two dimensional QCD with adjoint matter in the arbitrary Fock sector. Apart from their straightforward interpretation, these solutions can serve as a useful and economical basis to study the complete, *i.e.* coupled, set of "QCD_2 equations" providing a long sought approach to obtain a spectrum of reduced theories.

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