FLUID/GRAVITY CORRESPONDENCE AND HOLOGRAPHIC MIXED GAUGE-GRAVITATIONAL ANOMALY*

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We study, in the framework of the fluid/gravity correspondence, the anomaly induced current of a magnetic field and a vortex in a relativistic fluid. We use a holographic model with pure gauge and mixed gauge-gravitational Chern–Simons terms in the action, and confirm the results obtained within the Kubo formulae formalism [K. Landsteiner, E. Megías, F. Pena-Benitez, *Phys. Rev. Lett.* **107**, 021601 (2011); K. Landsteiner, E. Megías, L. Melgar, F. Pena-Benitez *J. High Energy Phys.* **09**, 121 (2011)]. The results are obtained in several frames, and we study the relation between these frames and the boundary conditions when solving the dynamical equations.

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1. Introduction

The modern understanding of hydrodynamics is as an effective field theory [1]. The equations of motion are the (anomalous) conservation laws of the energy-momentum tensor and spin one currents. These are supplemented by expressions for the energy-momentum tensor and the current

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which are organized in a derivative expansion, the so-called constitutive relations. These relations write in general

$$\langle T^{\mu\nu}\rangle = (\epsilon+p)\,u^\mu u^\nu + pg^{\mu\nu} + \langle T^{\mu\nu}\rangle_{\text{diss+anom}}\,, \tag{1}$$

$$\langle J^{\mu}\rangle = nu^\mu + \langle J^{\mu}\rangle_{\text{diss+anom}}\,. \tag{2}$$

Here ϵ is the energy density, p the pressure, n the charge density and u^μ the local fluid velocity. In addition to the ideal hydrodynamical contributions, there are extra terms which lead to dissipative and anomalous effects. Some examples of dissipative coefficients are the shear viscosity η [2], bulk viscosity ζ [3] and electric conductivity. Recent studies showed that anomalies give rise to new non-dissipative transport phenomena at finite temperature and chemical potential. In particular, an external magnetic field in the fluid $B^\mu = \epsilon^{\mu\nu\rho\lambda}u_\nu\partial_\rho A_\lambda$ induces an electric current via the so-called chiral magnetic effect [4], and a vortex in the fluid $\omega^\mu = \epsilon^{\mu\nu\rho\lambda}u_\nu\partial_\rho u_\lambda$ induces also a current parallel to the axial vorticity vector, the so-called chiral vortical effect [5, 6]. These effects are governed by chiral anomalies. Up to this point, only pure gauge anomalies had been considered to be relevant for first order hydrodynamics, but very recently it has been pointed out that mixed gauge-gravitational anomalies contribute also to the chiral vortical effect [7–9]. In this paper, we will study the chiral magnetic and vortical effects in the strong coupling regime via the fluid/gravity correspondence.

2. Fluid dynamics and gravity

We define in this section a holographic system which realizes a single chiral U(1) symmetry with a gauge and mixed gauge-gravitational anomaly, and obtain the equations of motion at first order in the hydrodynamical expansion. A detailed explanation of the method is presented in [5, 10].

2.1. Holographic model and derivative expansion

We consider an Einstein–Maxwell model in 5 dim, supplemented with pure gauge and mixed gauge-gravitational Chern–Simons terms [7, 9, 11]

$$S = \frac{1}{16\pi G}\int d^5x\sqrt{-g}\Big[R + 2\Lambda - \frac{1}{4}F_{MN}F^{MN} + \epsilon^{MNPQR}A_M \times\Big(\frac{\kappa}{3}F_{NP}F_{QR} + \lambda R^A{}_{BNP}R^B{}_{AQR}\Big)\Big] + S_{\text{GH}} + S_{\text{CSK}}\,. \tag{3}$$

In addition to the usual Gibbons–Hawking boundary term, S_{GH}, a second boundary term S_{CSK} is needed if we want the model to reproduce the gravitational anomaly at general hypersurface. The covariant form of the anomalous current is $(16\pi G)J^\mu = -\sqrt{-\gamma}F^{r\mu}|_\partial$, and its divergence leads to the standard form of the anomaly for chiral fermions [12]. This we use to fix the parameters $\kappa = -G/(2\pi)$ and $\lambda = -G/(48\pi)$. The bulk equations of motion of the model admit an AdS Reissner–Nordström black-brane solution

$$ds^2 = -r^2 f(r) u_\mu u_\nu dx^\mu dx^\nu + r^2 P_{\mu\nu} dx^\mu dx^\nu - 2u_\mu dx^\mu dr\,, \tag{4}$$

where $P_{\mu\nu} = u_\mu u_\nu + \eta_{\mu\nu}$, and the blackening factor and gauge field write

$$f(r) = 1 - \frac{M}{r^4} + \frac{Q^2}{r^6}\,, \qquad A_r(r) = 0\,, \qquad A_\mu(r) = A_\mu^{(b)} - \frac{\sqrt{3}Q}{r^2} u_\mu\,. \tag{5}$$

To study the chiral magnetic effect, we consider an external gauge field $A_\mu^{(b)}$ which leads to a background magnetic field B_μ. Eqs. (4)–(5) are a boosted version of the black-brane solution expressed in Eddington–Finkelstein coordinates, and they are a solution of the e.o.m. only when u_μ, M, Q and $A_\mu^{(b)}$ are independent of the space-time coordinates x^μ. If one assumes that these quantities are slow varying functions of x^μ, then one can find a solution for the metric and gauge field valid order by order in a derivative expansion (see *e.g.* [5, 10] for details). It is useful to separate the metric and gauge field in scalar, vector and tensor sectors with respect to u_μ. The anomalous contributions to the constitutive relations at first order in the derivative expansion come only from the vector sector, so in the following, we will neglect the other sectors. The metric and gauge field write

$$ds^2 = r^2 j_\sigma(r) \left(P_\mu^\sigma u_\nu + P_\nu^\sigma u_\mu\right) dx^\mu dx^\nu + \ldots\,, \qquad A_\mu = a_\nu(r) P_\mu^\nu + \ldots\,, \tag{6}$$

where dots indicate scalar and tensor contributions. The functions $j_\mu(r)$ and $a_\mu(r)$ admit a large r (near boundary) expansion of the form $j_\mu(r) = \sum_{\bar{n}=0}^{\infty} j_\mu^{(\bar{n})}/r^{\bar{n}}$ and $a_\mu(r) = \sum_{\bar{n}=0}^{\infty} a_\mu^{(\bar{n})}/r^{\bar{n}}$.

2.2. Einstein–Maxwell equations

The equations of motion in the vector sector at order n in the derivative expansion write[1]

$$\boldsymbol{J}_\mu^{(n)}(r) = \partial_r \left(r^5 \partial_r j_\mu^{(n)}(r) + 2\sqrt{3} Q a_\mu^{(n)}(r) \right)\,, \tag{7}$$

$$\boldsymbol{V}_\mu^{(n)}(r) = \partial_r \left(r^3 f(r) \partial_r a_\mu^{(n)}(r) + 2\sqrt{3} Q j_\mu^{(n)}(r) \right)\,. \tag{8}$$

[1] Following the notation in [5], superscripts (n) should not be confused with barred superscripts $(\bar{n})$, where the latter refers to the order in the near boundary expansion.

The form of the homogeneous part is the same at any order. Using the model of (3), the sources at first order write

$$\boldsymbol{J}_\mu^{(1)}(r) \;=\; 3r^2 u^\nu \partial_\nu u_\mu + \frac{96}{r^5}\left(Mr^2 - 5Q^2\right)\lambda B_\mu + \frac{16\sqrt{3}Q}{r^7}\left(20Mr^2 - 63Q^2\right)\lambda\omega_\mu \,, \tag{9}$$

$$\begin{aligned}\boldsymbol{V}_\mu^{(1)}(r) \;=\; & -\frac{\sqrt{3}}{r^2}\left(P_\mu^\nu \partial_\nu Q + Q u^\nu \partial_\nu u_\mu\right) - \frac{16\sqrt{3}Q}{r^3}\kappa B_\mu - \frac{48Q^2}{r^5}\kappa\omega_\mu \\ & -\frac{48}{r^{11}}\left(4M^2 r^4 - 16MQ^2 r^2 + 15Q^4\right)\lambda\omega_\mu \,. \end{aligned} \tag{10}$$

The first terms in the r.h.s. of (9) and (10) are responsible for dissipative effects. The terms proportional to B_μ and ω_μ are anomalous, and they come from the CS in the action. The dissipative terms and the one proportional to $\kappa\omega_\mu$ were already obtained in [5, 10]. Here we get new terms coming from the external magnetic field B_μ, and the gauge-gravitational anomaly[2] $\sim \lambda$. Finally, the non-ideal contributions to the constitutive relations in the vector sector can be computed at first order in the hydrodynamical expansion as

$$\langle T_{\mu\nu}\rangle^{(1)} = \frac{1}{4\pi G} j_\sigma^{(1,\bar{4})}(P_\mu^\sigma u_\nu + P_\nu^\sigma u_\mu) + \ldots \,, \qquad \langle J_\mu\rangle^{(1)} = -\frac{1}{8\pi G} a_\mu^{(1,\bar{2})} \,. \tag{11}$$

$\langle T_{\mu\nu}\rangle$ receives extra contributions from the scalar and tensor sectors [13].

3. Fluid frames and boundary conditions

Hydrodynamic constitutive relations depend on the definition of the fluid velocity, and this specifies a particular frame. In the case of the anomalous conductivities, it was shown in [14] how the Landau frame conductivities computed by Son and Surowka [6] can be obtained from a combination of the charge and energy transport coefficients. Each frame is related to some specific choice of boundary conditions when solving the equations of motion. Eqs. (7)–(8) are two differential equations of second order, so one needs four boundary conditions. Three of them are common for all the frames

$$j_\mu(r\to\infty) = 0 \,, \qquad a_\mu(r\to\infty) = 0 \,, \qquad a_\mu(r_{\rm h}) = \text{finite} \,, \tag{12}$$

and they are chosen to guarantee that the field theory metric and background field are not modified, and that they are regular at the outer horizon $r_{\rm h}$ [5]. There is some freedom for the fourth boundary condition, and we will discuss here three possibilities. The fluid velocity is defined in the Landau frame to be proportional to the energy flux

$$\left\langle T^{0i}\right\rangle = (\epsilon + p)u^i \,. \tag{13}$$

[2] Gravitational anomaly effects were studied recently in [11] using a similar approach.

This means that the non-ideal contributions to the energy-momentum tensor vanish in this frame, $\langle T^{0i}\rangle_{\text{diss+anom}} = 0$, and from (11) one can see that this frame is obtained with the condition $j_\mu^{(\bar{4})} = 0$. Likewise, the fluid velocity in the Eckart frame is defined such that it is proportional to the charge current

$$\left\langle J^i \right\rangle = n u^i \,, \tag{14}$$

which is equivalent to $\langle J^i\rangle_{\text{diss+anom}} = 0$, and from (11) the Eckart frame condition is $a_\mu^{(\bar{2})} = 0$. Transport phenomena related to the generation of an energy (charge) current are not directly visible in the Landau (Eckart) frame, rather they are absorbed in the definition of the fluid velocity. It is, therefore, more convenient to go to another frame in which we demand that the definition of the fluid velocity is not influenced when switching on an external magnetic field or having a vortex in the fluid: this is the laboratory rest frame[3]. We have checked that the anomalous conductivities in this frame are obtained when imposing the boundary condition, $j_\mu(r_\text{h}) = 0$. This could indicate that this frame is related to an entropy current J_S^i, so that [15][4]

$$\left\langle J_S^i \right\rangle_{\text{anom}} = 0 \,. \tag{15}$$

We leave a further analysis to future work [17]. The result for the anomalous chiral magnetic and vortical effects in a specific frame, F, can be written as

$$\langle T_{\mu\nu}\rangle^{(1)}_{\text{anom}} = \left(\sigma^\epsilon_B\right)^F \left(B_\mu u_\nu + B_\nu u_\mu\right) + \left(\sigma^\epsilon_V\right)^F \left(\omega_\mu u_\nu + \omega_\nu u_\mu\right) \,, \tag{16}$$

$$\langle J_\mu\rangle^{(1)}_{\text{anom}} = \left(\sigma_B\right)^F B_\mu + \left(\sigma_V\right)^F \omega_\mu \,. \tag{17}$$

The values for the conductivities in three different frames are summarized in Table I. The terms in the conductivities with only μ dependence come from the chiral anomaly, and the ones containing T^2 are induced by the gauge-gravitational anomaly. These expressions are in perfect agreement with the literature [5, 6, 9–11, 14]. They coincide with the result at weak coupling [7, 8], and this we take as a strong hint towards a non-renormalization theorem for the anomalous conductivities, at least when gauge fields are absent[5].

We get these values by an explicit computation using the boundary conditions specified above. Note, however, that the expressions in two different frames can be related by a shift in the fluid velocity, $u_\mu \to u_\mu + \delta u_\mu$, *e.g.* the Eckart frame is obtained from the Lab frame by a shift with $\delta u_\mu = -\frac{1}{n}\left(\sigma_B B_\mu + \sigma_V \omega_\mu\right)$. This serves as a cross-check of our results.

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3 The transport coef. in the Lab frame were derived from Kubo formulae in [7–9, 14].
4 See [11, 16] for a convenient definition of the local entropy current in fluid/gravity.
5 See [18] for a recent study in this direction.

TABLE I

Anomalous conductivities contributing to the constitutive relations (16)–(17) with the model of Sec. 2, in three different frames. The results in the Landau and Eckart frames are written in terms of the conductivities in the Lab rest frame.

Conductivities	Laboratory rest frame	Landau frame	Eckart frame
$(\sigma_B)^F$	$\sigma_B = \frac{\mu}{4\pi^2}$	$\sigma_B - \frac{n}{\epsilon+P}\sigma_B^\epsilon$	0
$(\sigma_V)^F$	$\sigma_V = \frac{\mu^2}{8\pi^2} + \frac{T^2}{24}$	$\sigma_V - \frac{n}{\epsilon+P}\sigma_V^\epsilon$	0
$(\sigma_B^\epsilon)^F$	$\sigma_B^\epsilon = \sigma_V$	0	$\sigma_B^\epsilon - \frac{\epsilon+P}{n}\sigma_B$
$(\sigma_V^\epsilon)^F$	$\sigma_V^\epsilon = \frac{\mu^3}{12\pi^2} + \frac{\mu T^2}{12}$	0	$\sigma_V^\epsilon - \frac{\epsilon+P}{n}\sigma_V$

Stability and causality issues of the hydrodynamic equations demand the knowledge of second order hydrodynamics [5, 10, 19]. A classification of the terms contributing to this order was obtained in [20]. A computation of the second order coefficients within the fluid/gravity correspondence with chiral and gauge-gravitational anomalies including external electromagnetic fields will be presented in a forthcoming paper [17].

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REFERENCES

[1] L.D. Landau, E.M. Lifshitz, H.G. Schopf (Ed.), P. Ziesche (Ed.), *Textbook on Theoretical Physics. Vol. 2: Classical Field Theory*, Butterworth-Heinemann, 2 edition, 1987, (in German); W. Israel, J.M. Stewart, *Ann. Phys.* **118**, 341 (1979); R. Baier *et al.*, *J. High Energy Phys.* **0804**, 100 (2008).

[2] G. Policastro, D.T. Son, A.O. Starinets, *Phys. Rev. Lett.* **87**, 081601 (2001).

[3] P.B. Arnold, C. Dogan, G.D. Moore, *Phys. Rev.* **D74**, 085021 (2006).

[4] K. Fukushima, D.E. Kharzeev, H.J. Warringa, *Phys. Rev.* **D78**, 074033 (2008); A. Rebhan, A. Schmitt, S.A. Stricker, *J. High Energy Phys.* **1001**, 026 (2010).

[5] J. Erdmenger, M. Haack, M. Kaminski, A. Yarom, *J. High Energy Phys.* **0901**, 055 (2009).

[6] D.T. Son, P. Surowka, *Phys. Rev. Lett.* **103**, 191601 (2009).

[7] K. Landsteiner, E. Megias, F. Pena-Benitez, arXiv:1110.3615 [hep-ph]; arXiv:1207.5808 [hep-th]; K. Landsteiner, E. Megias, L. Melgar, F. Pena-Benitez, *J. Phys. Conf. Ser.* **343**, 012073 (2012); *Fortschr. Phys.* **60**, 1064 (2012).

[8] K. Landsteiner, E. Megias, F. Pena-Benitez, *Phys. Rev. Lett.* **107**, 021601 (2011).

[9] K. Landsteiner, E. Megias, L. Melgar, F. Pena-Benitez, *J. High Energy Phys.* **1109**, 121 (2011).

[10] S. Bhattacharyya, V.E. Hubeny, S. Minwalla, M. Rangamani, *J. High Energy Phys.* **0802**, 045 (2008); N. Banerjee *et al.*, *J. High Energy Phys.* **1101**, 094 (2011).

[11] S. Chapman, Y. Neiman, Y. Oz, *J. High Energy Phys.* **1207**, 128 (2012).

[12] R.A. Bertlmann, *Anomalies in Quantum Field Theory*, Clarendon Press, Oxford U.K., 1996.

[13] M. Haack, A. Yarom, *J. High Energy Phys.* **0810**, 063 (2008).

[14] I. Amado, K. Landsteiner, F. Pena-Benitez, *J. High Energy Phys.* **1105**, 081 (2011).

[15] R. Loganayagam, arXiv:1106.0277 [hep-th].

[16] C. Eling, A. Meyer, Y. Oz, *J. High Energy Phys.* **1208**, 088 (2012).

[17] K. Landsteiner, E. Megias, F. Pena-Benitez, work in progress.

[18] S. Golkar, D.T. Son, arXiv:1207.5806 [hep-th]; K. Jensen, R. Loganayagam, A. Yarom, arXiv:1207.5824 [hep-th]; D. Hou, H. Liu, H.-c. Ren, arXiv:1210.0969 [hep-th].

[19] A. Muronga, *Phys. Rev.* **C69**, 034903 (2004); S. Bhattacharyya, *J. High Energy Phys.* **1207**, 104 (2012); G.D. Moore, K.A. Sohrabi, *J. High Energy Phys.* **1211**, 148 (2012) [arXiv:1210.3340 [hep-ph]].

[20] D.E. Kharzeev, H.U. Yee, *Phys. Rev.* **D84**, 045025 (2011).