ENERGY LOSS IN UNSTABLE QUARK-GLUON PLASMA WITH EXTREMELY PROLATE MOMENTUM DISTRIBUTION*

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The energy loss per unit path length of a highly energetic parton scattering elastically in a weakly coupled quark-gluon plasma is studied as an initial value problem. The approach is designed to study unstable plasmas but in the case of an equilibrium plasma the well known result is reproduced. An extremely prolate plasma, where the momentum distribution is infinitely elongated along one direction, is considered here. The energy loss is shown to be strongly time and directionally dependent and its magnitude can much exceed the energy loss in equilibrium plasma.

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1. Introduction

When a highly energetic parton travels through a quark-gluon plasma (QGP), it loses its energy due to, in particular, elastic interactions with plasma constituents. This is called *collisional energy loss* which for equilibrium QGP is well understood [1]. The quark-gluon plasma produced in

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relativistic heavy-ion collisions, however, reaches a state of local equilibrium only after a short but finite time interval, and during this period the momentum distribution of plasma partons is anisotropic. Consequently, the system is unstable (for a review see [3]). Collisional energy loss has been computed for an anisotropic QGP in Ref. [2], but the fact that unstable systems are explicitly time dependent as unstable modes exponentially grow in time was not taken into account.

We have developed an approach, see [4, 5] for a preliminary account, where energy loss is studied as an initial value problem. The approach is applicable to plasma systems evolving quickly in time. We compute the energy loss by treating the parton as an energetic classical particle with SU(N_c) color charge. For an equilibrium plasma, the known result is recovered and for an unstable plasma, the energy loss is shown to have contributions which exponentially grow in time. In Refs. [4], we have calculated the energy loss in a two-stream system which is unstable due to longitudinal chromoelectric modes and found that it manifests strong time and directional dependence. In Ref. [5] and here, we focus on an extremely prolate quark-gluon plasma with momentum distribution infinitely elongated in the beam direction. Such a system is unstable due to transverse chromomagnetic modes and the spectrum of collective excitations can be obtained in explicit analytic form. The system has thus nontrivial dynamics but the computation of energy loss is relatively simple. After a brief presentation of our approach, we show some of our results.

2. Formalism

Using the Wong equations [6], which describe the motion of classical parton in a chromodynamic field, one writes down the parton's energy as

$$\frac{dE(t)}{dt} = gQ^a \boldsymbol{E}_a(t, \boldsymbol{r}(t)) \cdot \boldsymbol{v}\,, \tag{1}$$

where g is the QCD coupling constant, gQ^a is the parton color charge, $\boldsymbol{E}_a(t, \boldsymbol{r})$ is the chromoelectric field and $\boldsymbol{v}$ is the parton's velocity which is assumed to be constant and $\boldsymbol{v}^2 = 1$. Since we deal with an initial value problem, we apply to the field not the usual Fourier transformation but the *one-sided Fourier transformation* defined as

$$f(\omega, \boldsymbol{k}) \;=\; \int_0^\infty dt \int d^3r e^{i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} f(t, \boldsymbol{r})\,, \tag{2}$$

$$f(t, \boldsymbol{r}) \;=\; \int_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i(\omega t - \boldsymbol{k}\cdot\boldsymbol{r})} f(\omega, \boldsymbol{k})\,, \tag{3}$$

where the real parameter $\sigma > 0$ is chosen is such a way that the integral over ω is taken along a straight line in the complex ω-plane, parallel to the real axis, above all singularities of $f(\omega, \boldsymbol{k})$. Introducing the current generated by the parton $\boldsymbol{j}_a(t, \boldsymbol{r}) = gQ^a \boldsymbol{v}\delta^{(3)}(\boldsymbol{r} - \boldsymbol{v}t)$, Eq. (1) can be rewritten as

$$\frac{dE(t)}{dt} = gQ^a \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i(\omega-\bar{\omega})t}\, \boldsymbol{E}_a(\omega, \boldsymbol{k}) \cdot \boldsymbol{v}\,, \tag{4}$$

where $\bar{\omega} \equiv \boldsymbol{k} \cdot \boldsymbol{v}$. The next step is to compute the chromoelectric field $\boldsymbol{E}_a$. Applying the one-sided Fourier transformation to the linearized Yang–Mills equations, which represent QCD in the Hard Loop approximation, we get

$$E_a^i(\omega, \boldsymbol{k}) = -i \left(\Sigma^{-1}\right)^{ij} (\omega, \boldsymbol{k}) \left[\omega j_a^j(\omega, \boldsymbol{k}) + \epsilon^{jkl} k^k B_{0a}^l(\boldsymbol{k}) - \omega D_{0a}^j(\boldsymbol{k})\right], \tag{5}$$

where $\boldsymbol{B}_0$ and $\boldsymbol{D}_0$ are the initial values of the chromomagnetic field and the chromoelectric induction, and $D_a^i(\omega, \boldsymbol{k}) = \varepsilon^{ij}(\omega, \boldsymbol{k}) E_a^j(\omega, \boldsymbol{k})$ with $\varepsilon^{ij}(\omega, \boldsymbol{k})$ being chromodielectric tensor; $\Sigma^{ij}(\omega, \boldsymbol{k}) \equiv -\boldsymbol{k}^2 \delta^{ij} + k^i k^j + \omega^2 \varepsilon^{ij}(\omega, \boldsymbol{k})$.

Substituting the expression (5) into Eq. (4), we get the formula

$$\begin{aligned} \frac{dE(t)}{dt} &= gQ^a v^i \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi i} \int \frac{d^3k}{(2\pi)^3} e^{-i(\omega-\bar{\omega})t} \left(\Sigma^{-1}\right)^{ij} (\omega, \boldsymbol{k}) \\ &\quad \times \left[\frac{i\omega g Q^a v^j}{\omega - \bar{\omega}} + \epsilon^{jkl} k^k B_{0a}^l(\boldsymbol{k}) - \omega D_{0a}^j(\boldsymbol{k})\right]. \end{aligned} \tag{6}$$

As seen, the integral over ω is controlled by the poles of the matrix $\Sigma^{-1}(\omega, \boldsymbol{k})$ which represent the collective modes of the system.

When the plasma is in equilibrium, all modes are damped and the poles of $\Sigma^{-1}(\omega, \boldsymbol{k})$ are located in the lower half-plane of complex ω. Consequently, the contributions to the energy loss corresponding to the poles of $\Sigma^{-1}(\omega, \boldsymbol{k})$ exponentially decay in time. The only stationary contribution is given by the pole $\omega = \bar{\omega} \equiv \boldsymbol{k} \cdot \boldsymbol{v}$. Therefore, Eq. (6) provides

$$\frac{dE}{dt} = -ig^2 C_R \int \frac{d^3k}{(2\pi)^3}\, \frac{\bar{\omega}}{\boldsymbol{k}^2} \left[\frac{1}{\varepsilon_{\mathrm{L}}(\bar{\omega}, \boldsymbol{k})} + \frac{\boldsymbol{k}^2 \boldsymbol{v}^2 - \bar{\omega}^2}{\bar{\omega}^2 \varepsilon_{\mathrm{T}}(\bar{\omega}, \boldsymbol{k}) - \boldsymbol{k}^2}\right], \tag{7}$$

where the color factor C_R, which equals $\frac{N_c^2-1}{2N_c}$ for a quark and N_c for a gluon, results from the averaging over colors of the test parton. The formula (7) agrees with the standard result [1].

To compare the energy loss in an unstable plasma to that in an equilibrium one, we have computed the integral in Eq. (7) numerically using cylindrical coordinates, which will also be used for the prolate system. Since the integral is known to be logarithmically divergent, it has been taken over a finite domain such that $-k_{\rm max} \le k_{\rm L} \le k_{\rm max}$ and $0 \le k_{\rm T} \le k_{\rm max}$. The energy loss in an equilibrium plasma of massless constituents can be expressed through the Debye mass, which we write as

$$\mu^2 \equiv g^2 \int \frac{d^3p}{(2\pi)^3} \frac{f(\boldsymbol{p})}{|\boldsymbol{p}|} \,. \tag{8}$$

When the plasma is unstable, the matrix $\Sigma^{-1}(\omega, \boldsymbol{k})$ has poles in the upper half-plane of complex ω, and the contributions to the energy loss from these poles grow exponentially in time. Using the linearized Yang–Mills equations, the initial values $\boldsymbol{B}_0$ and $\boldsymbol{D}_0$ are expressed through the current and

$$\frac{dE(t)}{dt} = g^2 C_R v^i v^l \int\limits_{-\infty+i\sigma}^{\infty+i\sigma} \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} e^{-i(\omega-\bar{\omega})t} \left(\Sigma^{-1}\right)^{ij}(\omega, \boldsymbol{k})$$
$$\times \left[\frac{\omega \delta^{jl}}{\omega - \bar{\omega}} - \left(k^j k^k - \boldsymbol{k}^2 \delta^{jk}\right) \left(\Sigma^{-1}\right)^{kl}(\bar{\omega}, \boldsymbol{k}) + \omega\, \bar{\omega}\, \varepsilon^{jk}(\bar{\omega}, \boldsymbol{k}) \left(\Sigma^{-1}\right)^{kl}(\bar{\omega}, \boldsymbol{k}) \right] , \tag{9}$$

which gives the energy loss of a parton in an unstable quark-gluon plasma.

When the anisotropy of the momentum distribution of plasma constituents is controlled by a single (unit) vector $\boldsymbol{n}$, it is not difficult to invert the matrix Σ. Following [7], we introduce the vector $\boldsymbol{n}_{\rm T}$ defined as $n_{\rm T}^i \equiv (\delta^{ij} - k^i k^j/\boldsymbol{k}^2) n^j$ and we use the basis of four symmetric tensors: $A^{ij}(\boldsymbol{k}) \equiv \delta^{ij} - k^i k^j/\boldsymbol{k}^2$, $B^{ij}(\boldsymbol{k}) \equiv k^i k^j/\boldsymbol{k}^2$, $C^{ij}(\boldsymbol{k}, \boldsymbol{n}) \equiv n_{\rm T}^i n_{\rm T}^j/\boldsymbol{n}_{\rm T}^2$ and $D^{ij}(\boldsymbol{k}, \boldsymbol{n}) \equiv k^i n_{\rm T}^j + k^j n_{\rm T}^i$. Since the matrix Σ is symmetric, it can be decomposed as $\Sigma = aA + bB + cC + dD$ and the inverse matrix equals

$$\Sigma^{-1} = \frac{1}{a} A + \frac{-a(a+c)\, B + \left(-d^2 \boldsymbol{k}^2 \boldsymbol{n}_{\rm T}^2 + bc\right)\, C + ad\, D}{a\left(d^2 \boldsymbol{k}^2 \boldsymbol{n}_{\rm T}^2 - b(a+c)\right)} \,. \tag{10}$$

The poles of Σ^{-1} determine collective excitations in the plasma system.

Substituting the matrix Σ^{-1} in the form (10) into Eq. (9), we get the energy-loss formula used in the subsequent section to compute the energy loss in an extremely prolate QGP.

3. Extremely prolate plasma

Anisotropy is a generic feature of the parton momentum distribution in heavy-ion collisions. At the early stage, when partons emerge from the incoming nucleons, the momentum distribution is strongly elongated along the beam — it is *prolate* — with $\langle p_{\rm T}^2\rangle \ll \langle p_{\rm L}^2\rangle$. Due to free streaming, the distribution evolves in the local rest frame to a form which is squeezed along the beam — it is *oblate* with $\langle p_{\rm T}^2\rangle \gg \langle p_{\rm L}^2\rangle$. We consider here the extremely prolate momentum distribution which can be written as

$$f(\boldsymbol{p}) = \delta\left(\boldsymbol{p}^2 - (\boldsymbol{p}\cdot\boldsymbol{n})^2\right)\, h(\boldsymbol{p}\cdot\boldsymbol{n})\,, \tag{11}$$

where $h(x)$ is any positive even function such that $\int d^3 f(\boldsymbol{p})$ is finite. The integral in Eq. (8) can be used to define a mass parameter for either isotropic or anisotropic momentum distributions, although in the later case μ^{-1} cannot be interpreted as the screening length.

With the distribution (11), one finds the matrix Σ which is then inverted. The poles of Σ^{-1} provide a spectrum of collective excitations which are

$$\omega_1^2(\boldsymbol{k}) \;=\; \tfrac{1}{2}\mu^2 + (\boldsymbol{k}\cdot\boldsymbol{n})^2\,, \qquad \omega_2^2(\boldsymbol{k}) = \tfrac{1}{2}\mu^2 + \boldsymbol{k}^2\,, \tag{12}$$

$$\omega_\pm^2(\boldsymbol{k}) \;=\; \tfrac{1}{2}\Big(\boldsymbol{k}^2 + (\boldsymbol{k}\cdot\boldsymbol{n})^2 \pm\sqrt{\boldsymbol{k}^4 + (\boldsymbol{k}\cdot\boldsymbol{n})^4 + 2\mu^2\boldsymbol{k}^2 - 2\mu^2(\boldsymbol{k}\cdot\boldsymbol{n})^2 - 2\boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{n})^2}\;\Big)\,. \tag{13}$$

The modes ω_1, ω_2 and ω_+, are always stable. The solution ω_-^2 is negative when $m^2\boldsymbol{k}^2 > m^2(\boldsymbol{k}\cdot\boldsymbol{n})^2 + \boldsymbol{k}^2(\boldsymbol{k}\cdot\boldsymbol{n})^2$. Writing $\omega_-^2 = -\gamma^2$, $0<\gamma\in\mathbb{R}$, the solutions are $\pm i\gamma$. The first is the Weibel unstable mode and the second is an overdamped mode. Collective excitations in the extremely prolate QGP were earlier studied in [8] using a method different than ours.

The energy loss in the extremely prolate system is controlled by the double pole at $\omega = 0$ and 8 single poles: $\omega = \pm\omega_1$, $\omega = \pm\omega_2$, $\omega = \pm\omega_+$, $\omega = \pm\omega_-$. Since the collective modes are known analytically, the integral over ω is computed analytically as well. The remaining integral over $\boldsymbol{k}$ is computed numerically using the cylindrical coordinates with the z axis along the vector $\boldsymbol{n}$. Since the integral is divergent (as is the case in equilibrium (7)), we choose a finite domain such that $-k_{\rm max} \le k_{\rm L} \le k_{\rm max}$ and $0 \le k_{\rm T} \le k_{\rm max}$ with $k_{\rm max} = 5\mu$. The values of remaining parameters are: $g=1$, $C_R = N_c = 3$. In Fig. 1, we show the parton's energy loss per unit time as a function of time for three different orientations of the parton's velocity $\boldsymbol{v}$ with respect to the z axis. The energy loss manifests a strong directional dependence and it exponentially grows in time, indicating the effect of instabilities. After a sufficiently long time, the magnitude of energy loss much exceeds that in equilibrium plasma which equals $0.18\, g^2\mu^2$ for $k_{\rm max} = 5\mu$.

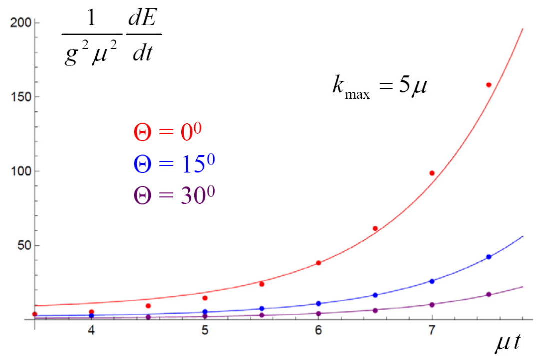

Fig. 1. The parton energy loss per unit time as a function of time for three angles Θ between the parton's velocity $\boldsymbol{v}$ and the axis z. The top (red) points correspond to $\Theta = 0$, the middle (blue) ones to $\Theta = \pi/12$ and the bottom (purple) points to $\Theta = \pi/6$. The solid lines represent the exponential fits to the computed points.

4. Conclusions

We have developed a formalism, where the energy loss of a fast parton in a plasma medium is found as the solution of an initial value problem. The formalism, which allows one to obtain the energy loss in an unstable plasma, is applied to an extremely prolate quark-gluon plasma with momentum distribution infinitely elongated in the z direction. This system is unstable due to chromomagnetic transverse modes. The energy loss per unit length of a highly energetic parton exponentially grows in time and exhibits a strong directional dependence. The magnitude of the energy loss can much exceed the equilibrium value.

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REFERENCES

[1] M. Le Bellac, *Thermal Field Theory*, Cambridge University Press, Cambridge 2000, Sec. 8.1.

[2] P. Romatschke, M. Strickland, *Phys. Rev.* **D71**, 125008 (2005).

[3] St. Mrówczyński, *Acta Phys. Pol. B* **37**, 427 (2006).

[4] M.E. Carrington, K. Deja, St. Mrówczyński, *Acta Phys. Pol. B Proc. Supp.* **5**, 947 (2012); **5**, 343 (2012).

[5] M.E. Carrington, K. Deja, St. Mrówczyński, `arXiv:1301.4563 [hep-ph]`.

[6] S.K. Wong, *Nuovo Cim.* **A65**, 689 (1970).

[7] P. Romatschke, M. Strickland, *Phys. Rev.* **D68**, 036004 (2003).

[8] P.B. Arnold, J. Lenaghan, G.D. Moore, *J. High Energy Phys.* **0308**, 002 (2003).