PHASE TRANSITIONS IN THE p-SPIN MODELS ON SCALE-FREE HYPERNETWORKS*

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Ferromagnetic and glassy phase transitions in p-spin models on scale-free hypernetworks are investigated, with Ising spins located in the nodes and p-spin exchange interactions corresponding to hyperedges. Monte Carlo simulations show that ferromagnetic transition at non-zero temperature is possible in such models which exhibits certain characteristics of the first-order phase transition. However, the ground state is, in general, degenerate and at low temperatures, depending on the network topology, the model apart from the ferromagnetic state can stay in one of few or even infinitely many disordered states. These states are degenerate with the ferromagnetic one and have structure resembling that of a spin glass. The presence of the first-order ferromagnetic transition and the degeneracy of the ground state is confirmed by analytic calculations in the mean-field approximation. The critical temperatures for the ferromagnetic transition obtained in the mean-field approach and from numerical simulations are in reasonable agreement.

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1. Introduction

In the last fifteen years theory and applications of complex networks have become an important topic in statistical physics [1, 2]. In particular, dynamical systems on complex networks, with interacting units located in the nodes and with the edges corresponding to interactions between pairs of units, have attracted considerable attention [2, 3]. For example, order–disorder phase transitions in the Ising model on scale-free (SF) networks was studied both numerically [4] and theoretically in the mean field (MF) approximation [5–9]. A possible generalization of the concept of networks

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are hypernetworks: in networks, pairs of nodes are connected by edges, while in hypernetworks, groups of more than two nodes are connected by hyperedges [10]. As in the case of networks, the topology of connections in hypernetworks may be complex: *e.g.*, SF hypernetworks are characterized by distributions of hyperdegrees (number of hyperedges k_i attached to a given edge i) obeying a power scaling law $P(k_i) \propto k_i^{-\alpha}$ [11]. By analogy with networks, complex hypernetworks offer the possibility to investigate many-body interactions (corresponding to hyperedges) among units in the nodes. As an example, ferromagnetic p-spin models without time-reversal symmetry on certain SF hypernetworks (with hyperedges corresponding to exchange p-spin interactions) were investigated by Monte Carlo (MC) simulations [12], and signatures of the ferromagnetic as well as glassy phase transitions were found. The purpose of this paper is to extend the study of Ref. [12] to p-spin models on a broader class of SF hypernetworks, and to investigate the thermodynamic properties of the above-mentioned phase transitions both numerically and theoretically, using the MF approximation.

2. The model

A general Hamiltonian for the p-spin model with N two-state spins, $\sigma_i = \pm 1, i = 1, 2, \ldots N$, located in the nodes of a hypernetwork is

$$H = -\langle k\rangle^{-1} \sum_{\{i_1,i_2,\ldots i_p\}} J_{i_1,i_2,\ldots i_p} \sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_p}\,, \tag{1}$$

where $\langle k\rangle$ is the mean hyperdegree of the nodes and the summation runs over all p-combinations (combinations of p indices without repetitions) from the set of N node indices. The exchange integrals are assumed $J_{i_1,i_2,\ldots i_p} = JN_{i_1,i_2,\ldots i_p}$, $J > 0$, where $N_{i_1,i_2,\ldots i_p}$ is the number of distinct hyperedges connecting the nodes $i_1, i_2, \ldots i_p$ (thus $J_{i_1,i_2,\ldots i_p} = 0$ if there are no such hyperedges). It is assumed that the model at temperature T obeys the Glauber thermal-bath dynamics, with the transition rates between two spin configurations which differ by a single flip of one spin, *e.g.*, that in the node i, in the form

$$w_i(\sigma_i) = \frac{1}{2}\left[1 - \sigma_i \tanh\left(\frac{I_i}{T}\right)\right], \tag{2}$$

where

$$I_i = \frac{J}{\langle k\rangle} \sum_{\{j_1,j_2,\ldots j_{p-1}\}} \sigma_{j_1}\sigma_{j_2}\ldots\sigma_{j_{p-1}} \tag{3}$$

is a local field acting on the spin i, and the sum in Eq. (3) runs over all different hyperedges attached to the node i.

Multi-spin ring interactions contribute to the magnetic properties of solid ^{3}He [13]. Models with all-to-all as well as short-range p-spin interactions with quenched disorder were used in spin-glass theory [14–17]. The ferromagnetic models with $p = 4$ (the plaquette model) were considered on two- and three-dimensional regular cubic and square lattices, and in the latter case ferromagnetic and glassy phase transitions (possibly first-order) as well as metastability were observed [18–22]. In this paper, ferromagnetic p-spin models on complex hypernetworks are investigated.

As an example of complex hypernetworks, in this paper SF hypernetworks will be considered. They can be constructed as evolving hypernetworks using an extension of the algorithm proposed in Ref. [11]. The algorithm starts with p nodes connected by $m_{\rm h}$ hyperedges, each of which connects all initial p nodes. In each step of the construction, m new nodes are added to the hypernetwork ($1 \leq m < p$) which are then connected by $m_{\rm h}$ new hyperedges to $p - m$ randomly chosen existing nodes according to the following preferential attachment rule: the probability to attach a new hyperedge to the existing node i is proportional to the hyperdegree k_i of this node, $p_i = k_i \left(\sum_j k_j\right)^{-1}$, where the summation runs over all existing nodes. The algorithm stops after a desired number N of nodes in the hypernetwork is reached. In Ref. [11] only the case with $m_{\rm h} = 1$ and $m = p - 1$ was considered, however, it is straightforward to generalize the result of Wang *et al.* and to show that the distribution of hyperdegrees in the resulting hypernetwork obeys a power scaling law, $P(k) \propto k^{-\alpha}$, with the exponent $\alpha = 1 + \frac{p}{p-m}$ which is independent of $m_{\rm h}$; in particular, $\alpha = p + 1 = m + 2$ for $m = p - 1$. In contrast with the case studied in Ref. [11], the above-mentioned extended algorithm allows multiple connections of the same nodes with different hyperedges. Finally, the spins are placed in the nodes and the hyperedges are treated as p-spin exchange interactions among the connected nodes. In Ref. [12] p-spin models with $p = 3$ and $m_{\rm h} = 1$ were investigated by means of MC simulations; here, the cases with p even and $m_{\rm h} \geq 1$ are studied both numerically and theoretically in the MF approximation.

3. Mean field approximation

3.1. General considerations

The Master equation for the probability that at time t the system is in the spin configuration $(\sigma_1, \sigma_2, \ldots \sigma_N)$ is

$$\begin{aligned}
&\frac{d}{dt}P\left(\sigma_1,\sigma_2,\ldots,\sigma_j,\ldots\sigma_N;t\right) = \\
&-\sum_{j=1}^{N} w_j\left(\sigma_j\right) P\left(\sigma_1,\sigma_2,\ldots,\sigma_j,\ldots\sigma_N;t\right) \\
&+\sum_{j=1}^{N} w_j\left(-\sigma_j\right) P\left(\sigma_1,\sigma_2,\ldots,-\sigma_j,\ldots\sigma_N;t\right) . \qquad (4)
\end{aligned}$$

Multiplying both sides of Eq. (4) by σ_i and performing an ensemble average, denoted by $\langle\ \rangle$, yields

$$\frac{d\langle\sigma_i\rangle}{dt} = -\langle\sigma_i\rangle + \left\langle \tanh\left(\frac{I_i}{T}\right)\right\rangle . \qquad (5)$$

As in the case of the Ising model on complex networks [5–9], the order parameter for the p-spin model on a hypernetwork is weighted magnetization $S = (N\langle k\rangle)^{-1}\sum_{i=1}^{N} k_i\sigma_i$. Its stationary value as well as the critical temperature for the possible order–disorder transition can be evaluated in the MF approximation. For this purpose, let us treat the spins σ_i linked by each hyperedge as independent random variables and approximate them with their average values $\langle\sigma_i\rangle$. Then, the order parameter S and the local field I_i can be approximated by their MF values

$$S \approx \langle S\rangle = (N\langle k\rangle)^{-1}\sum_{i=1}^{N} k_i\langle\sigma_i\rangle , \qquad (6)$$

$$I_i \approx \langle I_i\rangle = \frac{J}{\langle k\rangle}\sum_{\{j_1,j_2,\ldots j_{p-1}\}} \langle\sigma_{j_1}\rangle\langle\sigma_{j_2}\rangle\ldots\langle\sigma_{j_{p-1}}\rangle . \qquad (7)$$

In analogy with the case of networks, the nodes of the hypernetwork can be divided into classes according to their hyperdegrees k [5, 6]. Then, the average values of all spins located in the nodes belonging to the class with hyperdegree k are assumed equal and denoted as $\langle\sigma_k\rangle$. Under such assumptions, the sum over the nodes of the hypernetwork can be replaced by a sum over the classes of nodes with different hyperdegrees. For example, the MF value of the order parameter becomes

$$\langle S\rangle = \sum_{k=k_{\min}}^{k_{\max}} \frac{kp_k}{\langle k\rangle}\langle\sigma_k\rangle , \qquad (8)$$

where $k_{\min}$ and $k_{\max}$ are the minimum and maximum hyperdegrees of nodes, respectively. Similarly, taking into account that the probability that a node i is linked by a hyperedge to a node with hyperdegree k is $kp_k/\sum_l lp_l = kp_k/\langle k\rangle$ the MF value of the local field in Eq. (3) is

$$\begin{aligned}\langle I_i\rangle &= \frac{Jk_i}{\langle k\rangle}\sum_{k_1,k_2,\ldots,k_{p-1}=k_{\min}}^{k_{\max}} \frac{k_1p_{k_1}}{\langle k\rangle}\langle\sigma_{k_1}\rangle\frac{k_2p_{k_2}}{\langle k\rangle}\langle\sigma_{k_2}\rangle\ldots\frac{k_{p-1}p_{k_{p-1}}}{\langle k\rangle}\langle\sigma_{k_{p-1}}\rangle \\ &= \frac{Jk_i}{\langle k\rangle}\langle S\rangle^{p-1}\,. \end{aligned} \tag{9}$$

Multiplying both sides of Eq. (5) by k_i, performing the sum over all nodes and replacing it by the sum over all classes of nodes as in Eq. (8), the equation for the continuous-time dynamics of $\langle S\rangle$ is finally obtained

$$\frac{d\langle S\rangle}{dt} = -\langle S\rangle + \sum_{k=k_{\min}}^{k_{\max}}\frac{kp_k}{\langle k\rangle}\tanh\left(\frac{Jk\langle S\rangle^{p-1}}{\langle k\rangle T}\right)\,, \tag{10}$$

and the stable fixed points $\langle S\rangle_0$ of Eq. (10) correspond to the values of the order parameter in the stable (disordered or ordered) phases of the model at given temperature T.

3.2. Application to p-spin models on scale-free hypernetworks

Henceforth we will focus on SF hypernetworks constructed as described in Sec. 2, so that $k_{\min} = m_{\mathrm{h}}$. Let us assume that the hyperdegree k is a continuous rather than discrete variable, so that the distribution of node hyperdegrees is $p_k \to p(k) = Ak^{-\alpha}$, where A is a normalization constant which can be obtained from the condition $\int_{m_{\mathrm{h}}}^{k_{\max}} Ak^{-\alpha}dk = 1$. For a finite number of nodes N, the maximum hyperdegree $k_{\max}$ can be estimated from the condition $\int_{k_{\max}}^{\infty} Ak^{-\alpha}dk = N^{-1}$ [7] (then, it will be practically impossible to find a node with hyperdegree $k > k_{\max}$). These two conditions yield $k_{\max} = m_{\mathrm{h}}(N+1)^{\frac{1}{\alpha-1}}$ and $A = (\alpha-1)m_{\mathrm{h}}^{\alpha-1}\left(1+N^{-1}\right)$, thus $\langle k\rangle = \int_{m_{\mathrm{h}}}^{k_{\max}} Ak^{-\alpha+1}dk = \frac{A}{\alpha-2}\left(m_{\mathrm{h}}^{-\alpha+2} - k_{\max}^{-\alpha+2}\right)$. The equilibria $\langle S\rangle_0$ of Eq. (10) are, after replacing summation with integration, solutions of the equation

$$\langle S\rangle_0 = \frac{A}{\langle k\rangle}\int_{m_{\mathrm{h}}}^{k_{\max}} k^{-\alpha+1}\tanh\left(\frac{Jk\langle S\rangle_0^{p-1}}{\langle k\rangle T}\right)dk\,. \tag{11}$$

This result generalizes that, for networks with $p = 2$ [5, 6].

For $p \geq 2$ and high T, the only solution of Eq. (11) is $\langle S \rangle_0 = 0$, corresponding to disordered (paramagnetic) phase (Fig. 1). For $p = 2$, this solution becomes unstable, and two symmetric non-zero solutions appear at $T < T_{\mathrm{c}} = J\langle k^2 \rangle / \langle k \rangle^2$, where $\langle k^2 \rangle$ is the second moment of the distribution of the node hyperdegrees, corresponding to the ferromagnetic phase [5–9]. In contrast, for $p > 2$ the solution with $\langle S \rangle_0 = 0$ is always stable and may correspond to paramagnetic or glassy phases, depending on the temperature (Fig. 1). Besides, there is a critical temperature T_{c}, which cannot be simply expressed analytically, such that for $T < T_{\mathrm{c}}$ and $p = 3, 5, 7 \ldots$ there is one stable solution with $\langle S \rangle_0 > 0$, and for $p = 4, 6, 8 \ldots$ there are two symmetric stable solutions $\pm \langle S \rangle_0$, where $\langle S \rangle_0 > 0$, corresponding to the ferromagnetic phase (Fig. 1); the difference between the cases with p odd and even reflects the lack of time-reversal symmetry in the Hamiltonian (1) for p odd (*i.e.*, the two configurations with all spins flipped usually have different energy). It can be seen that for $p > 2$ the MF approximation predicts first-order transition to the ferromagnetic state as the temperature is decreased (Fig. 1), as in the case of p-spin models on regular three-dimensional lattices [18–22], and that the ordered phase should coexist with a disordered (glassy) one with $\langle S \rangle_0 = 0$.

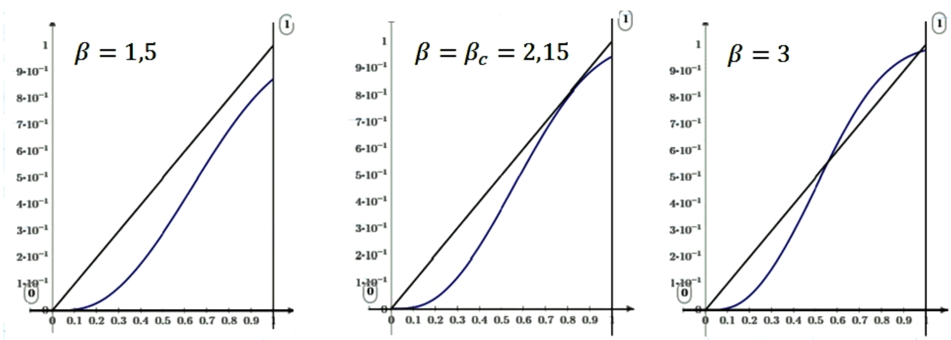

Fig. 1. Left-hand (straight line) and right-hand (curve) side of Eq. (11) as functions of $\langle S \rangle_0$ for $N = 10000$, $p = 4$, $m = 2$ ($\alpha = 3$), $m_{\mathrm{h}} = 2$ and different inverse temperatures $\beta = T^{-1}$ (see the legend).

In order to qualitatively understand the result of Eq. (11), let us note that for $p > 2$ the ground state of the system under consideration is, in general, degenerate: apart from the ferromagnetic state, there are multiple disordered (glassy) states with the same energy. In the case of $m_{\mathrm{h}} = 1$, $m = 1$, the degeneracy of the ground state is equal to the number of different spin configurations which minimize the energy of p spins coupled with a single hyperedge. This is so because the sign of the spin in a newly added node σ_{i_p} should be such that the product of spins in a newly created hyperedge is

$\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_p} = +1$, where the spins σ_{i_1}, $\sigma_2, \ldots$, $\sigma_{i_{p-1}}$ are fixed and belong to the existing hyperedge; thus, only the configuration of p spins coupled by the first hyperedge can be chosen out of several configurations which minimize their energy (in particular, this can be the configuration with all spins parallel, which leads to the ferromagnetic ground state of the system). For $m_{\mathrm{h}} = 1$ and $2 \leq m \leq p-1$, the degeneracy of the ground state is infinite in the thermodynamic limit, since there are several configurations of spins $\sigma_{i_{p-m+1}}$, $\sigma_{i_{p-m+2}}, \ldots$, σ_{i_p} in the newly added nodes for which the product of spins in a newly created hyperedge is $\sigma_{i_1}\sigma_{i_2}\ldots\sigma_{i_p} = +1$, where the spins σ_{i_1}, $\sigma_2, \ldots$, $\sigma_{i_{p-m}}$ are fixed and belong to the existing hyperedge; the ferromagnetic state is, of course, one of these ground states.

For $m_{\mathrm{h}} > 1$, the spins in the old nodes (early added and thus with higher hyperdegrees) tend to align in parallel in the ground state. Otherwise, the spins in the newly added nodes $\sigma_{i_{p-m+1}}$, $\sigma_{i_{p-m+2}}, \ldots$, σ_{i_p} could be frustrated, since they could be randomly linked with new hyperedges to groups of $p-m$ spins $\sigma^{(j)}_{i_1}$, $\sigma^{(j)}_{i_2}, \ldots$, $\sigma^{(j)}_{i_{p-m}}$, $j = 1, 2 \ldots m_{\mathrm{h}}$ (connected or not with the existing hyperedges) such that the product of spins is $\sigma^{(j)}_{i_1}\sigma^{(j)}_{i_2}\ldots\sigma^{(j)}_{i_{p-m}} = \pm 1$, depending on j, and it would be impossible to choose one configuration of the spins in the newly added nodes such that $\sigma^{(j)}_{i_1}\sigma^{(j)}_{i_2}\ldots\sigma^{(j)}_{i_{p-m}}\sigma_{i_{p-m+1}}\ldots\sigma_{i_p} = +1$ for all $j = 1, 2 \ldots m_{\mathrm{h}}$. As a result, for $m = 1$ the only ground state is the ferromagnetic one. In contrast, for $2 \leq m \leq p-1$ at the last step of the construction process of the network it is possible to add new spins $\sigma_{i_{p-m+1}}$, $\sigma_{i_{p-m+2}}, \ldots$, σ_{i_p} to the hypernetwork consisting of $N-m$ spins in the ferromagnetic state (*e.g.*, that with all spins up) in such a way that not all new spins are parallel to the $N-m$ old ones, but the product of the new spins is $\sigma_{i_{p-m+1}}\sigma_{i_{p-m+2}}\ldots\sigma_{i_p} = +1$, thus the energy of the system remains minimum. The latter argument applies to most nodes with low hyperdegree, which form a significant part of the set of nodes, since they are usually added at the end of the construction process of the hypernetwork. Thus, the degeneracy of the ground state is only partly lifted for $m_{\mathrm{h}} > 1$ and $2 \leq m \leq p-1$, and at low temperatures there are stable glassy ground states characterized by a small value of the order parameter, apart from the ferromagnetic state with $|\langle S\rangle_0| \approx 1$. It should be emphasised that in all cases the glassy ground states, though disordered, have minimum energy and do not exhibit spin frustration.

4. Monte Carlo simulations

In order to study the possible phase transitions and their thermodynamic nature in the system described by the Hamiltonian (1) using MC simulations the dependence of the order parameter S, the susceptibility χ_{S} and the fourth-order Binder cumulant B [23] are observed as functions of the temperature. The susceptibility is evaluated as

$$\chi_{\mathrm{S}} \propto \beta\overline{(\langle S^2\rangle - \langle S\rangle^2)}\,, \tag{12}$$

where $\beta = T^{-1}$, the brackets denote the time average over many MC simulation steps, and the bar denotes average over different realizations of the hypernetwork. The presence of the phase transition is characterized by the occurrence of the maximum of the curve χ_{S} *vs.* T at the critical temperature T_{c}. The Binder cumulant

$$B = \frac{1}{2}\overline{\left(3 - \frac{\langle S^4\rangle}{\langle S^2\rangle^2}\right)} \tag{13}$$

is a tool often used to characterize the order of the phase transition: for the second-order transition B decreases monotonically from $B = 1$ at $T = 0$ to $B = 0$ at $T \to \infty$ and is positive for any temperature, while for the first-order transition it exhibits a sharp negative minimum at the temperature close to the transition point. In all simulations ferromagnetic initial conditions with $\sigma_i = +1$, $i = 1, 2 \ldots N$ were assumed to prefer transition to the ordered state with decreasing temperature; however, due to high degeneracy of the ground state in most cases, the system can also settle at one of the glassy states characterized by small value of the order parameter S.

In Fig. 2 (a), (b) the dependence of S and χ_{S} on β is shown for a hypernetwork with large N, $p = 4$, $m = 1$ and different m_{h}. For $m_{\mathrm{h}} = 2, 3$ the order parameter rises fast to $S = 1$ for $\beta > \beta_{\mathrm{c}} \approx 1.5$, and there is a corresponding sharp peak of the susceptibility. This indicates transition to ferromagnetic state, which is the only ground state. Besides, for slightly smaller β there is a region of slow increase of S, with a corresponding small maximum of χ_{S}. The origin of this region, which occurs only for large enough N (Fig. 3 (a)), is unclear: it may correspond to the appearance of partial ordering or glassy transition in the system. For $m_{\mathrm{h}} = 1$, the order parameter increases slowly with β, and only the small maximum of χ_{S} can be seen. This is probably because the transition can be either to the ferromagnetic ground state or to a degenerate glassy ground state. In Fig. 2 (c), (d) the dependence of S and χ_{S} on β is shown for a hypernetwork with large N, $p = 4$, $m = 3$ and different m_{h}. In this case, the ground state is degenerate for any m_{h}, thus the order parameter does not rise to unity. Moreover, the susceptibility exhibits a peak only for $m_{\mathrm{h}} > 1$, indicating the presence of the phase transition.

In Fig. 3 (a), (b) the dependence of S and B on β is shown for hypernetworks with increasing N, $p = 4$, $m_{\mathrm{h}} = 3$ and $m = 1$. For small N, the order parameter seems to rise discontinuously from zero at the critical temperature, and the Binder cumulant has a negative minimum. However, for larger N the above-mentioned region of slow increase of S appears, and

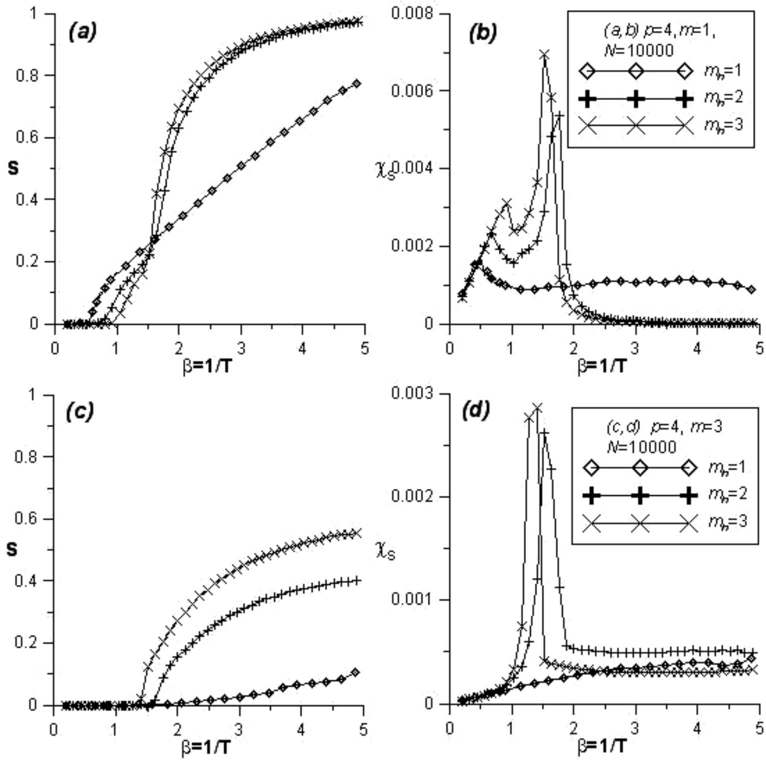

Fig. 2. The order parameter S and the susceptibility χ_S *vs.* the inverse temperature β obtained from MC simulations of the p-spin models with $p = 4, N = 10000$ on different SF hypernetworks (see the legend).

the minimum of the cumulant B seems to disappear. Moreover, the curves B *vs.* β do not cross at one (critical) value of temperature. Thus, in this case the transition shows only certain signatures of the first-order transition, as predicted by the MF theory, and its order cannot be definitively determined. The case with $m = 2$ is similar to that with $m = 1$ (not shown). In Fig. 3 (c), (d) the dependence of S and B on β is shown for hypernetworks with increasing N, $p = 4$, $m_{\rm h} = 3$ and $m = 3$. In this case, the transition is rather second-order, as suggested by the monotonic dependence of the Binder cumulant on temperature; the curves B *vs.* β for different N, again, do not cross at one point.

In Fig. 4, the critical inverse temperatures $\beta_{\rm c}$ obtained from MC simulations (from the location of the maxima of the susceptibility χ_S) are compared with those evaluated from Eq. (11) in the MF approximation. The agreement between numerical and theoretical results is only qualitative. It should be mentioned that quantitative discrepancy between the critical temperatures

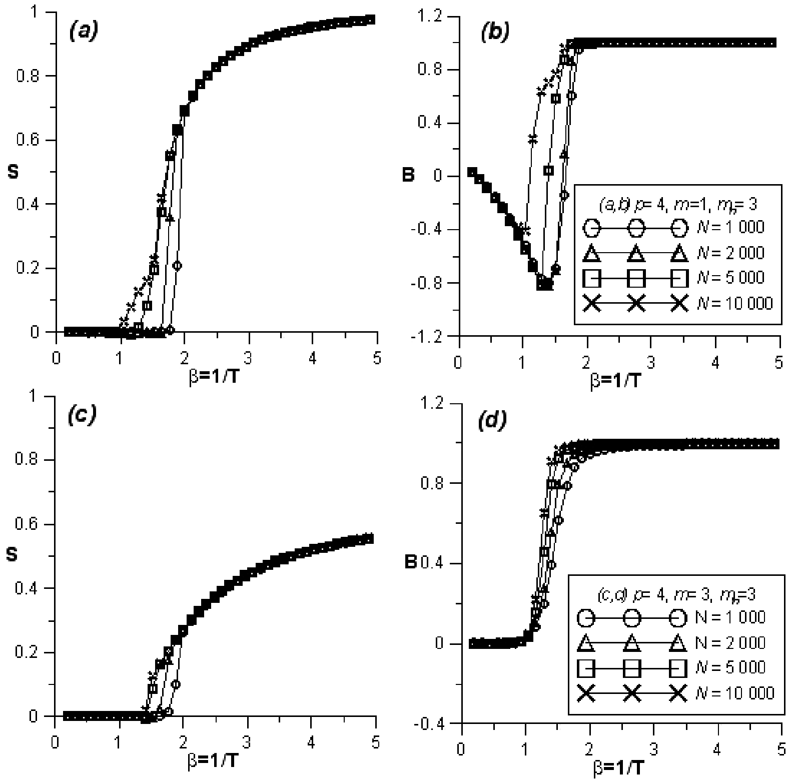

Fig. 3. The order parameter S and the Binder cumulant B *vs.* the inverse temperature β obtained from MC simulations of the p-spin models with $p = 4$, $m_{\mathrm{h}} = 3$ on SF hypernetworks with different m and N (see the legend).

for the ferromagnetic transition obtained from the MC simulations and in the MF approximation is often observed in the Ising models on SF networks [6–9]. The latter discrepancy is attributed to the correlations between node degrees in the network. In the case of p-spin models on hypernetworks, another source of quantitative discrepancy is the approximation of independent spins in Eq. (7).

Another interesting point is that Eq. (11) predicts the decrease of β_{c} (*i.e.*, increase of T_{c}) with N in models on SF hypernetworks with the exponent $2 < \alpha < 3$ in the power scaling law for the hyperdegree distribution (the cases with $m = 1$ in Fig. 4). This is, again, in analogy with the Ising model on SF networks [4–9]. Similar dependence of β_c on N is obtained in some cases from MC simulations (the case with $m = 1$, $m_{\mathrm{h}} = 3$ in Fig. 4 (b)), but is probably absent in other cases (the case with $m = 1$, $m_{\mathrm{h}} = 2$ in Fig. 4 (a)). On the other hand, from the MC simulations weak dependence of β_{c} on N

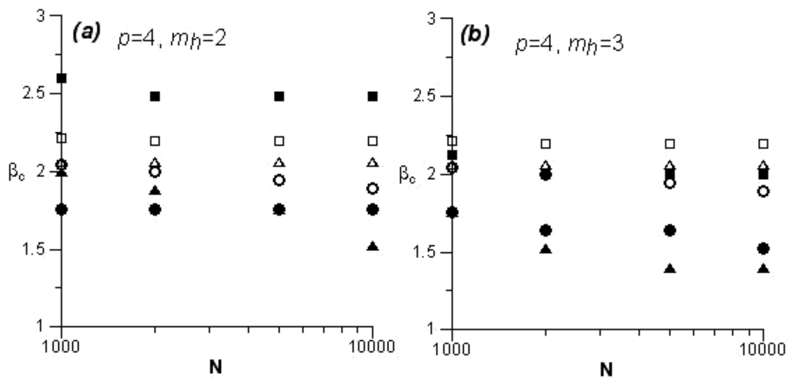

Fig. 4. The inverse critical temperature of the p-spin model β_c obtained from MC simulations (filled symbols) and MF approximation (empty symbols) *vs.* the number of nodes N in the hypernetwork for $p = 4$, $m = 1$ (circles), $m = 2$ (squares), $m = 3$ (triangles), and (a) $m_h = 2$, (b) $m_h = 3$.

can be obtained also in models on SF hypernetworks with $\alpha > 3$ (the cases with $m = 3$ in Fig. 4 (a), (b)). The dependence of T_c on the network size can be responsible for the lack of the crossing point of the curves B *vs.* β for different N in Fig. 3 (b), (d).

5. Summary and conclusions

Possible phase transitions were investigated in the p-spin models on SF hypernetworks, with two-state spins located in the nodes and with hyperedges corresponding to p-spin ferromagnetic exchange interactions. In such systems, the ground state is, in general, degenerate, and apart from the ferromagnetic state there are multiple spin-glass like states with the same minimum energy. Theoretical investigation using the MF approximation suggests that in such systems at low temperature first-order ferromagnetic phase transition is possible, as well as transition to one of the glassy ground states. MC simulations using the Glauber dynamics confirm the presence of the phase transition for most of topologies of the SF hypernetworks, which for some topologies can exhibit certain signatures of the first-order transition, *e.g.*, negative dip of the Binder cumulant, while for other ones is rather second-order. The critical temperatures evaluated in the MF approximation and obtained from MC simulations show only qualitative agreement.

Complex hypernetworks form a natural environment to study systems with complex many-body interactions. In this paper, it was shown, using the problem of phase transitions in the p-spin models on SF hypernetworks as an example, that such systems can be investigated both numerically and analytically, with methods similar to those in the theory of interacting systems on complex networks (*e.g.*, the MF approximation). They can exhibit more diverse behaviour than systems on networks (*e.g.*, degeneracy of the ground state, phase transitions with some signatures of the first-order transition). Hence, investigation of interacting systems on hypernetworks can become a promising trend in the research on complex systems, as it was in the case of interacting systems on complex networks and their generalizations.

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