NON-EQUILIBRIUM GHOSTS*

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We discuss how to introduce Faddeev–Popov ghosts to the Keldysh–Schwinger formalism describing equilibrium and non-equilibrium statistical systems of quantum fields such as the quark–gluon plasma which is considered.

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1. Introduction

In field theories obeying a gauge symmetry, the number of fields exceeds the number of physical degrees of freedom. To get rid of unphysical degrees of freedom in a manifestly Lorentz covariant way, one introduces the fictitious fields known as Faddeev–Popov ghosts which play a crucial role in non-Abelian field theories where unphysical degrees of freedom interact with physical ones. In vacuum field theory — we use the term to contrast it with the *statistical field theory* — the propagator of free ghosts has a simple form of massless scalar field [1] but in the Keldysh–Schwniger formalism [2, 3], which is applicable to equilibrium and non-equilibrium systems, the situation is more complicated. The Green's functions of the Keldysh–Schwinger formalism are of much richer structures as they carry information not only about microscopic degrees of freedom of the system but about its statistical features as well. And it is unclear how to proceed with ghosts — whether these unphysical particles are constituents of the system of gauge fields or should be merely included in scattering matrix elements.

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The complete analysis of the problem is presented in [4]. Here, only a brief summary is given. We consider a system of quarks and gluons which is, in general, out of equilibrium but the system is assumed to be translationally invariant. It is thus homogeneous (in coordinate space) but the momentum distribution is arbitrary. In particular, the system can be strongly anisotropic. The translational invariance greatly simplifies our analysis, as each two-point function depends on its two arguments only through their difference.

2. Keldysh–Schwinger formalism

Since the Yang–Mills fields are of our special interest, the Keldysh–Schwinger formalism is presented in terms of Green's functions of the gauge vector field $A^a_\mu(x)$. The main object of the approach is the contour-ordered Green's function defined as

$$i\mathcal{D}^{ab}_{\mu\nu}(x,y) \stackrel{\rm def}{=} \frac{\mathrm{Tr}\left[\rho(t_0)\,\tilde{T} A^a_\mu(x) A^b_\nu(y)\right]}{\mathrm{Tr}[\rho(t_0)]}\,, \tag{1}$$

where the trace is understood as a summation over a complete set of states of the system $\mathrm{Tr}[\ldots] = \sum_\alpha \langle\alpha|\ldots|\alpha\rangle$, $\rho(t_0)$ is a density operator at time t_0. The time arguments x_0 and y_0 are complex with an infinitesimal positive or negative imaginary part which locates them on the upper or lower branch of the contour shown in Fig. 1. The real time t_0 is smaller than the real parts of x_0 and y_0, and the real time $t_{\rm max}$ is greater than the real parts of x_0 and y_0. The times t_0 and $t_{\rm max}$ are usually shifted to $-\infty$ and $+\infty$, respectively. The time ordering operation $\tilde{T}$ is performed along the contour.

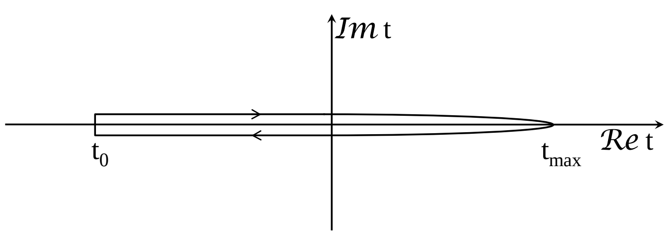

Fig. 1. The time contour of the Keldysh–Schwinger formalism.

The contour Green's function carries information about microscopic interactions in the system under consideration and its statistical properties. It involves four Green's functions with real time arguments $\mathcal{D}^c, \mathcal{D}^a, \mathcal{D}^>$, and $\mathcal{D}^<$. The function $\mathcal{D}^c$ describes a particle disturbance propagating forward in time, and an antiparticle disturbance propagating backward in time. The meaning of $\mathcal{D}^a$ is analogous but particles are propagated backward in time and antiparticles forward. The functions $\mathcal{D}^{\lessgtr}$ play a role of the phase-space densities of (quasi-)particles, so they can be treated as quantum analogs of the classical distribution functions.

The free Green's functions D can be found solving the equations of motion and in the Feynman gauge the functions read

$$\left(D^{ab}_{\mu\nu}\right)^{>}(p) = \frac{i\pi}{E_p} g_{\mu\nu}\delta^{ab}\Big[\delta(p_0 - E_p)\big(n_g(\boldsymbol{p}) + 1\big) + \delta(p_0 + E_p) n_g(-\boldsymbol{p})\Big]\,, \quad (2)$$

$$\left(D^{ab}_{\mu\nu}\right)^{<}(p) = \frac{i\pi}{E_p} g_{\mu\nu}\delta^{ab}\Big[\delta(p_0 - E_p) n_g(\boldsymbol{p}) + \delta(p_0 + E_p)\big(n_g(-\boldsymbol{p})+1\big)\Big]\,, \quad (3)$$

$$\left(D^{ab}_{\mu\nu}\right)^{\substack{c\\a}}(p) = \mp g_{\mu\nu}\delta^{ab}\Big[\frac{1}{p^2 \pm i0^+} \mp \frac{i\pi}{E_p}\Big(\delta(p_0 - E_p) n_g(\boldsymbol{p}) + \delta(p_0 + E_p) n_g(-\boldsymbol{p})\Big)\Big]\,, \quad (4)$$

where $n_g(\boldsymbol{p})$ is a distribution function of gluons which are assumed to be unpolarized with respect to spin and color degrees of freedom.

The free Green's functions of a fermion field can be derived in a similar way by solving the appropriate equations of motion, see *e.g.* [5]. One could also find the Green's functions of ghost fields solving the equations of motion but it is fairly unclear what is the distribution function of ghosts. The Slavnov–Taylor identity, which is derived in the next section, allows one to resolve the ambiguity.

3. Generating functional and Slavnov–Taylor identities

Following [3], we construct the generating functional of the Keldysh–Schwinger formalism in two steps. In the first one, we write

$$W_0\left[J,\chi,\chi^*\right] = N_0 \int\limits_{\substack{A(-\infty+i0^+,\boldsymbol{x})=A'(\boldsymbol{x})\\ A(-\infty-i0^+,\boldsymbol{x})=A''(\boldsymbol{x})}} \mathcal{D}A(x) \int\limits_{\substack{c(-\infty+i0^+,\boldsymbol{x})=c'(\boldsymbol{x})\\ c(-\infty-i0^+,\boldsymbol{x})=c''(\boldsymbol{x})}} \mathcal{D}c(x) \times \int\limits_{\substack{c^*(-\infty+i0^+,\boldsymbol{x})=c^{*\prime}(\boldsymbol{x})\\ c^*(-\infty-i0^+,\boldsymbol{x})=c^{*\prime\prime}(\boldsymbol{x})}} \mathcal{D}c^*(x) \exp\left[i\int_C d^4x\, \mathcal{L}_{\rm eff}(x)\right], \quad (5)$$

where the time integral is along the contour and the effective Lagrangian is

$$\mathcal{L}_{\rm eff} = -\frac{1}{4}F_a^{\mu\nu}F^a_{\mu\nu} + \bar{\psi}(i\gamma_\mu D^\mu - m)\psi - \frac{1}{2\alpha}\left(\partial^\mu A^a_\mu\right)^2 - c^*_a\left(\partial^\mu\partial_\mu\delta^{ab} - g\partial^\mu f^{abc}A^c_\mu\right)c_b + J^\mu_a A^a_\mu + \chi^*_a c_a + \chi_a c^*_a\,. \quad (6)$$

The first two terms constitute the fundamental QCD Lagrangian, the third one fixes the general covariant gauge and the subsequent one with c^* and c being the ghost Grassmann fields allows one to properly count the volume of a gauge orbit [1]. The remaining three terms describe interactions of the fields A, c and c^* with external sources J, χ^* and χ. The sources of ghosts are Grassmannian. The terms of interaction of quark fields with external sources are missing in Eq. (6). Since we are mostly interested in the gauge fields, the quarks are ignored all together from now on.

The generating functional of Keldysh–Schwinger formalism is obtained from the functional (5) by integrating it over the boundary fields $A'(\boldsymbol{x})$, $A''(\boldsymbol{x})$, $c'(\boldsymbol{x})$, $c''(\boldsymbol{x})$, $c^{*\prime}(\boldsymbol{x})$, $c^{*\prime\prime}(\boldsymbol{x})$ weighted with the density matrix ρ which describes the system of fields at $t=-\infty$. The matrix is not really physical because of the unphysical degrees of freedom of gauge fields and of the ghosts but our results do not depend on a form of the density matrix. The complete generating functional equals

$$W\left[J,\chi,\chi^*\right] \;=\; N\int DA'(mbx)\,DA''(\boldsymbol{x})\,Dc'(\boldsymbol{x})\,Dc''(\boldsymbol{x})\,Dc^{*\prime}(\boldsymbol{x})\,Dc^{*\prime\prime}(\boldsymbol{x}) \\ \times\rho\big[A'(\boldsymbol{x}),c'(\boldsymbol{x}),c^{*\prime}(\boldsymbol{x})\big|A''(\boldsymbol{x}),c''(\boldsymbol{x}),c^{*\prime\prime}(\boldsymbol{x})\big]\;W_0\left[J,\chi,\chi^*\right]\,. \tag{7}$$

The constant N is chosen in such a way that $W[J=0,\chi=0,\chi^*=0]=1$.

Contrary to the vacuum field theory, the generating functional of the Keldysh–Schwinger formalism cannot be expressed in a closed explicit form even for a free theory because of the unspecified density operator. Nevertheless, the functional (7) provides various relations among the Green's functions, in particular, the Slavnov–Taylor identities [6].

The general Slavnov–Taylor identity [6] results from the invariance of the generating functional with respect to the infinitesimal gauge transformations $A^a_\mu \to A^a_\mu + f^{abc}\omega^b A^c_\mu - \frac{1}{g}\partial_\mu\omega^a$, where $|\omega|\ll 1$. We assume that the gauge transformation does not work at $t=-\infty$, that is $\omega(t=-\infty,\boldsymbol{x})=0$, and consequently, the density matrix ρ remains unchanged. Requiring invariance of the generating functional with respect to the gauge transformation, we get the general Slavnov–Taylor identity

$$\Bigg\{ i\partial^\mu_{(z)}\frac{\delta}{\delta J^\mu_d(z)} - \int\limits_C d^4x\, J^\mu_a(x)\left(\partial^{(x)}_\mu\delta^{ab} + igf^{abc}\frac{\delta}{\delta J^\mu_c(x)}\right) \\ \times M^{-1}_{bd}\left[\frac{1}{i}\frac{\delta}{\delta J}\bigg|x,z\right]\Bigg\} W\left[J,\chi^*,\chi\right] = 0\,, \tag{8}$$

which holds in the Feynman gauge; M^{-1} is essentially the ghost Green's function. Differentiating the general relation (8) with respect to $J^\nu_e(y)$ and putting $\chi = \chi^* = J = 0$, we obtain

$$\partial^{\mu}_{(z)} \mathcal{D}^{ab}_{\mu\nu}(z,y) = \partial^{(y)}_{\nu} \Delta_{ab}(y,z)\,, \tag{9}$$

which relates to each other the contour Green's functions of interacting gluons and free ghosts. Locating the time arguments y_0 and z_0 on the upper or lower branch of the contour shown in Fig. 1, we get the relations for the Green's functions of real arguments. Since the system under study is translationally invariant, the Fourier transformed identity (9) gets

$$-p^{\mu} \mathcal{D}^{ab}_{\mu\nu}(p) = p_{\nu} \Delta_{ab}(-p)\,, \tag{10}$$

which relates the longitudinal part of the gluon Green's function to the free ghost function. Equation (10) also expresses the well-known fact that the longitudinal part of the gluon Green's function is not modified by interaction and consequently the polarization tensor is purely transversal.

With the explicit expressions of the gluon functions (2, 3, 4), the relation (10) provides the Green's functions of free ghosts

$$\begin{aligned}
\Delta^{>}_{ab}(p) &= -\delta^{ab}\frac{i\pi}{E_p}\Big[\delta(E_p - p_0)\big(n_g(\boldsymbol{p})+1\big) + \delta(E_p+p_0)n_g(-\boldsymbol{p})\Big] \\
\Delta^{<}_{ab}(p) &= -\delta^{ab}\frac{i\pi}{E_p}\Big[\delta(E_p - p_0)n_g(\boldsymbol{p}) + \delta(E_p+p_0)\big(n_g(-\boldsymbol{p})+1\big)\Big]\,, \\
\Delta^{\substack{c\\a}}_{ab}(p) &= \pm\delta^{ab}\Big[\frac{1}{p^2 \pm i0^+} \mp \frac{i\pi}{E_p}\Big(\delta(p_0 - E_p)n_g(\boldsymbol{p}) + \delta(p_0+E_p)n_g(-\boldsymbol{p})\Big)\Big]\,.
\end{aligned}$$

As seen, the gluon distribution function $n_g(\boldsymbol{p})$, which describes the *physical* gluons, enters the Green's functions of *unphysical* ghosts.

4. Gluon polarization tensor

As an application of the Green's functions of the free ghosts, we discuss here the retarded polarization tensor of a quark–gluon plasma. Our computation is performed within the hard loop approximation which is discussed in the context of anisotropic systems in [7]. The retarded polarization tensor is an important characteristic of a plasma system, as it carries information about its chromodynamic properties like collective excitations or screening lengths.

The polarization tensor of QCD is obtained by summing up four contributions shown in Fig. 2 where the curly, plain and doted lines denote, respectively, gluon, quark and ghost fields. After subtracting the vacuum effect, one gets

$$\Pi^{\mu\nu}_{ab}(k) = g^2\delta^{ab}\int \frac{d^3p}{(2\pi)^3}\frac{f(\boldsymbol{p})}{E_p}\frac{g^{\mu\nu}(k\cdot p)^2 - (k^{\mu}p^{\nu}+p^{\mu}k^{\nu})(k\cdot p) + k^2p^{\mu}p^{\nu}}{(k\cdot p + i0^+)^2}\,,$$

where $f(\boldsymbol{p}) \equiv n_q(\boldsymbol{p}) + \bar{n}_q(\boldsymbol{p}) + 2N_c n_g(\boldsymbol{p})$. As seen, the tensor is symmetric with respect to Lorentz indices $\Pi_{ab}^{\mu\nu}(k) = \Pi_{ab}^{\nu\mu}(k)$ and transverse $k_\mu \Pi_{ab}^{\mu\nu}(k) = 0$, as required by the gauge invariance.

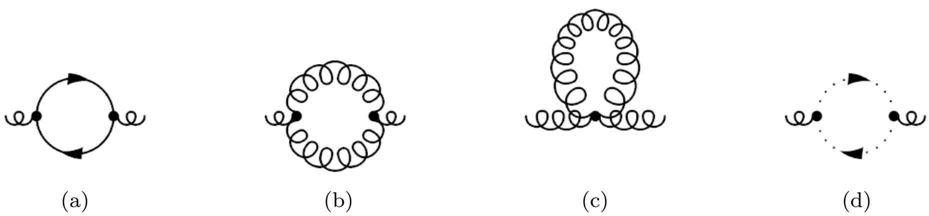

Fig. 2. The one-loop contributions to the gluon polarization tensor.

5. Conclusions

The transversality of the computed polarization tensor, which appears automatically, clearly shows that the derived Green's functions of ghosts work properly. This opens a possibility to perform other real-time calculations in the Feynman gauge which are usually much simpler than those in physical gauges like the Coulomb one.

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