AdS/CFT AND THE AXIAL SECTOR OF LARGE-N YANG–MILLS THEORY*

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In holographic models of large-N gauge theories, the pure-glue axial sector is described in terms of a massless pseudoscalar field, dual to the topological density operator $\text{Tr}F^{\mu\nu}\tilde{F}_{\mu\nu}$. I will outline how the duality can be used to compute observables such as axial glueball masses, as well as correlation functions and transport coefficients in the axial sector. I will consider 5-dimensional phenomenological holographic models for pure Yang–Mills (YM) theory, and focus on the particular set of CP-odd observables connected to the topological density operator. This provides a simple case study of how the holographic correspondence in association with other techniques can provide quantitative results and (possibly) predictions.

1. The topological charge operator in the Yang–Mills theory and its gravity dual description

In the large-N YM theory, the action including the θ-term is most conveniently written as

$$\mathcal{L}_{\text{YM}} = N\left[\frac{1}{4\lambda}\text{Tr}F_{\mu\nu}F^{\mu\nu} + \frac{\theta}{32\pi^2 N}\text{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu}\right], \qquad \tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma},$$

(1)

where $\lambda = g_{\text{YM}}^2 N$ is 't Hooft coupling, which one keeps finite as $N \to \infty$, and $\theta \in [0, 2\pi]$. The fact that the θ-term is suppressed by $1/N$ with respect to the gauge kinetic term indicates that, in the large-N limit, the contribution of topological charge on glue dynamics can be neglected [1].

In the gauge/gravity duality (see [2] for a review), a large-N gauge field theory (*boundary theory*) is mapped to a gravitational theory in a higher-dimensional curved space-time (*bulk theory*). For a theory which is

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conformal in the UV, the space-time is asymptotically AdS$_5$, with metric $ds^2 \to (\ell^2/r^2)\left(dr^2 + dx_\mu dx^\mu\right)$ as $r \to 0$. The field theory can be thought of as living on the conformal boundary of AdS, at $r = 0$. The non-compact coordinate r which parametrizes the distance from the boundary corresponds to the energy scale in the field theory.

The main ingredient for concrete calculations is the field/operator correspondence: *to each boundary theory gauge-invariant operator $O(x)$, there corresponds a bulk field $\Phi(x, r)$. The boundary value $\Phi(x, 0)$ represents a source for $O(x)$ in the field theory.* According to this prescription, in order to describe the Yang–Mills operators with lower dimension, we need two bulk spin-0 fields: a scalar field $\lambda(x, r)$ (the *dilaton*) dual to the Yang–Mills kinetic operator $\mathrm{Tr}F^2$, and representing the running Yang–Mills coupling; a pseudoscalar field $a(x, r)$ (the *axion*) dual to $\mathrm{Tr}F\tilde{F}$, whose boundary value is the UV θ-angle. Beside these fields, the dynamical 5-dimensional metric is dual to the gauge theory stress tensor.

Here, we follow a bottom-up phenomenological approach to the holographic description of pure YM theory [3]: we keep only the fields discussed above, and ignore other operators of higher dimension.

Since in the large-N limit the topological term gives a negligible contribution to the glue dynamics, we will assume a five-dimensional action of the form

$$S_{\text{bulk}} = N^2 S_{\text{bkg}}\left[g_{\mu\nu}, \lambda\right] + \int d^5 x\, \sqrt{-g} Z(\lambda)\frac{(\partial a)^2}{2}\,. \tag{2}$$

Here, S_{bkg} contains the leading-order dilaton action and Einstein–Hilbert action which determine the background geometry. We will generically consider five-dimensional gravity solutions which are confining [3] and have a deconfinement transition at finite temperature [4]. The second term in (2) encodes the dynamics of the axion, which will be treated as a probe, *i.e.* it does not affect the metric-dilaton background. $Z(\lambda)$ is a function that describes how the axion couples to the background, and has to be determined phenomenologically. Since the θ-term has an exact shift symmetry in the large-N limit, in which instanton effects are negligible, this shift symmetry should be reproduced in the bulk action for a, which therefore contains no potential term.

The Poincaré invariant vacuum will be described by the background solution

$$ds^2 = b^2(r)\left(dr^2 + \eta_{\mu\nu}dx^\mu dx^\nu\right), \qquad \lambda = \lambda(r). \tag{3}$$

A non-trivial scale factor $b(r)$ signals the breaking of conformal invariance away from the UV. Once the functions $b(r)$ and $\lambda(r)$ are determined from S_{bkg}, the axion linear field equations are derived from (2)

$$\partial_r\left(A(r)\partial_r a(r, x)\right) + A(r)\partial^\mu \partial_\mu a(r, x) = 0, \qquad A(r) \equiv b^3(r)Z(\lambda(r)). \tag{4}$$

2. Axial sector observables

There are several gauge theory observables connected to the CP-odd operator $\mathrm{Tr}F\tilde{F}$, that one can map to the gravity dual picture. Below, we discuss some of these observables and review how they are calculated using the bulk axion.

Vacuum topological susceptibility

In the large-N Yang–Mills, the vacuum energy is composed by a leading $O(N^2)$ θ-independent term, and by a subleading θ-dependent contribution [1]

$$\mathcal{E}(\lambda, \theta) \approx N^2 E_0(\lambda) - \frac{\chi_{\mathrm{top}}}{2}\theta^2 \,, \tag{5}$$

where χ_{top} is a constant called vacuum *topological susceptibility*. In the gravity dual, the vacuum energy is computed by the action (2) evaluated on the solution of the field equations. The relevant θ-dependent contribution comes from the axion field, for which the homogeneous solution of (4) reads

$$a(r) = a_0 + a_1 \int_0^r \frac{dr'}{A(r')} \,. \tag{6}$$

As discussed in the previous section, the boundary value a_0 is identified with the source of $\mathrm{Tr}F\tilde{F}$, *i.e.* $a_0 = \theta$, up to an overall coefficient which we choose to set to unity. The coefficient a_1 is fixed by a regularity condition at the infrared end of space, $r \to r_{\mathrm{IR}}$, where we require $a(r) \to 0$. Evaluating the axion action in (2) on this solution, we find a θ-dependence as in (5) with

$$\chi_{\mathrm{top}} = \left(\int_0^{r_{\mathrm{IR}}} \frac{dr'}{A(r')} \right)^{-1} \,. \tag{7}$$

Axial glueball spectrum

The spectrum of physical gauge-invariant states in the field theory is mapped in the gravity dual to the spectrum of normalizable modes around a background solution. In the large-N limit, we have two decoupled towers of spin-0 excitations corresponding to dimension-four operators: 0^{++} glueballs, which are created by the operator $\mathrm{Tr}F^2$, and 0^{-+} glueballs created by $\mathrm{Tr}F\tilde{F}$. According to the AdS/CFT prescription, the latter are in one-to-one correspondence with the tower of normalizable eigenmodes of the axion

radial equation, obtained from equation (4) when one imposes a harmonic space-time dependence

$$-a''(r) - \frac{A'(r)}{A(r)}a'(r) = m_n^2 a(r)\,, \qquad a(x,r) = a(r)e^{ik_\mu x^\mu}\,, \qquad k^\mu k_\mu = -m_n^2\,.$$
(8)

In holographic duals of confining gauge theories, the above equation gives rise to a discrete and gaped spectrum for a very general choice of $Z(\lambda)$.

Thermal topological susceptibility

At finite temperature, gravity duals of confining gauge theories display a first order deconfinement transition to a black hole geometry, of the form

$$ds^2 = b^2(r)\left[\frac{dr^2}{f(r)} - f(r)dt^2 + dx^i dx_i\right]\,, \qquad f(r_{\rm h}) = 0\,,$$
(9)

where $r_{\rm h}$ is the black hole horizon. Above the critical temperature $T_{\rm c}$, the axion field equation is modified by the presence of the black hole. In the homogeneous case, we have

$$\partial_r\left(A(r)f(r)\partial_r a(r)\right) = 0 \quad\Rightarrow\quad a(r) = a_0 + a_1\int_0^r \frac{dr'}{f(r')A(r')}\,.$$
(10)

In this case, the only solutions which are regular at the black hole horizon $r_{\rm h}$ have $a_1 = 0$. The bulk axion action (the second term in (2)) evaluated on these solutions vanishes, leading to the result that the topological susceptibility is identically zero in the deconfined phase [4]. This is in agreement with lattice results [5].

Chern–Simons diffusion coefficient

In the deconfined phase, the relaxation to equilibrium of the topological charge density ${\rm Tr}F\tilde{F}$ is governed by a transport coefficient, the *Chern–Simons diffusion constant* $\Gamma_{\rm CS}$. This can be defined from the retarded two-point function of ${\rm Tr}F\tilde{F}$ as

$$\Gamma_{\rm CS}(T) = \lim_{\omega\to 0}\frac{2T}{\omega}\,{\rm Im}\,\langle q(\omega)q(-\omega)\rangle_{\rm ret}\,, \qquad q(\omega) = \frac{1}{32\pi^2}\int d^3x dt\, e^{i\omega t}{\rm Tr}F\tilde{F}\,,$$
(11)

and it plays an important role in the chiral magnetic effect [6]. It can be computed holographically by considering axion fluctuations in the deconfined phase, obeying infalling boundary conditions at the black hole horizon.

The result is [7]

$$\Gamma_{\text{CS}} = \frac{sT}{N^2} \frac{Z(\lambda(r_{\text{h}}))}{2\pi} \,, \tag{12}$$

where s is the entropy density, r_{h} is the BH horizon position, and $Z(\lambda)$ is the axion wavefunction normalization appearing in (2).

Like similar real-time quantities, Γ_{CS} is very hard to obtain using lattice techniques.

3. Explicit model and comparison with data

To explicitly compute the observables in the holographic theory, we first need to specify a metric-dilaton background, and to chose a function $Z(\lambda)$.

We take as a background the model discussed in [8], which is confining, has realistic 0^{++} and 2^{++} glueball towers with linear Regge behaviour, mimics UV asymptotic freedom for the 't Hooft coupling, and at finite T reproduces the equation of state of pure Yang–Mills theory calculated on the lattice.

We choose the function $Z(\lambda)$ that governs the axion dynamics according to the following criteria: (1) it should give a finite vacuum topological susceptibility. This constrains the behaviour of $Z(\lambda)$ as $\lambda \to 0$; (2) the 0^{-+} glueball tower should have the same asymptotic linear slope as the 0^{++} and 2^{++} towers (glueball universality). This requires $Z(\lambda) \sim \lambda^4$ as $\lambda \to \infty$. A simple parametrization that satisfies both criteria is

$$Z(\lambda) = Z_0 \left(1 + c_1\lambda + c_4\lambda^4\right) \,. \tag{13}$$

Here, Z_0, c_1 and c_4 are phenomenological parameters that can be fixed by matching with lattice calculations some of the observables discussed in the previous section.

For any fixed value of (c_1, c_4), Z_0 can be fixed using (7) to match the lattice value of the topological susceptibility, $\chi_{\text{lat}} = (191 \text{ MeV})^4$ [5, 9].

The known lattice values [10] of the two lowest 0^{-+} glueball masses (in units of the 0^{++} ground state) fix a combination of c_1 and c_4. Taking $c_1 = 0$, $c_4 = 0.26$ gives a good fit of these values, as shown in the table below [8].

	5d model	Lattice hep-lat/9901004
$m_{0^{-+}}/m_{0^{++}}$	1.50	1.50(4)
$m_{0^{*-+}}/m_{0^{++}}$	2.10	2.11(6)

These observables do not lift the degeneracy between parameters: as shown in [7], there is a one-dimensional curve in the space (c_1, c_4) that fits equally well the first two glueball masses. Also, higher glueball states are essentially insensitive to the point chosen on this curve.

On the other hand, the specific values of c_1, c_4 do matter for the Chern–Simons diffusion coefficient Γ_{CS} as shown in Fig. 1, taken from [7]. According to the holographic model, the magnitude of Γ_{CS} can drastically increase close to the transition temperature if the values of c_1, c_4 are large enough, and this could possibly lead to an observable chiral magnetic effect. A quantity which, unlike Γ_{CS}, can be computed on the lattice *and* is sensitive to the values (c_1, c_4) is the full Euclidean two-point correlation function of $\mathrm{Tr} F \tilde{F}$. Comparing a direct calculation of this object on the lattice to the analog result obtained using the holographic model, would potentially allow to fix $Z(\lambda)$ and to predict Γ_{CS}.
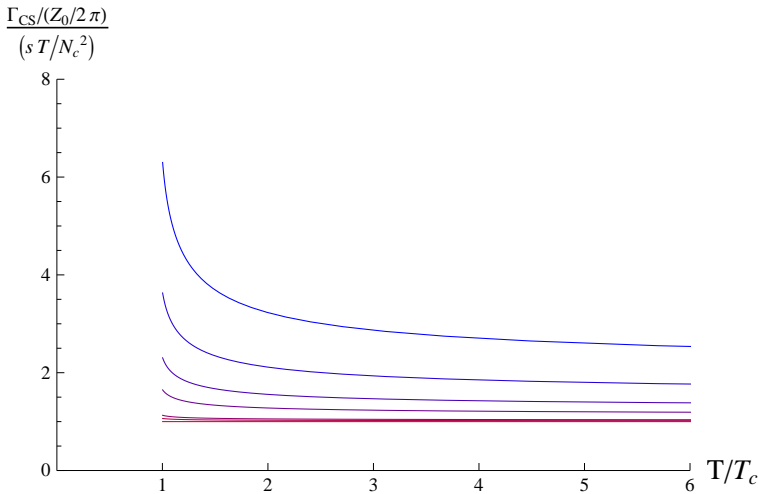

Fig. 1. Holographic determination of $\Gamma_{\mathrm{CS}}/(sT/N_c^2)$ as a function of T/T_c, normalized to the $T \to \infty$ value $Z_0/2\pi$, for different choices of the parameters (c_1, c_4) in (13). From the bottom (red) curve to the top (blue) curve, $(c_1, c_4) = (0, 0.26), (0.5, 0.87), (1, 2.2), (5, 24), (10, 75), (20, 230), (40, 600)$.

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