ELEMENTS OF A NON-HERMITIAN QUANTUM THEORY WITHOUT HERMITIAN CONJUGATION — SCALAR PRODUCT AND SCATTERING*

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The description of — in a Hermitian setting — seemingly nonlocal and nonperturbative phenomena such as confinement or superconductivity is most conveniently performed by generalizing quantum theory to a non-Hermitian regime where these phenomena appear perturbative and local. The short presentation provides a clue how this can be done on the basis of Lorentz covariance while preserving the analyticity of the theory. After deriving with the help of Lorentz covariance a quantum scalar product without making any use of metric or complex conjugation, we sketch how the formalism of scattering theory can be extended analytically to a non-Hermitian regime.

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1. Introductory remarks

To us, there are mainly three points becoming gradually clear after about 20 years of intensive reseach effort to generalize quantum theory (QT) to a non-Hermitian (NH) setting (see *e.g.* Refs. [1–14] and references therein): (1) The description of physical systems within an idealized Hermitian setting is at odds with experimental reality; (2) Various strong statements[1] made

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1 *E.g.* that confinement cannot be generated by scalar bosons or that the quartic coupling of a Higgs scalar has — due to stability reasons — to be positive [4].

naively within a Hermitian setting do not hold in an NH setting; (3) The advanced sector of NHQT required by Lorentz covariance [2] and analyticity is in a Hermitian setting obtained by applying to the retarded sector a Hermitian conjugation joint with a non-local, non-analytic metric [15].

2. Setup of non-Hermitian Quantum Theory (NHQT)

2.1. Covariance and conservation of complex energy in the complex plane

In the first place, one should recall that the relativistic Klein–Gordon equation being essentially the wave equation is a differential equation of second order in the time coordinate which can be decomposed [2, 3, 6] into two first order equations. For a — without loss of generality — time independent eventually NH Hamilton operator H, we have

$$0 = \left((i\,\hbar)^2 \,\frac{\partial^2}{\partial t^2} - H^2 \right) |\psi(t)\rangle = \left(i\,\hbar\,\frac{\partial}{\partial t} - H \right) \left(i\,\hbar\,\frac{\partial}{\partial t} + H \right) |\psi(t)\rangle \;. \qquad (1)$$

The right solution of the Klein–Gordon equation $|\psi(t)\rangle = |\psi^{\,(+)}(t)\rangle + |\psi^{\,(-)}(t)\rangle$ is therefore obtained by superimposing additively the solutions $|\psi^{(+)}(t)\rangle$ and $|\psi^{(-)}(t)\rangle$ of the retarded and advanced Schrödinger equation, respectively

$$0 = \left(i\,\hbar\,\frac{\partial}{\partial t} - H \right) \left|\psi^{(+)}(t)\right\rangle , \qquad 0 = \left(i\,\hbar\,\frac{\partial}{\partial t} + H \right) \left|\psi^{(-)}(t)\right\rangle . \qquad (2)$$

The respective left eigen-solution $\langle\!\langle\psi^{(+)}(t)|$ and $\langle\!\langle\psi^{(-)}(t)|$ of these two equations is the right eigen-solution $|\psi^{(+)}(t))\equiv\langle\!\langle\psi^{(+)}(t)|^{\mathrm{T}}$ and $|\psi^{(-)}(t))\equiv\langle\!\langle\psi^{(-)}(t)|^{\mathrm{T}}$ of the respective so-called “transposed retarded” and “transposed advanced” Schrödinger equation (here ‘T’ denotes transpositon!), *i.e.*

$$0 = \left(i\,\hbar\,\frac{\partial}{\partial t} - H^{\mathrm{T}} \right) \left|\psi^{(+)}(t)\right) , \qquad 0 = \left(i\,\hbar\,\frac{\partial}{\partial t} + H^{\mathrm{T}} \right) \left|\psi^{(-)}(t)\right) . \qquad (3)$$

In using the notation $\psi^{(\pm)}_{\mathrm{R}}(z,t) \equiv \langle\!\langle z|\psi^{(\pm)}(t)\rangle \equiv (\!(\psi^{(\pm)}(t)|z)^{\mathrm{T}}$ and $\psi^{(\pm)}_{\mathrm{L}}(z,t) \equiv (\!(z|\psi^{(\pm)}(t)) \equiv \langle\!\langle\psi^{(\pm)}(t)|z\rangle^{\mathrm{T}}$, the non-relativistic one-dimensional limit of Eqs. (2) and (3) reads in spatial representation

$$\pm\, i\,\hbar\,\frac{\partial}{\partial t}\,\psi^{(\pm)}_{\mathrm{R}}(z,t) \;=\; \left(-\frac{\hbar^2}{2\,M}\frac{\partial^2}{\partial z^2} + V(z) \right) \psi^{(\pm)}_{\mathrm{R}}(z,t)\,, \qquad (4)$$

$$\pm\, i\,\hbar\,\frac{\partial}{\partial t}\,\psi^{(\pm)}_{\mathrm{L}}(z,t) \;=\; \left(-\frac{\hbar^2}{2\,M}\frac{\partial^2}{\partial z^2} + V(z)^{\mathrm{T}} \right) \psi^{(\pm)}_{\mathrm{L}}(z,t)\,. \qquad (5)$$

On the basis of these equations, it is easy to show that there hold the following two continuity equations

$$\frac{\partial\,\rho^{(+)}(z,t)}{\partial t} = -\frac{\partial j^{(+)}(z,t)}{\partial z}\,, \qquad \frac{\partial\,\rho^{(-)}(z,t)}{\partial t} = -\frac{\partial j^{(-)}(z,t)}{\partial z} \tag{6}$$

for the retarded and advanced energy densities $\rho^{(+)}(z,t)$ and $\rho^{(-)}(z,t)$ and the respective energy current densities $j^{(\pm)}(z,t)$ defined as follows:

$$\rho^{(\pm)}(z,t) = \psi_{\mathrm{L}}^{(\mp)}(z,t)^{\mathrm{T}} \cdot \psi_{\mathrm{R}}^{(\pm)}(z,t)\,, \tag{7}$$

$$j^{(\pm)}(z,t) = \frac{1}{\pm i\hbar} \times \left(\psi_{\mathrm{L}}^{(\mp)}(z,t)^{\mathrm{T}} \cdot \frac{\hbar^2}{2\,M}\frac{\partial\psi_{\mathrm{R}}^{(\pm)}(z,t)}{\partial z} - \frac{\partial\psi_{\mathrm{L}}^{(\mp)}(z,t)^{\mathrm{T}}}{\partial z} \cdot \frac{\hbar^2}{2\,M}\,\psi_{\mathrm{R}}^{(\pm)}(z,t)\right). \tag{8}$$

The continuity equations (6) can be integrated along some suitable contour connecting two points z_1 and z_2 in the complex z-plane yielding

$$\frac{\partial}{\partial t}\int_{z_1}^{z_2} dz\rho^{(\pm)}(z,t) = -\left(j^{(\pm)}(z_2,t) - j^{(\pm)}(z_1,t)\right)\,. \tag{9}$$

Any integration contour with $j^{(\pm)}(z_2,t) = j^{(\pm)}(z_1,t)$ defines an eventually NHQT with a time-independent scalar product [1, 6]

$$\int_{z_1}^{z_2} dz\rho^{(\pm)}(z,t) = \int_{z_1}^{z_2} dz\psi_{\mathrm{L}}^{(\mp)}(z,t)^{\mathrm{T}} \cdot \psi_{\mathrm{R}}^{(\pm)}(z,t) = \mathrm{const} \tag{10}$$

replacing the well known scalar product of Max Born.

2.2. Elements of non-Hermitian scattering theory

Without loss of generality, we consider now one-dimensional scattering at a time-independent eventually NH potential $V(z)$. For such a potential, the Schrödinger equations (4) and (5) can be solved by a separation ansatz $\psi_{\mathrm{R}}^{(\pm)}(z,t) = \exp(\pm Et/(i\hbar))\,\phi_{\mathrm{R}}^{(\pm)}(z)$ and $\psi_{\mathrm{L}}^{(\pm)}(z,t) = \exp(\pm Et/(i\hbar))\,\phi_{\mathrm{L}}^{(\pm)}(z)$ yielding the time-independent Schrödinger equations

$$E\,\phi_{\mathrm{R}}^{(\pm)}(z) \;=\; \left(-\frac{\hbar^2}{2\,M}\frac{\partial^2}{\partial z^2} + V(z)\right)\phi_{\mathrm{R}}^{(\pm)}(z)\,, \tag{11}$$

$$E\,\phi_{\mathrm{L}}^{(\pm)}(z) \;=\; \left(-\frac{\hbar^2}{2\,M}\frac{\partial^2}{\partial z^2} + V(z)^{\mathrm{T}}\right)\phi_{\mathrm{L}}^{(\pm)}(z)\,, \tag{12}$$

and according to Eqs. (8), the time-independent energy current densities

$$j^{(\pm)}(z) = \frac{1}{\pm\, i\hbar}\left(\phi_{\mathrm{L}}^{(\mp)}(z)^{\mathrm{T}}\cdot\frac{\hbar^2}{2\,M}\frac{\partial\phi_{\mathrm{R}}^{(\pm)}(z)}{\partial z} - \frac{\partial\phi_{\mathrm{L}}^{(\mp)}(z)^{\mathrm{T}}}{\partial z}\cdot\frac{\hbar^2}{2\,M}\,\phi_{\mathrm{R}}^{(\pm)}(z)\right). \tag{13}$$

Here, we will discuss merely retarded scattering. Hence, we use in the following the abbreviations $j(z)\equiv j^{(+)}(z)$, $\phi^{(+)}(z)\equiv\phi_{\mathrm{R}}^{(+)}(z)$, $\phi^{(-)}(z)\equiv\phi_{\mathrm{L}}^{(-)}(z)$, $\phi^{(+)}(z)'\equiv\partial\phi_{\mathrm{R}}^{(+)}(z)/\partial z$ and $\phi^{(-)}(z)'\equiv\partial\phi_{\mathrm{L}}^{(-)}(z)/\partial z$. Advanced scattering results are nonetheless easily derivable from their retarded counterparts. In the region of vanishing potential, *i.e.* $V(z)=0$, the solution of the Schrödinger Eqs. (11) and (12) is of plane-wave type with $k_0\equiv\sqrt{2ME/\hbar^2}$

$$\phi^{(\pm)}(z) = \exp(\pm i\,k_0 z)\;c^{(\pm)}(k_0) + \exp(\mp i\,k_0 z)\;c^{(\pm)}(-k_0)\,, \tag{14}$$

$$\begin{aligned}\Rightarrow\; j(z) &= \frac{1}{i\hbar}\left(\phi^{(-)}(z)^{\mathrm{T}}\cdot\frac{\hbar^2}{2\,M}\phi^{(+)}(z)' - \phi^{(-)}(z)'^{\,\mathrm{T}}\cdot\frac{\hbar^2}{2\,M}\,\phi^{(+)}(z)\right)\\ &= c^{(-)}(k_0)^{\mathrm{T}}\cdot\frac{\hbar k_0}{M}\,c^{(+)}(k_0) - c^{(-)}(-k_0)^{\mathrm{T}}\cdot\frac{\hbar k_0}{M}\,c^{(+)}(-k_0)\,.\end{aligned} \tag{15}$$

In the following, we want to consider retarded scattering between two points $z_<$ and $z_>$ of vanishing potential, *i.e.* $V(z_<)=V(z_>)=0$. Wave functions and their derivatives at $z_>$ and $z_<$ are related by transfer matrices $T^{(\pm)}$

$$\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\phi^{(\pm)}\,(z_>)\\ \sqrt{\frac{\hbar^2}{2M}}\phi^{(\pm)}\,(z_>)'\end{pmatrix} = \begin{pmatrix}T_{11}^{(\pm)} & T_{12}^{(\pm)}\\ T_{21}^{(\pm)} & T_{22}^{(\pm)}\end{pmatrix}\begin{pmatrix}\sqrt{\frac{\hbar^2}{2M}}\,\phi^{(\pm)}\,(z_<)\\ \sqrt{\frac{\hbar^2}{2M}}\,\phi^{(\pm)}\,(z_<)'\end{pmatrix}, \tag{16}$$

or, alternatively,

$$\begin{pmatrix}a_1^{(\pm)}\\ e_2^{(\pm)}\end{pmatrix} = \tilde{T}^{(\pm)}\begin{pmatrix}e_1^{(\pm)}\\ a_2^{(\pm)}\end{pmatrix} = \begin{pmatrix}\tilde{T}_{11}^{(\pm)} & \tilde{T}_{12}^{(\pm)}\\ \tilde{T}_{21}^{(\pm)} & \tilde{T}_{22}^{(\pm)}\end{pmatrix}\begin{pmatrix}e_1^{(\pm)}\\ a_2^{(\pm)}\end{pmatrix}, \tag{17}$$

with

$$e_1^{(\pm)} \equiv e^{\pm ik_0 z_<}\sqrt{\frac{\hbar k_0}{M}}\;c_<^{(\pm)}(k_0)\,,\qquad e_2^{(\pm)}\equiv e^{\mp ik_0 z_>}\sqrt{\frac{\hbar k_0}{M}}\;c_>^{(\pm)}(-k_0)\,, \tag{18}$$

$$a_1^{(\pm)} \equiv e^{\pm ik_0 z_>}\sqrt{\frac{\hbar k_0}{M}}\;c_>^{(\pm)}(k_0)\,,\qquad a_2^{(\pm)}\equiv e^{\mp ik_0 z_<}\sqrt{\frac{\hbar k_0}{M}}\;c_<^{(\pm)}(-k_0)\,. \tag{19}$$

Simple algebra establishes the following relation between $\tilde{T}^{(\pm)}$ and $T^{(\pm)}$

$$\tilde{T}^{(\pm)} = 1_2 + \frac{\sqrt{k_0}}{2}\begin{pmatrix}1 & \pm\frac{1}{ik_0}\\ 1 & \mp\frac{1}{ik_0}\end{pmatrix}\left(T^{(\pm)} - 1_2\right)\begin{pmatrix}1 & 1\\ \pm ik_0 & \mp ik_0\end{pmatrix}\frac{1}{\sqrt{k_0}}\,. \tag{20}$$

1_2 is the 2×2 unit matrix. Moreover, we assume the energy current densities at points $z_<$ and $z_>$ to be equal, *i.e.* $j(z_<) = j(z_>)$, yielding (see Eq. (15))

$$\begin{aligned} & c_>^{(-)}(k_0)^{\mathrm{T}} \cdot \frac{\hbar k_0}{M}\, c_>^{(+)}(k_0) - c_>^{(-)}(-k_0)^{\mathrm{T}} \cdot \frac{\hbar k_0}{M}\, c_>^{(+)}(-k_0) \\ & = c_<^{(-)}(k_0)^{\mathrm{T}} \cdot \frac{\hbar k_0}{M}\, c_<^{(+)}(k_0) - c_<^{(-)}(-k_0)^{\mathrm{T}} \cdot \frac{\hbar k_0}{M}\, c_<^{(+)}(-k_0)\,, \end{aligned} \tag{21}$$

or, equivalently, $a_1^{(-)\,\mathrm{T}} \cdot a_1^{(+)} - e_2^{(-)\,\mathrm{T}} \cdot e_2^{(+)} = e_1^{(-)\,\mathrm{T}} \cdot e_1^{(+)} - a_2^{(-)\,\mathrm{T}} \cdot a_2^{(+)}$. Inspecting $e_1^{(-)\,\mathrm{T}} \cdot e_1^{(+)} + e_2^{(-)\,\mathrm{T}} \cdot e_2^{(+)} = a_1^{(-)\,\mathrm{T}} \cdot a_1^{(+)} + a_2^{(-)\,\mathrm{T}} \cdot a_2^{(+)}$, we can define the S-matrix $S^{(+)}$ and transpose of its inverse $S^{(-)} = (S^{(+)-1})^{\mathrm{T}}$ by

$$\begin{pmatrix} a_1^{(\pm)} \\ a_2^{(\pm)} \end{pmatrix} = S^{(\pm)} \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix} = \begin{pmatrix} S_{11}^{(\pm)} & S_{12}^{(\pm)} \\ S_{21}^{(\pm)} & S_{22}^{(\pm)} \end{pmatrix} \begin{pmatrix} e_1^{(\pm)} \\ e_2^{(\pm)} \end{pmatrix}. \tag{22}$$

Making use of Eq. (17) and $S^{(-)\mathrm{T}} S^{(+)} = 1_2$, we obtain

$$S^{(\pm)} = \begin{pmatrix} S_{11}^{(\pm)} & S_{12}^{(\pm)} \\ S_{21}^{(\pm)} & S_{22}^{(\pm)} \end{pmatrix} = \begin{pmatrix} \left(\tilde{T}_{11}^{(\mp)\mathrm{T}}\right)^{-1} & \tilde{T}_{12}^{(\pm)} \tilde{T}_{22}^{(\pm)-1} \\ -\tilde{T}_{22}^{(\pm)-1} \tilde{T}_{21}^{(\pm)} & \tilde{T}_{22}^{(\pm)-1} \end{pmatrix}, \tag{23}$$

$$S^{(\mp)\mathrm{T}} = \begin{pmatrix} S_{11}^{(\mp)\mathrm{T}} & S_{21}^{(\mp)\mathrm{T}} \\ S_{12}^{(\mp)\mathrm{T}} & S_{22}^{(\mp)\mathrm{T}} \end{pmatrix} = \begin{pmatrix} \tilde{T}_{11}^{(\pm)-1} & -\tilde{T}_{11}^{(\pm)-1} \tilde{T}_{12}^{(\pm)} \\ \tilde{T}_{21}^{(\pm)} \tilde{T}_{11}^{(\pm)-1} & \left(\tilde{T}_{22}^{(\mp)\mathrm{T}}\right)^{-1} \end{pmatrix}. \tag{24}$$

The transmittivities T_1 and T_2 and reflectivities R_1 and R_2 are therefore

$$T_1 = 1 - R_1 = S_{11}^{(-)\mathrm{T}} S_{11}^{(+)} = \tilde{T}_{11}^{(+)-1} \left(\tilde{T}_{11}^{(-)\mathrm{T}}\right)^{-1} = \left(\tilde{T}_{11}^{(-)\mathrm{T}} \tilde{T}_{11}^{(+)}\right)^{-1}, \tag{25}$$

$$T_2 = 1 - R_2 = S_{22}^{(-)\mathrm{T}} S_{22}^{(+)} = \left(\tilde{T}_{22}^{(-)\mathrm{T}}\right)^{-1} \tilde{T}_{22}^{(+)-1} = \left(\tilde{T}_{22}^{(+)} \tilde{T}_{22}^{(-)\mathrm{T}}\right)^{-1}. \tag{26}$$

3. Simple application: scattering at a delta-potential

For the scattering at a delta-potential $V(z) = g\, \delta(z-a)$ with g being eventually complex-valued, we choose $x_> = a + 0$ and $x_< = a - 0$. The delta-potential is represented by the following transfer matrices

$$T^{(+)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\, g\, \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}, \qquad T^{(-)} = \begin{pmatrix} 1 & 0 \\ \sqrt{\frac{2M}{\hbar^2}}\, g^T\, \sqrt{\frac{2M}{\hbar^2}} & 1 \end{pmatrix}. \tag{27}$$

Invoking these transfer matrices into Eq. (20), we obtain

$$\tilde{T}^{(+)}_{11} = \tilde{T}^{(-)\mathrm{T}}_{22} = \left(S^{(-)\mathrm{T}}_{11}\right)^{-1} = \left(S^{(-)\mathrm{T}}_{22}\right)^{-1} = 1 + \frac{1}{2i}\sqrt{\frac{2M}{\hbar^2 k_0}}\, g\, \sqrt{\frac{2M}{\hbar^2 k_0}}\,, \quad (28)$$

$$\tilde{T}^{(-)\mathrm{T}}_{11} = \tilde{T}^{(+)}_{22} = \left(S^{(+)}_{11}\right)^{-1} = \left(S^{(+)}_{22}\right)^{-1} = 1 - \frac{1}{2i}\sqrt{\frac{2M}{\hbar^2 k_0}}\, g\, \sqrt{\frac{2M}{\hbar^2 k_0}}\,, \quad (29)$$

and, consequently,

$$\begin{aligned} T_1 &= T_2 = 1 - R_1 = 1 - R_2 \\ &= \left[\left(1 - \frac{1}{2i}\sqrt{\frac{2M}{\hbar^2 k_0}}\, g\, \sqrt{\frac{2M}{\hbar^2 k_0}}\right)\left(1 + \frac{1}{2i}\sqrt{\frac{2M}{\hbar^2 k_0}}\, g\, \sqrt{\frac{2M}{\hbar^2 k_0}}\right)\right]^{-1} . \quad (30)\end{aligned}$$

This is — without involving any complex conjugation — the standard result which will be for one scattering channel obviously real-valued, positive and within the invervall $[0, 1]$, if $(Mg/k_0)^2$ is real-valued and non-negative.

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