

PSEUDOSPIN AND SPIN SYMMETRIES IN
THE DIRAC EQUATION FOR CONFINING
POTENTIALS WITH THE APPLICATION TO
THE COULOMB POTENTIAL IN 1+1 DIMENSIONS*

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In this paper, we revise the main features of pseudospin and spin symmetries of the Dirac equation with scalar and vector potentials and mention several of its applications to strong interacting physical systems. We present some recent results in which these symmetries are applied to Coulomb potentials in the Dirac equation in 1+1 dimensions, including also pseudoscalar potentials. These potentials are linear in $|x|$ and may be applied in confining quark models. We explore all possible bound solutions, both for fermions and antifermions, and show the relation between spin and pseudospin symmetries by means of charge-conjugation and γ^5 chiral transformations.

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1. Introduction

Pseudospin symmetry has been a topic in nuclear physics since the late 60s, when it was introduced to explain the near degeneracy of some single-

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particle levels near the Fermi surface. The subject was revived in 1997 when Ginocchio was able to relate it with a symmetry of the Dirac equation with scalar S and vector V mean-field potentials such that $V = -S + C$, where C is a constant. However, this symmetry cannot be realized exactly in nuclei since the potential $V + S$ provides the binding of nucleons in nuclei, but it can be realized for confining potentials as harmonic oscillator or linear potentials [1–3]. A related symmetry, the spin symmetry, was used to explain the suppression of spin–orbit splittings in states of mesons with a heavy and a light quark. For a review of these symmetries and their applications, see Ref. [4]. In this paper, we will review briefly the origin of spin and pseudospin symmetries in the Dirac equation, its generators, both for general potentials and radial potentials, and their quantum numbers. Finally, we report about the main conclusions of a recent work, in which these symmetries are applied to Coulomb potentials in the 1+1 Dirac equation, including scalar and vector potentials as well as pseudoscalar potentials. These are linear confining potentials so the results obtained may be applied in quark models in 3+1 dimensions which use these kind of potentials. We also explore the relationship between those symmetries by means of charge-conjugation and γ^5 chiral transformations already established in [2].

2. Spin and pseudospin symmetries in the Dirac equation

The time-independent Dirac equation for a spin 1/2 particle with mass m , energy E , under the action of scalar, S , and vector, V , potentials reads

$$\begin{aligned}
 H\psi &= [\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}c + \beta (mc^2 + S) + V] \psi = E\psi, \\
 \boldsymbol{\alpha} &= \begin{pmatrix} 0 & \boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad (1)
 \end{aligned}$$

and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli matrices. Following closely Bell and Rueg's paper [5], we project the spinor ψ into its components $\psi_{\pm} = P_{\pm}\psi$, with $P_{\pm} = [(I \pm \beta)/2] \psi$, yielding $\psi_{+}^{\top} = (\phi \ 0)$ and $\psi_{-}^{\top} = (0 \ \chi)$, ϕ and χ being two-component spinors. Using P_{\pm} on the Dirac equation (1), we get

$$c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\psi_{-} + (\Sigma + mc^2)\psi_{+} = E\psi_{+}, \quad c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}\psi_{+} + (\Delta - mc^2)\psi_{-} = E\psi_{-}, \quad (2)$$

where $\Sigma = V + S$ and $\Delta = V - S$.

If $\Delta = 0$ ($V = S$) and after multiplying it by $c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}}$, the second equation in (2) becomes $\hat{\mathbf{p}}^2/(E/c^2 + m)\psi_{+} = (E - mc^2 - \Sigma)\psi_{+}$, which is invariant under the transformation $\delta\psi_{+} = \frac{\boldsymbol{\epsilon} \cdot \tilde{\boldsymbol{\sigma}}}{2i}\psi_{+}$ [5], where $\tilde{\boldsymbol{\sigma}}$ is the 4×4 spin matrix. Since $\psi_{-} = (c\boldsymbol{\alpha} \cdot \hat{\mathbf{p}})/(E + mc^2)\psi_{+}$, and defining $\delta\psi = \boldsymbol{\epsilon} \cdot \mathbf{S}/(2i)\psi$, we can write the generators of this symmetry, called *spin symmetry*, as

$$\mathbf{S} = \tilde{\sigma}P_+ + \sigma_s P_- = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma}_s \end{pmatrix}, \tag{3}$$

where $\boldsymbol{\sigma}_s = \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} \tilde{\sigma} \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} / \hat{\mathbf{p}}^2$. These generators commute with the Hamiltonian in (1) when $V = S$ and form an SU(2) algebra, *i.e.*, $[\mathcal{S}_i, \mathcal{S}_j] = 2i\varepsilon_{ijk}\mathcal{S}_k$. The physical significance of this symmetry can be understood by looking at the second-order differential equation for ψ_+ when scalar and vector potentials are radial

$$\hat{\mathbf{p}}^2 \psi_+ + \frac{\frac{2}{r} \Delta' \mathbf{L} \cdot \mathbf{S} \psi_+ - \hbar^2 \Delta' \frac{\partial \psi_+}{\partial r}}{E - \Delta + mc^2} = \frac{1}{c^2} (E - \Delta + mc^2) (E - \Sigma - mc^2) \psi_+, \tag{4}$$

where $\Delta' = d\Delta/dr$ and $\mathbf{S} = (\hbar/2) \tilde{\sigma}$, $\mathbf{L} = \mathbf{r} \times \hat{\mathbf{p}}$. From (4) is clear that spin symmetry means *the disappearance of the spin-orbit coupling in a relativistic theory*. For radial potentials, there is another SU(2) symmetry connected to the orbital angular momentum, whose generators are $\mathcal{L} = \mathbf{L}P_+ + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} \mathbf{L} \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} / \hat{\mathbf{p}}^2 P_-$. One has $\mathcal{L}^2 \psi = \hbar^2 \ell(\ell + 1)\psi$, where ℓ is the orbital angular momentum quantum number of ψ_+ , even though ψ_+ and ψ_- have different orbital angular momentum quantum numbers. The orbital angular momentum of the lower component, $\tilde{\ell}$, is given by $\tilde{\ell} = \ell - \kappa/|\kappa|$, where $\kappa = -(\ell + 1)$ if $j = \ell + \frac{1}{2}$, $\kappa = \ell$ if $j = \ell - \frac{1}{2}$, meaning that levels with the quantum numbers $(n, l, j = l - 1/2)$ and $(n, l, j = l + 1/2)$ are degenerate. The results above would still be true if Δ were just a constant.

If $\Sigma = 0$ ($V = S$) or a constant, one can repeat the arguments of the previous section for the spinor ψ_- , whose second-order equation would be a Schroedinger-like equation. The corresponding symmetry, the *pseudospin symmetry*, has the SU(2) generators $\tilde{\mathcal{S}} = \tilde{\sigma}P_- + \sigma_s P_+ = \gamma^5 \mathbf{S}$. Now, it is the spin-orbit coupling for ψ_- which disappears, as can be seen from

$$\hat{\mathbf{p}}^2 \psi_- + \frac{\frac{2}{r} \Sigma' \mathbf{L} \cdot \mathbf{S} \psi_- - \hbar^2 \Sigma' \frac{\partial \psi_-}{\partial r}}{E - \Sigma - mc^2} = \frac{1}{c^2} (E - \Delta + mc^2) (E - \Sigma - mc^2) \psi_-. \tag{5}$$

Again, in this case, there is another SU(2) symmetry, whose generators are $\tilde{\mathcal{L}} = \mathbf{L}P_- + \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} \mathbf{L} \boldsymbol{\alpha} \cdot \hat{\mathbf{p}} / \hat{\mathbf{p}}^2 P_+ = \gamma^5 \mathcal{L}$. In this case, the orbital angular momentum $\tilde{\ell}$ is a good quantum number, *i.e.*, $\tilde{\mathcal{L}}^2 \psi = \hbar^2 \tilde{\ell}(\tilde{\ell} + 1)\psi$, such that the doublets $(n', l + 2, j = l - 1/2)$ $(n, l, j = l + 1/2)$ are degenerate. In nuclei, $n' = n - 1$, and it was precisely the observation of the near-degeneracy of such levels in nuclei that led to the pseudospin concept.

The 1+1 dimensional time-independent Dirac equation for a fermion of mass m , energy E , under the action of vector V , scalar, S , and pseudoscalar, V_p , potentials can be written as

$$H\psi = E\psi, \quad H = c\sigma_1 p + \sigma_3 mc^2 + \frac{I + \sigma_3}{2} \Sigma + \frac{I - \sigma_3}{2} \Delta + \sigma_2 V_p, \quad (6)$$

using the potentials Σ and Δ defined before and where I denotes the 2×2 unit matrix. This equation is covariant under $x \rightarrow -x$ if V_p changes sign and V and S are unchanged. The charge-conjugation operation is given by the transformation $\psi_c = \sigma_1 \psi^*$ and the Dirac equation becomes $H_c \psi_c = -E\psi_c$, with $H_c = c\sigma_1 p + \sigma_3 mc^2 - \frac{I + \sigma_3}{2} \Delta - \frac{I - \sigma_3}{2} \Sigma + \sigma_2 V_p$, *i.e.*, it changes the sign of E , V and V_p , and therefore turns Σ into $-\Delta$ and Δ into $-\Sigma$.

The chiral operator for a Dirac spinor is $\gamma^5 = \sigma_1$. Under the *discrete chiral transformation* [2], the spinor is transformed as $\psi_\chi = \gamma^5 \psi$ and the transformed Hamiltonian $H_\chi = \gamma^5 H \gamma^5$ is $H_\chi = c\sigma_1 p - \sigma_3 mc^2 + \frac{I + \sigma_3}{2} \Delta + \frac{I - \sigma_3}{2} \Sigma + \sigma_2 V_p$. Thus the chiral transformation changes the sign of the mass, the scalar and pseudoscalar potentials, thus turning Σ into Δ and *vice versa*.

If we now write the two-component spinor ψ in terms of its upper (ψ_+) and lower (ψ_-) components, *i.e.*, $\psi^\top = (\psi_+ \ \psi_-)$, the Dirac equation gives rise to two coupled first-order differential equations for these components

$$-i\hbar c\psi'_- + mc^2\psi_+ + \Sigma\psi_+ - iV_p\psi_- = E\psi_+, \quad (7)$$

$$-i\hbar c\psi'_+ - mc^2\psi_- + \Delta\psi_- + iV_p\psi_+ = E\psi_-, \quad (8)$$

where the prime denotes differentiation with respect to x . From (7) and (8), one can see that the components have opposite parities and also get

$$-\frac{\hbar^2}{2m}\psi''_+ + \frac{(E + mc^2)\Sigma + V_p^2 + \hbar cV'_p}{2mc^2}\psi_+ = \frac{E^2 - m^2c^4}{2mc^2}\psi_+, \quad (9)$$

for $E \neq -mc^2$ and $\Delta = 0$, whereas for $\Sigma = 0$ and with $E \neq mc^2$, one gets

$$-\frac{\hbar^2}{2m}\psi''_- + \frac{(E - mc^2)\Delta + V_p^2 - \hbar cV'_p}{2mc^2}\psi_- = \frac{E^2 - m^2c^4}{2mc^2}\psi_-. \quad (10)$$

The solutions of (7) and (8) with $E = \mp mc^2$ respectively are called isolated solutions [1]. In 1+1 dimensions, the potential generated by a point charge at the origin, the Coulomb potential, is linear in $|x|$ [6]. We will consider either Σ or Δ to be equal to $k_1|x|$ and $V_p = k_2x$. Equation (9) is written as

$$-\frac{\hbar^2}{2m}\psi''_+ + \left(\frac{1}{2}Ax^2 + B|x|\right)\psi_+ = \frac{E^2 - m^2c^4 - \hbar ck_2}{2mc^2}\psi_+, \quad (11)$$

where $A = k_2^2/(mc^2)$, $B = k_1(E + mc^2)/(2mc^2)$, whereas Eq. (10) becomes

$$-\frac{\hbar^2}{2m}\psi''_- + \left(\frac{1}{2}A'x^2 + B'|x|\right)\psi_- = \frac{E^2 - m^2c^4 + \hbar ck_2}{2mc^2}\psi_-, \quad (12)$$

where $A' = k_2^2/mc^2$, $B' = k_1(E - mc^2)/2mc^2$.

The cases with $k_2 = 0$ or $k_1 = 0$ have already been studied (see [7] and [1]). For $k_2 = 0$, one gets as solutions Airy functions for $x > 0$ [8], $\psi_+(z) \propto \text{Ai}(z)$, with $z = ax + b$, $a = [(E + mc^2)k_1/\hbar^2c^2]^{1/3}$, $b = -(E - mc^2)a/k_1$. Solutions for the entire real axis are obtained by requiring that they have a definite parity, which implies $\psi_+(0) = 0$ for odd solutions and $\psi'_+(0) = 0$ for even solutions, giving rise to the conditions $\text{Ai}(b) = 0$ and $\text{Ai}'(b) = 0$, respectively. From these, one is able to obtain the energy eigenvalues for the even and odd solutions. When $\Sigma = 0$, solutions for ψ_- are again Airy functions $\text{Ai}(z')$ with $z' = a'x + b'$, $a' = [(E - mc^2)k_1/(\hbar^2c^2)]^{1/3}$, $b' = -(E + mc^2)a'/k_1$. The spectra for both $\Delta = 0$ and $\Sigma = 0$ cases have only positive energy states for $k_1 > 0$ and negative energy states for $k_1 < 0$.

When $k_1 = 0$, one gets the so-called Dirac oscillator in 1+1 dimensions studied in [2]. The spectrum has both positive and negative energies for a particular value of k_2 , such that, defining the dimensionless strength $\kappa_2 = k_2/(\hbar m^2c^3)$, the energy eigenvalues are given by $E_n^2 = m^2c^4[1 + \kappa_2 + |\kappa_2|(2n + 1)]$, with n integer (see [2] for details).

For the general case of $k_1 \neq 0$ and $k_2 \neq 0$ ($\Delta = 0$), the bound solutions of equation (11) are given by parabolic cylinder functions, denoted by $D_\nu(y)$, where $y = \alpha x + \beta_\Sigma$, $\alpha = \sqrt{2|k_2|}/(\hbar c)$, $\beta_\Sigma = \alpha k_1(E + mc^2)/(2k_2^2)$, $\nu = (E^2 - m^2c^4)/(2\hbar c|k_2|) - 1/2(k_2/|k_2| + 1) + k_1^2/|k_2|^3(E + mc^2)^2/(8\hbar c)$. The eigenenergies are obtained, as before, by the requirements $D_\nu(\beta_\Sigma) = 0$ and $D'_\nu(\beta_\Sigma) = 0$ corresponding to odd and even solutions respectively.

For $\Sigma = 0$ solutions, Eq. (12), we have similar solutions $D_\mu(y')$ with $y' = \alpha x + \beta_\Delta$, $\beta_\Delta = \alpha k_1(E - mc^2)/(2k_2^2)$, $\mu = (E^2 - m^2c^4)/(2\hbar c|k_2|) + 1/2(k_2/|k_2| - 1) + k_1^2/|k_2|^3(E - mc^2)^2/8\hbar c$.

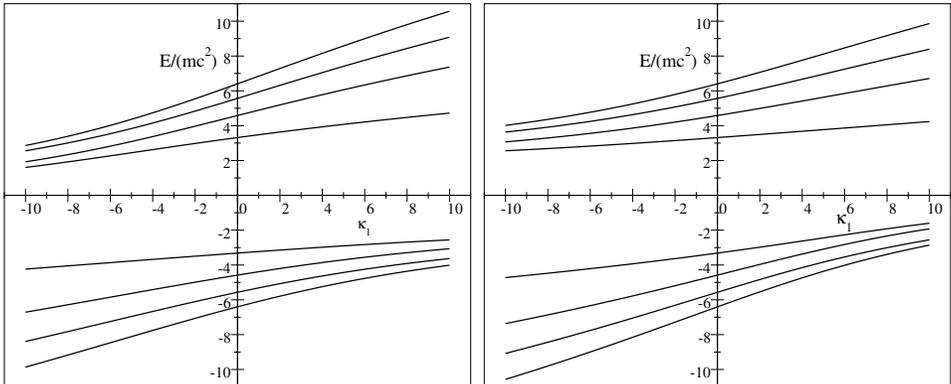


Fig. 1. First four energy levels as functions of κ_1 when $\kappa_2 = 5$ and $\Delta = 0$ (left panel) and $\kappa_2 = -5$ and $\Sigma = 0$ (right panel).

In Fig. 1 the solutions for $\Delta = 0$ and $\Sigma = 0$ for two symmetric values of κ_2 are plotted. For $k_1 = 0$, we get the Dirac oscillator solutions, and the solutions presented correspond to the values $n = 0, 1, 2, 3$ of its quantum number, which also defines the parity of the solutions: for even or odd n , one has even or odd solutions, respectively.

3. Discussion and conclusions

We note that in Fig. 1 the two plots are identical if we reverse both the vertical and horizontal axes, *i.e.*, for $\kappa_2 = -5$, $\Sigma = 0$, we get the same solutions as for $\kappa_2 = 5$ and $\Delta = 0$ if we reverse the signs of the energy and of κ_1 . This is because β_Σ turns into β_Δ when $k_1 \rightarrow -k_1$ and $E \rightarrow -E$ and, on the other hand, ν turns into μ when $k_2 \rightarrow -k_2$ and $E \rightarrow -E$, while it is left unchanged by the change of the k_1 sign. These are exactly the transformations induced by charge-conjugation, *i.e.*, performing the changes $E \rightarrow -E$, $\Delta \rightarrow -\Sigma$, $V_p \rightarrow -V_p$. The chiral transformation γ^5 changes the combination $E \pm mc^2$ into $E \mp mc^2$, changes the sign of k_2 , and also turns Δ into Σ and *vice versa*. This is equivalent to the charge conjugation transformation described above.

It is also interesting to remark that the introduction of a pseudoscalar potential in addition to the vector and scalar potentials has the affect of allowing positive and negative energy solutions for the same set of values of (k_1, k_2) . This happens also for scalar and vector Coulomb potentials [9] but not for pure scalar and vector confining potentials as is the present case. These conclusions can be applied to the one-dimensional linear potential problem in a 3+1 Dirac equation and in the spherically symmetric problem also in 3+1 dimensions, provided one has s states. As such, these results might be of relevance to quarkonium phenomenology in 3+1 dimensions.

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