METASTABLE STATES IN THE PARALLEL ISING MODEL*

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We study the parallel version of the Ising model, introduced as a model for opinion formation. We first recall some results about the statistical analysis of the serial and fully parallel version. We introduce the dilution (or asyncronism) of the updating rule and show that the chequerboard patterns that appear in the fully parallel version are unstable with respect to dilution, but exhibit finite-size effects and long-lasting metastable states.

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1. Introduction

There is a quite large number of studies about opinion formation in uniform societies [1]. Many such models adopt an approach similar to that of the Ising model [2]. In such cases, one has two opinions, say A and B or -1 and 1, and one is interested in the establishment of a majority (magnetic phase transitions) or in the effects of borders, or in the influence of some leader (social impact theory) [3]. This opinion space can be seen as the first ingredient of these models.

The second ingredient is how to model the response to an external influence. It is common to classify the attitude of people (agents) as either conformist or contrarian (also known as nonconformist). A conformist tends to agree with his neighbours and a contrarian to disagree. It is also to map

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this attitude onto Ising terms: conformist agents correspond to ferromagnetic coupling and contrarians to antiferromagnetic ones [1]. The presence of contrarian agents in a society have been studied in models related to Ising and the voter model [4–10].

The Ising model applied to opinion formation shares some characteristics with the bootstrap percolation [11] and the voter model [12, 13].

In bootstrap percolation, if the number of neighbours with an polarizing opinion exceeds a fixed threshold, the individual under examination adopts that opinion with a given probability p. The system is not symmetric under the exchange of opinions. The main characteristic of bootstrap percolation, to be discussed in the following, is the relation between metastability and a system size. It can be shown [14] that the probability threshold p_c depends logarithmically on the system size, making numerical investigations aiming at finding such a threshold hopeless. So, while the only stable asymptotic state is, in principle, the uniform one, where everybody adopts the polarizing opinion, for any finite site, one sees an apparent threshold, which is only a metastable effect.

In the voter model, a site chosen at random takes one of the values among the neighbouring spins. In the nonlinear voter model [15], this choice depends on the number of adopters of each opinion. The linear version always ends in a homogeneous phase, while the nonlinear one allows for clusters of different opinions. The main difference with the Ising model is the presence of absorbing states: once that (locally) an homogeneous phase has been reached, there is no way for which a different opinion can arise, while in the Ising model with noninfinite couplings (finite temperatures) this can always happen. Thus, the nonlinear voter model can be considered analogous to the Ising model with two-site and multiple-site (plaquette) terms, where the term that couples the given site with all neighbours is infinite (thus producing locally absorbing phases). This model is presented in a companion paper [16], and the absorbing states were used in other opinion models [9], where one can also find a discussion about which model is more representative of a real society.

In the standard Ising model, metastability is also present. For instance, stating from an uniform state with a small magnetic field promoting the other phase, and for coupling values in the ferromagnetic phase, one can observe long transients [17]. The transition is determined by the appearance of a large enough droplet of the opposite phase.

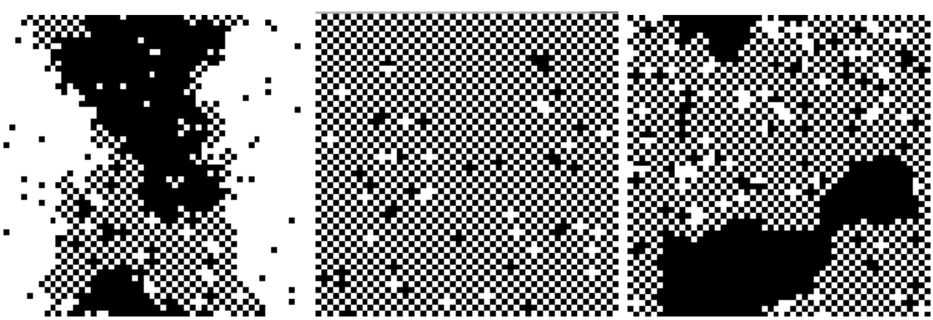
The third ingredient is the connectivity, *i.e.*, how the neighbourhood of a given individual is formed. Traditionally, magnetic systems have been studied either on regular lattices, on trees or with random connections, whose behaviour is similar to that of the mean-field approach. In recent years, much attention has been devoted to other network topologies, and so researchers have studied the Ising model on several kind of networks, from small-world [18] to scale-free [19] *etc.* In social terms, we can speak of "hubs", that have a great influence, "leaves" whose influence is minimal and long-range interactions mediated by Internet for instance, or local ones in friends' cliques.

With respect to the standard Ising model, there is no need to impose a monotonous behaviour of the probability with the number of neighbours adopting a given opinion: a hipster may decide to follow the minority. Moreover, the response character becomes a site variable, that may be different from individual to individual. In these terms, the simple ferromagnetic Ising model represents a uniform society of conformists with local symmetric interactions.

The final element is the update scheduling, that may be completely asynchronous, like in standard Monte Carlo simulations, or completely parallel, like in Cellular Automata, or something in the middle. It is not clear which scheme is the most representative. Human interactions are continuous, but also clocked by days, elections, *etc.* An effect that is favoured by parallelism is synchronization in the presence of complex dynamics. A macroscopic irregular behaviour (chaos) implies a coherent, but irregular motion of many elements.

There have not been many studies about the parallel Ising model [20–22], and none (to our knowledge) that examined the role of partial asynchronicity, which, however, is known to be able to induce phase transitions in deterministic cellular automata [23].

In the fully parallel Ising model, for large values of J, a chequerboard pattern, typical of the ferromagnetic phase but oscillating in time appears in addition to the continuous phases of opposite magnetization, Fig. 1 (a). A similar phenomenon is present for $J < 0$, where homogeneous oscillating



(a) (b) (c)

Fig. 1. Typical patterns in a 50×50 lattice, for $J = 1$, where white denotes negative spins and black positive ones. (a) The three phases in the fully parallel Ising model starting from a disordered configuration. (b) Stability (with fluctuations) of the chequerboard pattern for small dilutions ($d = 0.02$). (c) Droplets growing for larger dilutions ($d = 0.045$).

phases appear for large enough values of $|J|$. Actually, for lattices of even size, there is complete symmetry between the ferromagnetic (conformist) phase and the antiferromagnetic (contrarian) one, as long as no external field is imposed.

We are interested here in the stability of these "wrong phase" patterns. In particular, we are interested in how their stability is lost for partially asynchronous updating (see Fig. 1).

2. The Ising model with sequential and parallel dynamics

Our model is composed of spins (sites) s_i that can take two values ($s_i = -1, 1$) on a lattice or graph defined by the adjacency matrix $a_{ij} = 0, 1$, where $a_{ij} = 1$ (0) in the presence (absence) of a direct connection from site j to site i (*i.e.*, site j influences site i if $a_{ij} = 1$).

By changing the adjacency matrix, one can represent 1D, 2D and other regular lattices or disordered networks, with fixed or variable connectivity. The influence of neighbours is given by the normalized local field

$$h_i = \frac{1}{k_i} \sum_j a_{ij} s_j \, ,$$

where $k_i = \sum_j a_{ij}$ is the connectivity of site i.

A configuration of spins is denoted as $\boldsymbol{s} = (s_1, s_1, \ldots, s_N)$. The standard Ising model is defined by a rescaled Hamiltonian $H(\boldsymbol{s})$

$$H(\boldsymbol{s}) = -J \sum_i s_i h_i$$

for zero external magnetic field (the temperature is absorbed into the coupling). The quantity J characterizes the "response" of spin s_i with respect to local field h_i. For $J > 0$, this response is ferromagnetic, and the spin s_i tends to align to h_i, for $J < 0$, the response is antiferromagnetic.

The equilibrium probability distribution $P_{\text{eq}}(\boldsymbol{s})$ of a configuration $\boldsymbol{s}$ is

$$P_{\text{eq}}(\boldsymbol{s}) = \frac{\exp(-H(\boldsymbol{s}))}{\sum_{\boldsymbol{s}'} \exp(-H(\boldsymbol{s}'))} = \frac{1}{Z} \exp(-H(\boldsymbol{s})) \, .$$

The sum in the denominator in the middle expression is over all configurations $\boldsymbol{s}'$ and is the partition function Z. In practical cases, this probability distribution is computed using Monte Carlo simulations or analytical computations based on the Markov equation. We define the temporal probability distribution $P(\boldsymbol{s}, t)$. We start from a given configuration $\boldsymbol{s}_0$. A time step is divided into N tentative updates of each site. The update sequence may be random or ordered, the final distribution is independent of the kind of updating as long as the resulting Markov process is irreducible or ergodic (*i.e.*, it allows going from any configuration to any other one in a finite number of steps). The updating scheme may be important for the speed of convergence.

For each time step, one chooses a single spin i and updates its value (from s_i to s'_i) using the transition probability (heat bath dynamics)

$$\begin{aligned}\tau(s'_i|\boldsymbol{s}) &= \frac{\exp\left(Js'_i h_i\right)}{\exp\left(Js'_i h_i\right)+\exp\left(-Js'_i h_i\right)} \\ &= \frac{1}{1+\exp\left(-2Js'_i h_i\right)} \\ &= \frac{1}{2}\left(1+\tanh\left(Js'_i h_i\right)\right) \ . \end{aligned} \tag{1}$$

Notice that the updating rule does not depend on the previous value of the spin s_i.

The heat bath dynamics obeys the detailed balance

$$\frac{\tau(s'|\boldsymbol{s})}{\tau(s|\boldsymbol{s}')} = \exp\left(H(\boldsymbol{s}) - H\left(\boldsymbol{s}'\right)\right) \ ,$$

where $\boldsymbol{s}'$ is defined to be equal to $\boldsymbol{s}$ except at the site under consideration ($s'_j = s_j$ for $j \neq i$, and $s'_i = -s_i$).

The resulting Markov chain is

$$P\left(\boldsymbol{s}', t+\frac{1}{N}\right) = \frac{1}{N}\sum_{i=1}^{N} \tau\left(s'_i|\boldsymbol{s}\right)\left[P(\boldsymbol{s},t) + P(\boldsymbol{s}',t)\right] \ .$$

This equation expresses the fact that in sequential updating, each spin has in average the same probability $1/N$ of being chosen (either in order or randomly); in each individual step, a given configuration $\boldsymbol{s}'$ can only be generated by the same configuration $\boldsymbol{s}'$ or the one with an opposite value on site i ($\boldsymbol{s}$). The transition probability $\tau(s'_i|\boldsymbol{s})$ in the standard Ising model does not include the self-interaction of spin s_i with itself (which in the linear model is in any case irrelevant).

If the couplings are symmetric (which, in our case, implies that are all the same) and uniform (so that the adjacency matrix is irreducible), in the long time limit $P(\boldsymbol{s}, t)$ tends to $P_{\mathrm{eq}}(\boldsymbol{s})$ independently of the initial configuration $\boldsymbol{s}_0$, *i.e.* the Markov process is ergodic. This is not assured for asymmetric couplings (*e.g.*, spin glasses).

This model can be extended to the parallel case [20]. In this case, each spin is updated using the same transition probability τ of Eq. (1). The resulting Markov chain is

$$P\left(\boldsymbol{s}', t+1\right) = \sum_{\boldsymbol{s}} \left[\prod_{i=1}^{N} \tau\left(s_i' | \boldsymbol{s}\right)\right] P(\boldsymbol{s}, t)\,.$$

The equilibrium distribution $\tilde{P}_{\mathrm{eq}}(\boldsymbol{s})$ is now

$$\tilde{P}_{\mathrm{eq}}(\boldsymbol{s}) = \frac{1}{\tilde{Z}} \prod_{i=1}^{N} \cosh\left(\sum_j J_j s_j h_j\right)\,.$$

These two distributions show different properties [21]. In particular, let us consider, for instance, the case in which the lattice can be divided into two sub-lattices, one updated with respect to the other. This is the case for the two-dimensional square lattice, with nearest-neighbours connections. The sequential dynamics actually couples these two lattices, so that the asymptotic distribution is unique, while the parallel dynamics makes the two sub-lattices independent.

Let us illustrate this more in detail, using the one-dimensional case. In 1D, one sub-lattice is composed of even sites at even times plus odd sites at odd times, the other sites belonging to the other sub-lattice. For instance, in one dimension, site $s_i(t+1)$, with i and t even, only depends on sites $s_{i-1}(t)$ and $s_{i+1}(t)$. In the case of sequential updating, only one site is updated and the others are copied, and this latter operation couples the two sub-lattices.

For parallel updating, the two sub-lattices are however uncoupled, so the Markov chain is no more irreducible and the asymptotic distribution is not unique. In the case of broken symmetry (*i.e.*, when the asymptotic distribution is no more unique and small fluctuations in the initial condition can lead to different asymptotic states), one sub-lattice can "go" into an attractor (*e.g.*, assume a given magnetization), while the other can assume the opposite one. This is, for instance, the case of the Q2R model [24], which is a microcanonical parallel version of the 2D Ising model. In this case, a spin flips only if the local field h_i is zero.

2.1. Metastable states

We investigate here the Ising model with nearest-neighbours interaction on two-dimensional lattices $N \times N$, with N even. Due to the form of the heat bath dynamics, Eq. (1), in the fully parallel case the lattice decouples into two noninteracting sub-lattices.

In this case, for large couplings, there are three stable states: for $J > 0$, one is formed with all spins that take value one, one with all spins taking value -1, and two with a chequerboard alternation of -1 and 1s, where spins flip at each time step. For $J < 0$, the homogeneous phases are oscillating and the chequerboard ones are stable. With small couplings, the usual disordered state appears, see Fig. 1. Due to the symmetry in changing the sign of J and the value of all spins in one sub-lattice, in the following we only consider the case $J > 0$.

We can distinguish these states by looking at the two-site correlation c defined by

$$c = \frac{1}{2} \sum_{ij} s_{ij} \left(s_{i+1,j} + s_{i,j+1} \right) .$$

The quantity c takes value 1 for the homogeneous (all 1 or all -1 case), it takes value -1 for the chequerboard pattern and zero for a disordered pattern.

Starting with a random configuration, one can see in Fig. 2 (a) a typical time evolution of the quantity c for small coupling (high temperature) and in Fig. 2 (b) that for coupling values near the phase transition. In this case, the correlation c oscillates among values near -1 and 1, *i.e.*, the configurations are near homogeneous ones ($c = 1$) or chequerboard ones ($c = -1$).

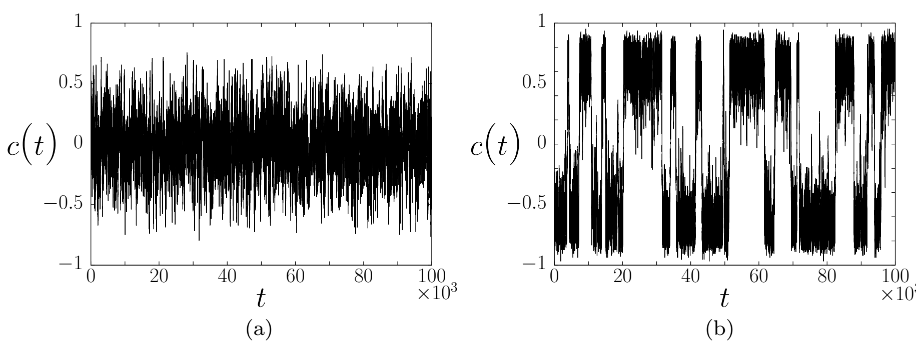


Fig. 2. Time behaviour of correlation c for $N = 16$ and coupling (a) $J = 0.40$, (b) $J = 0.45$.

An alternative view of this behaviour is illustrated in Fig. 3. In this case, one computes the probability distribution of c over a single evolution for a large time. For $J = 0.40$, the distribution is centred around $c = 0$, *i.e.*, disordered configurations dominate. For $J = 0.45$, one can see the appearance of peaks near $c = \pm 1$, but the distribution is still unique, *i.e.*, these states are metastable (as seen in Fig. 2 (b)). For $c = 0.5$, the symmetry is broken and the asymptotic distribution is no more unique: repeating the simulation, one observes either a peak near $c = 1$ or a peak near $c = -1$.

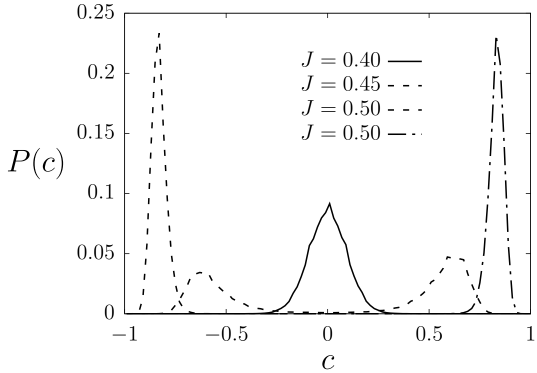


Fig. 3. Probability distribution of correlation c for $N = 32$ and several values of J. The distribution is computed over one evolution for 2×10^6 time steps after a transient of 10^4 time steps and using 100 bins. For $J = 0.50$, the probability distribution is not unique (we show two simulations with different asymptotic distributions). The jagged shape of the distribution is not due to low statistics: repeating the simulation with larger running time the distribution remains the same.

2.2. The effects of asynchronism

Let us now introduce a partial asynchronism, that we call dilution, for an even-side lattice. The control parameter is the fraction d of sites that are not updated, retaining their old values. The dilution couples the two sub-lattices.

We can extend the transition probability $\tau_d(s'|h,s,d)$ including the dilution probability so that

$$\tau_d(s'|h,s,d) = \begin{cases} \tau(s'|h) & \text{with probability } d\,, \\ \delta_{s',s} & \text{otherwise}\,, \end{cases}$$

i.e.,

$$s_{ij}(t+1) = \begin{cases} 1 & \text{with probability } \frac{1}{2}(1-d)\,[1+\tanh(Jh_{ij}(t))]\,, \\ -1 & \text{with probability } \frac{1}{2}(1-d)\,[1-\tanh(Jh_{ij}(t))]\,, \\ s_{ij}(t) & \text{otherwise, } i.e., \text{ with probability } d\,. \end{cases} \tag{2}$$

The Master equation of the system is

$$P\left(\boldsymbol{s}', t+1\right) = \sum_{\boldsymbol{s}} \left(\prod_i \tau_d\left(s'_i | h_i, s_i, d\right) \right) P(\boldsymbol{s}, t)\,.$$

The usual Ising model corresponds to $d \to 0$ (say, 1 spin updated per time step, neglecting the null moves), while the fully parallel version to $d = 1$.

The observables that depend only on single-site properties take the same values in parallel or sequential dynamics [20]. The presence of the chequerboard pattern however, is due to the strict parallelism of the model. By diluting the rule, *i.e.*, applying it only to a fraction d of sites, the chequerboard pattern should disappear. This is indeed the case, but for low values of the dilution d the chequerboard pattern is metastable.

We performed some experiments for $N = 8$, 16 and 32 and $J = 1$ (well inside the magnetized phase). We started with a chequerboard configuration, and updated it with a small dilution d. We monitored the correlation c and defined the escape time t_e from the chequerboard "basin" the time after which c becomes greater than zero. Due to the dilution, the effective time has to be rescaled. We define a rescaled time step as the number of updates, not counting the application of the identity of Eq. (2).

As shown in Fig. 4 (a), the distribution of the escaping time is exponential, and therefore the standard deviation is equal to the average escape time $\langle t_\mathrm{e} \rangle$. Despite the large width of the distribution, the average exit time is well-defined, as shown in Fig. 4 (b). From this last figure, one can see that there is an apparent divergence of the exit times for $d \simeq 0.038$–0.04. By plotting the exit times in logarithmic scale, as reported in Fig. 5 (a), one can see that indeed the divergence happens for $d = 0$. However, there are strong corrections to scaling, so that the divergence for $p_c = 0$ is clear only for $N = 8$ (the curve that approximates better a line for smaller values of p). Already with $N = 16$, one should perform extremely long simulations for putting this behaviour into evidence, and indeed in Fig. 5 (b), it is almost impossible to distinguish the best linear behaviour in the log–log plot, for the same computational time of $N = 8$. The vanishing of the threshold with the system size is the reason for using small lattices.

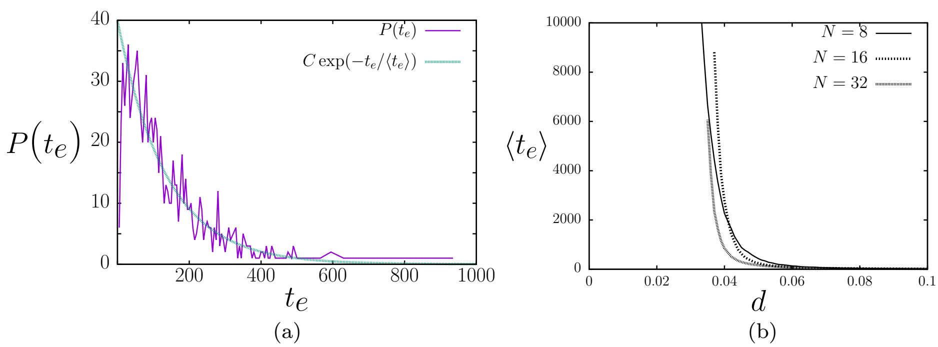


Fig. 4. (a) Histogram of the exit time t_e for $N = 8$, 1000 samples and $d = 0.06$. The distribution is exponential. Here $\langle t_\mathrm{e} \rangle \simeq 135$. (b) Average value of the exit time t_e for three values of N, 1000 samples.

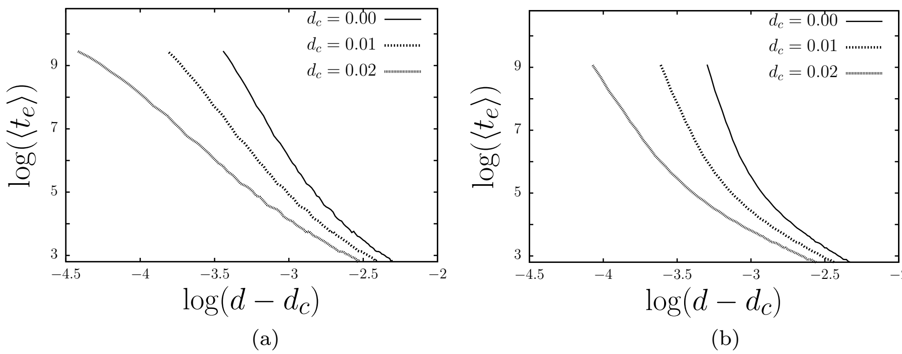


Fig. 5. Scaling behaviour of the correlation c with the dilution d for $N = 8$ (a) and $N = 16$ (b). Statistics over 1000 repetitions.

This finite-size behaviour is similar to that of bootstrap percolation, where larger lattices apparently show smaller critical threshold, but with less pronounced finite-size effects. We shall investigate in more details the relation between this metastability, arising by the coupling of the two sublattices in the parallel model, and that due to the presence of an external field in the standard serial model [17] in a forthcoming work.

3. Conclusions

We have presented some aspects related to the transition from the classical, serial Ising model to the parallel version. We have shown that the chequerboard patterns are unstable with respect to dilution, but that the transition shows finite-size effects and long-lasting metastable states, for small lattices. Metastability have important consequences for models of opinion formation: human groups are never infinite, and often of rather limited size.

We analysed mainly the role of updating, which in social terms may correspond to social interactions or political elections, for instance, which is a kind of parallel updating with a dilution term that may be constituted by people not participating to the choice. The chequerboard effect analysed here may represent, in social systems, a kind of nonhomogeneity that may persist for long times, according to the kind of updating. Thus, the main results of this paper may be rephrased saying that a high rate of social contacts (little dilution) promotes the maintenance of nonhomogeneity, which tends to disappear if only a fraction of the population really participate to the political debate and to elections. However, differently from the real case, in our model, people belonging to this fraction are randomly drafted, while in real life unsocial people tend to remain unsocial.

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