LIGHT QUARK MASS DIFFERENCES IN THE π^0–η–η' SYSTEM*

A.A. Osipov, B. Hiller, A.H. Blin, J. Moreira

CFisUC, Department of Physics, University of Coimbra
3004-516 Coimbra, Portugal

(Received June 30, 2016)

A generalized 3 flavor Nambu–Jona-Lasinio Lagrangian, including the explicit chiral symmetry breaking interactions which contribute at the same order in the large $1/N_{\rm c}$ counting as the $\rm U_A(1)$ 't Hooft flavor determinant, is considered to obtain the mixing angles in the π^0–η–η' system and related current quark mass ratios in close agreement with phenomenological values. At the same time, an accurate ordering and magnitude of the splitting of states in the low-lying pseudoscalar nonet is obtained.

DOI:10.5506/APhysPolBSupp.9.413

Properties related to the mixing in the π^0–η–η' system are a subject of continued interest, as they address the complexity of non-perturbative QCD subject to the combined effects of chiral symmetry breaking and the $\rm U_A(1)$ anomaly [1]. In addition, this system is a standard probe used in the determination of the values of the current quark masses [2–5], and is a process of considerable importance in the studies of weak [6] and strong CP violation [7].

We report on our results for the π^0–η–η' mixing angles [8] that rely on an effective multi-quark low-energy Lagrangian for QCD [9], operational at the scale of spontaneous breaking of chiral symmetry, of the order of $\Lambda_{\chi\rm SB} \sim 4\pi f_\pi$ [10], and generalizes the Nambu–Jona-Lasinio (NJL) model [11] (where this scale is also related to the gap equation and given by the ultra-violet cutoff Λ of the one-loop quark integral) as follows: generic vertices L_i of non-derivative type that contribute to the effective potential as $\Lambda \to \infty$

$$L_i \sim \frac{\bar{g}_i}{\Lambda^\gamma}\chi^\alpha \Sigma^\beta\,, \tag{1}$$

* Presented at "Excited QCD 2016", Costa da Caparica, Lisbon, Portugal, March 6–12, 2016.

where powers of Λ give the correct dimensionality of the interactions (below, we use also unbarred couplings, $g_i = \frac{\bar{g}_i}{\Lambda^\gamma}$); the L_i are C, P, T and chiral $\mathrm{SU(3)_L} \times \mathrm{SU(3)_R}$ invariant blocks, built of powers of the sources χ which at the end give origin to the explicit symmetry breaking (ESB) and have the same transformation properties as the U(3) Lie-algebra valued field $\Sigma = (s_a - ip_a)\frac{1}{2}\lambda_a$; here, $s_a = \bar{q}\lambda_a q$, $p_a = \bar{q}\lambda_a i\gamma_5 q$, and $a = 0, 1, \ldots, 8$, $\lambda_0 = \sqrt{2/3} \times 1$, λ_a being the standard SU(3) Gell-Mann matrices for $1 \leq a \leq 8$.

The interaction Lagrangian without external sources χ is well-known,

$$\begin{aligned} L_{\mathrm{int}} &= \frac{\bar{G}}{\Lambda^2}\mathrm{tr}\left(\Sigma^\dagger \Sigma\right) + \frac{\bar{\kappa}}{\Lambda^5}\left(\det \Sigma + \det \Sigma^\dagger\right) \\ &\quad + \frac{\bar{g}_1}{\Lambda^8}\left(\mathrm{tr}\,\Sigma^\dagger \Sigma\right)^2 + \frac{\bar{g}_2}{\Lambda^8}\mathrm{tr}\left(\Sigma^\dagger \Sigma \Sigma^\dagger \Sigma\right) . \end{aligned} \tag{2}$$

The second term is the 't Hooft determinant [12], the last two, the 8 quark (q) interactions [13] which complete the number of relevant vertices in 4D for dynamical chiral symmetry breaking [14]. The interactions dependent on the sources χ contain eleven terms [9][1]

$$L_\chi = \sum_{i=0}^{10} L_i\,, \tag{3}$$

$$\begin{aligned} L_0 &= -\mathrm{tr}\left(\Sigma^\dagger\chi + \chi^\dagger\Sigma\right) , & L_2 &= \frac{\bar{\kappa}_2}{\Lambda^3} e_{ijk} e_{mnl} \chi_{im} \Sigma_{jn} \Sigma_{kl} + \mathrm{h.c.}\,, \\ L_3 &= \frac{\bar{g}_3}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\Sigma^\dagger\chi\right) + \mathrm{h.c.}\,, & L_4 &= \frac{\bar{g}_4}{\Lambda^6}\mathrm{tr}\left(\Sigma^\dagger\Sigma\right)\mathrm{tr}\left(\Sigma^\dagger\chi\right) + \mathrm{h.c.}\,, \\ L_5 &= \frac{\bar{g}_5}{\Lambda^4}\mathrm{tr}\left(\Sigma^\dagger\chi\Sigma^\dagger\chi\right) + \mathrm{h.c.}\,, & L_6 &= \frac{\bar{g}_6}{\Lambda^4}\mathrm{tr}\left(\Sigma\Sigma^\dagger\chi\chi^\dagger + \Sigma^\dagger\Sigma\chi^\dagger\chi\right) , \\ L_7 &= \frac{\bar{g}_7}{\Lambda^4}\left(\mathrm{tr}\Sigma^\dagger\chi + \mathrm{h.c.}\right)^2 , & L_8 &= \frac{\bar{g}_8}{\Lambda^4}\left(\mathrm{tr}\Sigma^\dagger\chi - \mathrm{h.c.}\right)^2 . \end{aligned} \tag{4}$$

The N_c assignments are $\Sigma \sim N_\mathrm{c}$; $\Lambda \sim N_\mathrm{c}^0 \sim 1$; $\chi \sim N_\mathrm{c}^0 \sim 1$ [2]. We get that exactly the diagrams which survive as $\Lambda \to \infty$ also survive as $N_\mathrm{c} \to \infty$ and comply with the usual requirements.

At LO in $1/N_\mathrm{c}$, only the $4q$ interactions ($\sim G$) in Eq. (2) and L_0 contribute. The OZI rule violating vertices are always of the order of $\frac{1}{N_\mathrm{c}}$ with respect to the leading contribution. Non-OZI-violating Lagrangian pieces scaling as N_c^0 represent NLO contributions with one internal quark loop

[1] We omit L_1, L_9, L_{10} from the list, as they refer to Kaplan–Manohar ambiguity [15] within the model, which allows to set the couplings to 0.

[2] The counting for Λ is a direct consequence of the gap equation $1 \sim N_\mathrm{c} G \Lambda^2$.

in $N_{\rm c}$ counting; their couplings encode the admixture of a four-quark component $\bar{q}q\bar{q}q$ to the leading $\bar{q}q$ at $N_{\rm c} \to \infty$. Diagrams tracing OZI rule violation are: $\kappa, \kappa_2, g_1, g_4, g_7, g_8$; diagrams with admixture of 4 quark and 2 quark states are: g_2, g_3, g_5, g_6.

Putting $\chi = \frac{1}{2}\mathrm{diag}(m_u, m_d, m_s)$, the current quark masses, we obtain a consistent set of explicitly breaking chiral symmetry terms.

One ends up with 5 parameters needed to describe the LO contributions (the scale Λ, the coupling G, and the m_i) and 10 in NLO ($\bar{\kappa}, \bar{\kappa}_2, \bar{g}_1, \ldots, \bar{g}_8$).

The details of bosonization in the framework of functional integrals, which lead from $L = \bar{q}i\gamma^\mu\partial_\mu q + L_{\rm int} + L_\chi$ to the long distance effective mesonic Lagrangian can be found in [9, 13, 16], here, we only collect the result for the kinetic and mesonic pseudoscalar mass terms

$$\begin{aligned} L_{\rm kin} + L_{\rm mass} &= \frac{N_{\rm c}I_1}{8\pi^2}(\partial\phi_a)^2 + \frac{N_{\rm c}I_0}{4\pi^2}\phi_a^2 \\ &-\frac{N_{\rm c}I_1}{24\pi^2}\Big\{\Big[\phi_u^2\left(2M_u^2 - M_d^2 - M_s^2\right) + \phi_d^2\left(2M_d^2 - M_u^2 - M_s^2\right) \\ &+\phi_s^2\left(2M_s^2 - M_u^2 - M_d^2\right)\Big]\Big\} + \frac{1}{2}\,h_{ab}^{(2)}\phi_a\phi_b + \ldots\,, \end{aligned} \tag{5}$$

where $h_{ab}^{(2)}$ carries all the dependence on the model couplings and current quark masses, and M_i $\{i = u, d, s\}$ are the constituent quark masses obtained by solving the model's gap equations. The kinetic term requires a redefinition of meson fields $\phi_a = g\phi_a^R$, $g^2 = \frac{4\pi^2}{N_{\rm c}I_1} = \frac{(M_u+M_d)^2}{2f_\pi^2}$, which are related to the flavor $\{u, d, s\}$ and the strange–non-strange basis as $\phi_u = \phi_3 + \frac{\sqrt{2}\phi_0+\phi_8}{\sqrt{3}} = \phi_3 + \eta_{\rm ns}$, $\phi_d = -\phi_3 + \frac{\sqrt{2}\phi_0+\phi_8}{\sqrt{3}} = -\phi_3 + \eta_{\rm ns}$, $\phi_s = \sqrt{\frac{2}{3}}\phi_0 - \frac{2\phi_8}{\sqrt{3}} = \sqrt{2}\eta_{\rm s}$. Defining $m_\Delta = \frac{1}{2}(m_d - m_u)$, $m_\Sigma = \frac{1}{2}(m_d + m_u)$, $h_\Delta = \frac{1}{2}(h_d - h_u)$ and $h_\Sigma = \frac{1}{2}(h_d + h_u)$, one has, for example, for the matrix elements relevant for π^0–η and π^0–η' mixing

$$\begin{aligned} \sqrt{6}\left(h_{03}^{(2)}\right)^{(-1)} &= h_\Delta\left(2g_2h_\Sigma + \kappa + g_3m_\Sigma\right) \\ &+m_\Delta\big[g_3h_\Sigma + 2(\kappa_2 - g_8(m_s + 2m_\Sigma) \\ &-(g_5 - g_6)m_\Sigma)\big] \end{aligned} \tag{6}$$

and a quite similar expression for $(h_{38}^{(2)})^{(-1)}$. Note that both elements vanish in LO in $N_{\rm c}$ and that explicit m_i dependence occurs only in presence of ESB interactions at NLO. In their absence, the effects of ESB are present in the difference of the condensates $h_\Delta \neq 0$ if the conventional QCD mass term $m_u \neq m_d$. In this case, only the 't Hooft $\sim \kappa$ and the $8q \sim g_2$ contribute to (6).

The physical states π_0, η, η' are obtained by diagonalizing the symmetric meson mass matrix B_{ab} in the subspace $\{0,3,8\}$ of (5). Since one is free to transform between the states, we chose to represent the mixing angles with the strange–non-strange basis as reference[3] by the following transformation $\mathcal{S} = \mathcal{U}\mathcal{V}$

$$(\phi_3, \phi_0, \phi_8)\, S^{-1} S \begin{pmatrix} B_{33} & B_{03} & B_{38} \\ B_{03} & B_{00} & B_{08} \\ B_{38} & B_{08} & B_{88} \end{pmatrix} S^{-1} S \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix},$$

rotating first to this basis through the orthogonal involutory matrix $\mathcal{V}$

$$\begin{pmatrix} \phi_3 \\ \eta_{\rm ns} \\ \eta_{\rm s} \end{pmatrix} = \mathcal{V} \begin{pmatrix} \phi_3 \\ \phi_0 \\ \phi_8 \end{pmatrix}; \qquad \mathcal{V} = \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & \sqrt{2} & 1 \\ 0 & 1 & -\sqrt{2} \end{pmatrix},$$

and then using the unitary transformation $\mathcal{U}$ to obtain the physical states [4]

$$\begin{pmatrix} \pi^0 \\ \eta \\ \eta' \end{pmatrix} = \mathcal{U}(\epsilon_1, \epsilon_2, \psi) \begin{pmatrix} \phi_3 \\ \eta_{\rm ns} \\ \eta_{\rm s} \end{pmatrix}. \tag{7}$$

$\mathcal{U}$ is linearized in the π^0–η and π^0–η' mixing angles, since it is assumed that ϕ_3 couples weakly to the $\eta_{\rm ns}$ and $\eta_{\rm s}$ states, decoupling in the isospin limit, while the mixing for the η–η' system is strong[4]

$$\mathcal{U} = \begin{pmatrix} 1 & \epsilon_1 + \epsilon_2 \cos\psi & -\epsilon_2 \sin\psi \\ -\epsilon_2 - \epsilon_1 \cos\psi & \cos\psi & -\sin\psi \\ -\epsilon_1 \sin\psi & \sin\psi & \cos\psi \end{pmatrix}. \tag{8}$$

We checked this hypothesis in our model calculations by diagonalizing the mass matrix also exactly using the explicit analytical expressions for the *eigenvalues* of a symmetric 3×3 matrix M [17]

$$\begin{aligned} \lambda_1 &= \xi - \sqrt{\varsigma}\left(\cos[\varphi] + \sqrt{3}\sin[\varphi]\right), \\ \lambda_2 &= \xi - \sqrt{\varsigma}\left(\cos[\varphi] - \sqrt{3}\sin[\varphi]\right), \\ \lambda_3 &= \xi + 2\sqrt{\varsigma}\cos[\varphi], \end{aligned}$$

where we use the abbreviations: $\xi = \frac{\mathrm{tr}[M]}{3}, \mathcal{M} = M - \xi \boldsymbol{I}, \varsigma = \frac{1}{6}\sum_i \sum_j (\mathcal{M}_{ij})^2,$ $\vartheta = \frac{1}{2}\det[\mathcal{M}], \varphi = \frac{1}{6}\mathrm{ArcTan}\left[\frac{\sqrt{\varsigma^3 - \vartheta}}{\vartheta}\right]$. The *eigenvectors* can then be obtained by normalizing the vectors given by

$$\vec{v_i} = \left(\left(\vec{M}_1 - \lambda_i \hat{e}_1\right) \times \left(\vec{M}_2 - \lambda_i \hat{e}_2\right)\right)^*,$$

[3] We show that the decay constants transform as the states in this basis in our model [8].

[4] The usual redefinitions $\epsilon = \epsilon_2 + \epsilon_1 \cos\psi, \epsilon' = \epsilon_1 \sin\psi$ are adopted.

where $\vec{M}_j$ corresponds to the j column of M. These are then used to build the diagonalization matrix with the standard parametrization of the CKM matrix (with the abbreviation $C_{ij} \equiv \cos[\theta_{ij}]$, $S_{ij} \equiv \sin[\theta_{ij}]$, $\theta_{12} = -\epsilon$, $\theta_{13} = -\epsilon'$, $\theta_{23} = \psi$ and phase $= 0$)

$$U = \begin{bmatrix} C_{12}C_{13} & -S_{12}C_{23} - C_{12}S_{13}S_{23} & S_{12}S_{23} - C_{12}S_{13}C_{23} \\ S_{12}C_{13} & C_{12}C_{23} - S_{12}S_{13}S_{23} & -C_{12}S_{23} - S_{12}S_{13}C_{23} \\ S_{13} & C_{13}S_{23} & C_{13}C_{23} \end{bmatrix}.$$

The numerical deviations, as compared to (8), are within 2% for the cases considered. The results are shown in Tables I–III. One sees that the explicit symmetry breaking interactions of the generalized NJL Lagrangian considered are crucial to obtain the phenomenological quoted value for the ratio $\frac{\epsilon}{\epsilon'}$, Table IV. We obtain values for the ϵ mixing angle which lie within the results discussed in the literature. Unfortunately, the value for ϵ' is much less discussed. The values ϵ and ϵ' are reasonably close to the ones indicated in [3, 4] for the renormalization group invariant mass ratio m_u/m_d and current quark mass values in agreement with the presently quoted average values, $\frac{m_u}{m_d} = 0.46(5)$, $m_u = 2.15(15)$ MeV, $m_d = 4.70(20)$ MeV, $m_s = 93.5 \pm 2.5$ MeV [22].

TABLE I

Empirical fits (input marked by (*), also in Tables II, III) for set A (with NLO ESB interactions), and B (LO ESB). Masses in units of MeV, angle ψ in degrees.

Set	m_π^0	$m_\pi^\pm$	m_η	m_η'	m_K^0	$m_K^\pm$	f_π	f_K	ψ
A	136*	136.6	547*	958*	500	494*	92*	113*	39.7*
B	136*	137.0	477	958*	501	497*	92*	116*	39.7*

TABLE II

The couplings emerging from the fits have the following units: $[G] = \text{GeV}^{-2}$, $[\kappa] = \text{GeV}^{-5}$, $[g_1] = [g_2] = \text{GeV}^{-8}$, $[\kappa_2] = \text{GeV}^{-3}$, $[g_3] = [g_4] = \text{GeV}^{-6}$, $[g_5] = [g_6] = [g_7] = [g_8] = \text{GeV}^{-4}$. $\Lambda = 828.5^*; 835.7$ MeV in A, B.

Set	G	$-\kappa$	g_1	g_2	κ_2	g_3	g_4	g_5	g_6	g_7	g_8
A	10.48	116.8	3284	1237	6.24	2365	1182	160	712	580	44
B	9.79	137.4	2500*	117	0*	0*	0*	0*	0*	0*	0*

TABLE III

$\frac{m_u}{m_d} = 0.46^*$, current and constituent quark masses m_u, m_d, m_s, M_u, M_d, M_s in MeV, π^0–η, π^0–η' mixing angles ϵ and ϵ'.

Set	m_u	m_d	m_s	M_u	M_d	M_s	ϵ	ϵ'	$\frac{\epsilon}{\epsilon'}$
A	2.179	4.760	95*	372	375	544*	0.014*	0.0037*	3.78
B	3.774	8.246	194	373	380	573	0.022	0.0025	8.78

TABLE IV

ϵ and ϵ' values in the literature.

	ϵ	ϵ'	$\frac{\epsilon}{\epsilon'}$
[3] phen.	0.014	0.0037	3.78
[4] phen.	0.017 ± 0.002	0.004 ± 0.001	4.25 ± 1.17
[18] ChPt NLO	$0.014 \div 0.016$	—	—
[19] phen.	0.021	—	—
[20] Exp.	0.030 ± 0.002	—	—
[21] Exp.	0.026 ± 0.007	—	—

B.H. thanks the organizers for the kind invitation.

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