GRAVITATIONAL INSTANTONS, OLD AND NEW* **

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In memory of Jerzy Lewandowski (1959–2024)

This is a review of gravitational instantons — solutions to Riemannian Einstein or Einstein–Maxwell equations in four dimensions which yield complete metrics on non-compact four-manifolds, and which asymptotically 'look like' flat space. The review focuses on examples, and is based on lectures given by the author at the Cracow School of Theoretical Physics held in Zakopane in June 2024.

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1. Introduction

Gravitational instantons are solutions to the four-dimensional Einstein equations in the Riemannian signature which give complete metrics and asymptotically 'look like' flat space: If (M, g) is a gravitational instanton, then

$$\int_M |\mathrm{Riem}|^2 \mathrm{vol}_M < \infty \,,$$

where $|\mathrm{Riem}|^2$ is the squared g-norm of the Riemann tensor of g.

The study of gravitational instantons has been initiated by Hawking in his quest for Euclidean quantum gravity [31], and since then a lot of effort has been put to make the term 'look like' into a precise mathematical statement. While Euclidean quantum gravity does not any more aspire to a status of a fundamental theory, the study of gravitational instantons has influenced both theoretical physics and pure mathematics. This short

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review focuses on examples. It is based on lectures given by the author at the Cracow School of Theoretical Physics held in Zakopane in June 2024, and at the Banach Center — Oberwolfach Graduate Seminar *Black Holes and Conformal Infinities of Spacetime* held in Bedlewo in October 2024.

2. Examples

Some gravitational instantons arise as analytic continuations of Lorentzian black hole solutions to Einstein, or Einstein–Maxwell equations. If the imaginary time is turned into a periodic coordinate with the period given by the surface-gravity of Lorentzian black holes, then the resulting solutions are regular Riemannian metrics. Euclidean Schwarzschild and Kerr metrics belong to this category. Other gravitational instantons have no Lorentzian analogues, for example, because their Riemann curvature is anti-self-dual. The Eguchi–Hanson and anti-self-dual Taub-NUT solutions are such examples.

2.1. Euclidean Schwarzschild metric

The Schwarzschild metric is given by

$$g = -\left(1-\frac{2m}{r}\right)\mathrm{d}t^2 + \left(1-\frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) .$$

The apparent singularity at $r = 2m$ corresponds to an event horizon, and can be removed by a coordinate transformation. The singularity at $r = 0$ is essential as the squared norm of the Riemann tensor blows up as r^{-6}. An attempt to get rid of this singularity by removing the origin $r = 0$ from the space-time leads to a geodesically incomplete metric.

The Euclidean Schwarzschild metric [31] is obtained by setting $t = i\tau$, and restricting the range of r to $2m < r < \infty$. Set $\rho = 4m\sqrt{1-2m/r}$. Near $\rho = 0$, the metric takes the form

$$g \sim \mathrm{d}\rho^2 + \frac{\rho^2}{16m^2}\mathrm{d}\tau^2 + 4m^2\left(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2\right) .$$

This metric is flat and regular as long as the imaginary time τ is periodic with the period $8\pi m$. This period is inverse proportional to the Hawking temperature of the black hole radiation (Fig. 1). Although this was not how the Hawking temperature was first discovered, the instanton methods gave rise to a derivation simpler than the original calculation based on the Bogoliubov transformation [23, 24]. In a similar manner, the non-extreme Kerr black hole can be turned into the Euclidean Kerr instanton with the period of the imaginary time proportional to the inverse of the surface gravity. In the case of the extreme Kerr solution, the surface gravity vanishes and the extreme Kerr instanton does not exist.

Fig. 1. Stephen Hawking's gravestone.

2.2. Anti-self-dual Taub-NUT and ALF metric

Before introducing the next example, let us define the left-invariant one-forms $(\sigma_1, \sigma_2, \sigma_3)$ on $S^3 = SU(2)$ by

$$\sigma_1 + i\sigma_2 = \mathrm{e}^{-i\psi}(\mathrm{d}\theta + i\sin\theta \mathrm{d}\phi)\,, \qquad \sigma_3 = \mathrm{d}\psi + \cos\theta \mathrm{d}\phi\,,$$

where $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $0 \le \psi \le 4\pi$. They satisfy

$$\mathrm{d}\sigma_1 + \sigma_2 \wedge \sigma_3 = 0\,, \qquad \mathrm{d}\sigma_2 + \sigma_3 \wedge \sigma_1 = 0\,, \qquad \mathrm{d}\sigma_3 + \sigma_1 \wedge \sigma_2 = 0\,.$$

In terms of these one-forms, the flat metric on $\mathbb{R}^4$ is given by

$$g_{\mathbb{R}^4} = \mathrm{d}r^2 + \frac{1}{4}r^2\left({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right)\,. \tag{2.1}$$

The Taub-NUT instanton [31] is

$$g_{\mathrm{TN}} = \frac{1}{4}\frac{r+m}{r-m}\mathrm{d}r^2 + m^2\frac{r-m}{r+m}\sigma_3^2 + \frac{1}{4}\left(r^2 - m^2\right)\left(\sigma_1^2 + \sigma_2^2\right)\,. \tag{2.2}$$

Introducing a coordinate ρ by $r = m + \frac{\rho^2}{2m}$ shows that, near $r = m$, the metric (2.2) approaches the flat metric (2.1) and so $r = m$ is only a coordinate singularity. The Riemann curvature of the metric (2.2) is anti-self-dual (ASD); it satisfies

$$R_{abcd} = -\frac{1}{2}\varepsilon_{ab}{}^{pq}R_{cdpq}\,, \tag{2.3}$$

where $\varepsilon_{abcd} = \varepsilon_{[abcd]}$ is a chosen volume form on M. The ASD condition in particular implies the vanishing of the Ricci tensor. This follows from taking the trace of (2.3). It also shows that the metric (2.2) has no Lorentzian analogue, as the Riemann tensor of a metric in signature $(3,1)$ is ASD iff the metric is flat. For large r, the metric g_{TN} is the S^1 bundle over S^2 with the Chern number equal to 1 — this is the Hopf fibration with the total space S^3.

The ASD Taub-NUT example (2.2) motivates the following definition:

Definition 2.1 *A complete regular four-dimensional Riemannian manifold (M, g) which solves the Einstein equations is called ALF (asymptotically locally flat) if it approaches S^1 bundle over S^2 at infinity.*

The asymptotic form of an ALF metric is

$$\lim_{r\to\infty} g = (\mathrm{d}\tau + 2n\cos\theta \mathrm{d}\phi)^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2\right),$$

where the integer n is the Chern number of the S^1 bundle. If the S^1 bundle is trivial, so that $n = 0$, the ALF metric is called asymptotically flat (AF). Euclidean Schwarzschild and Euclidean Kerr metrics are AF. According to the Lorentzian black hole uniqueness theorems of Hawking, Carter, Robinson, and others [51], the Kerr family of solutions exhausts all AF solutions to the Einstein equations with $\Lambda = 0$. These theorems gave rise to the Riemannian 'black hole uniqueness' conjecture stating that the Euclidean Schwarzschild and Kerr are the only AF gravitational instantons [38]. This conjecture is now known to be false. We shall return to it in Section 4.

The ASD Taub-NUT instanton and other ALF metrics can be uplifted to the so-called Kaluza–Klein monopoles in 4 + 1-dimension [28, 48] with the product metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + g_{\mathrm{TN}}\,.$$

The Kaluza–Klein reduction of $\mathrm{d}s^2$ along the Killing vector $\partial/\partial\psi$ gives a monopole-type solution to the Einstein–Maxwell-dilaton theory in $(3 + 1)$ dimensions.

2.3. Eguchi–Hanson and the ALE metrics

The Eguchi–Hanson (EH) instanton [19, 20] is given by

$$g_{\mathrm{EH}} = \left(1 - \frac{a^4}{r^4}\right)^{-1}\mathrm{d}r^2 + \frac{1}{4}r^2\left(1 - \frac{a^4}{r^4}\right)\sigma_3^2 + \frac{1}{4}r^2\left(\sigma_1^2 + \sigma_2^2\right) \tag{2.4}$$

with $r > a$. Setting $\rho^2 = r^2\left[1 - (a/r)^4\right]$, we find that, near $r = a$, the metric is given by

$$g \sim \frac{1}{4}\left(\mathrm{d}\rho^2 + \rho^2\mathrm{d}\psi^2\right).$$

This metric is regular as long as the ranges of the angles are

$$0 \le \phi \le 2\pi\,, \qquad 0 \le \theta \le \pi\,, \qquad 0 \le \psi \le 2\pi\,.$$

Thus, although for $r \to \infty$, the Eguchi–Hanson metric approaches (2.1), given the allowed range of ψ, this metric is not asymptotically Euclidean, but corresponds to a quotient $\mathbb{R}^4/\mathbb{Z}_2$. The Eguchi–Hanson example motivates the following:

Definition 2.2 *A complete regular four-dimensional Riemannian manifold (M, g) which solves the Einstein equations is called ALE (asymptotically locally Euclidean) if it approaches $\mathbb{R}^4/\Gamma$ at infinity, where Γ is a discrete subgroup of $SO(4)$.*

The anti-self-dual ALE metrics are the best-understood class of gravitational instantons. This is due to the following:

Theorem 2.3 (Kronheimer [35, 36]) *For any Γ (cyclic A_N, dihedral D_N, dihedral, tetrahedral, octahedral, and icosahedral), there exists an ALE gravitational instanton.*

The Eguchi–Hanson metric corresponds to the case A_2, where $\Gamma = \mathbb{Z}_2$. It is not known [25, 42] whether there exist non-self-dual or anti-self-dual ALE Ricci-flat metrics.

3. Multi-centered metrics

Both the Taub-NUT and the Eguchi–Hanson metrics belong to the class of the so-called multi-centred gravitational instantons. These instantons arise as superpositions of fundamental solutions to the Laplace equation on $\mathbb{R}^3$ via the Gibbons–Hawking ansatz [22]. The verification of the Ricci-flat condition for this ansatz as well as its geometric characterisation is best achieved by using an equivalent formulation of the ASD Riemannian condition in terms of the hyper-Kähler structure. We shall give the necessary definitions and review the terminology in the next subsection. A more detailed discussion can be found in [17].

3.1. Mathematical detour: Hyper-Kähler metrics

We shall start with a definition

Definition 3.1 *An almost complex structure on a 4-manifold M is an endomorphism $I : TM \to TM$ such that $I^2 = -Id$.*

The almost complex structure gives rise to a decomposition

$$\mathbb{C} \otimes TM = T^{1,0}M \oplus T^{0,1}M\,, \quad \text{given by} \quad X = \tfrac{1}{2}[X - iI(X)] + \tfrac{1}{2}[X + iI(X)]$$

of the complexified tangent bundle into eigenspaces of I with eigenvalues $\pm i$. One says that I is a complex structure iff these eigenspaces are integrable in the sense of the Frobenius theorem, *i.e.*

$$[T^{1,0}M, T^{1,0}M] \subset T^{1,0}M\,. \tag{3.1}$$

A theorem of Newlander and Nirenberg justifies the terminology: I is a complex structure iff there exists a holomorphic atlas so that M is a two-dimensional complex manifold. For example, if $M = \mathbb{R}^4$ and

$$I = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$

then (3.1) holds and the complex atlas on $M = \mathbb{C}^2$ consists of complex coordinates $z_1 = x_1 + ix_3, z_2 = x_2 + ix_4$, and $T^{1,0}M = \text{span}\{\partial/\partial z_1, \partial/\partial z_2\}$.

We shall now assume that (M, g) is a Riemannian four-manifold with almost-complex structure I. We say that the metric g is

— Hermitian if $g(X, Y) = g(IX, IY)$;

— Kähler if I is a complex structure, and $\mathrm{d}\Omega = 0$, where $\Omega(X, Y) = g(X, IY)$;

— hyper-Kähler if it is Kähler w.r.t. three complex structures I_1, I_2, I_3 such that

$$I_1 I_2 = I_3\,, \qquad I_2 I_3 = I_1\,, \qquad I_3 I_1 = I_2\,.$$

For example, if $M = \mathbb{R}^4$, then the metric $g = |\mathrm{d}z_1|^2 + |\mathrm{d}z_2|^2$ is hyper-Kähler with

$$\Omega_1 = \frac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \mathrm{d}z_2 \wedge \mathrm{d}\bar{z}_2)\,, \qquad \Omega_2 + i\Omega_3 = \mathrm{d}z_1 \wedge \mathrm{d}z_2\,.$$

The importance of hyper-Kähler metrics in the study of gravitational instantons comes from the fact that locally, and with the choice of orientation which makes the Kähler forms self-dual (SD), the Riemann tensor of (M, g) anti-self-dual (ASD) iff (M, g) is hyper-Kähler. Therefore. the ASD gravitational instantons are complete hyper-Kähler metrics. Compact hyper-Kähler metrics are far more rare. There is the four-dimensional torus with a flat metric, and the elusive $K3$ surface whose existence follows from Yau's proof [57] of the Calabi conjecture. Finding the explicit closed form of a metric on a $K3$ surface is one of the biggest open problems in the field.

3.2. Gibbons–Hawking ansatz

Let (V, A) be, respectively, a function and a one-form on $\mathbb{R}^3$. The metric

$$g = V\left(\mathrm{d}x_1{}^2 + \mathrm{d}x_2{}^2 + \mathrm{d}x_3{}^2\right) + V^{-1}(\mathrm{d}\tau + A)^2\,, \tag{3.2}$$

is hyper-Kähler (and therefore ASD and Ricci flat) with the Kähler forms given by

$$\Omega_i = -(\mathrm{d}\tau + A) \wedge \mathrm{d}x_i + \tfrac{1}{2} V \epsilon_{ijk} \mathrm{d}x_j \wedge \mathrm{d}x_k \,, \qquad i = 1, 2, 3$$

iff the Abelian Monopole Equation

$$\mathrm{d}A = \star_3 \mathrm{d}V \tag{3.3}$$

holds (here $\star_k$ is the Hodge operator on $\mathbb{R}^k$ taken w.r.t. the flat metric and a chosen volume form). This equation follows from the closure condition $\mathrm{d}\Omega_i = 0$ and implies that the function V is harmonic on $\mathbb{R}^3$. The general Gibbons–Hawking ansatz (3.2) is characterised by the hyper-Kähler condition together with the existence of a Killing vector K which Lie-derives all Kähler forms. The Cartesian coordinates (x_1, x_2, x_3) in (3.2) arise as the moment maps, *i.e.* $K \lrcorner \, \Omega_i = \mathrm{d}x_i$.

The multi-centre metrics correspond to a choice

$$V = V_0 + \sum_{m=1}^{N} \frac{1}{|\, \boldsymbol{x} - \boldsymbol{x}_m \,|} \,, \tag{3.4}$$

where V_0 is a constant, and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_N$ are position vectors of N points in $\mathbb{R}^3$. The special cases of (3.4) are

— $V_0 = 0, N = 1$ give the flat metric;

— $V_0 = 0, N = 2$ give the Eguchi–Hanson metric (2.4) albeit in a different coordinate system; $V_0 = 0$ and $N > 2$ correspond to the general A_N ALE instantons;

— $V_0 \neq 0, N = 1$ give the Taub-NUT metric (2.2). $V_0 \neq 0, N > 1$ correspond to the A_N ALF instantons.

4. The Chen–Teo instanton

The Riemannian black hole uniqueness conjecture we alluded to in Section 2.2 is now known to be wrong. Chen and Teo [7, 8] have constructed a five-parameter family of toric (*i.e.* admitting two commuting Killing vectors) Riemannian Ricci flat metrics interpolating between the ALE three-centre Gibbons–Hawking metrics with centres on one axis and Euclidean Plebański–Demiański solutions [47]. The Chen–Teo family contains a two-parameter sub-family of AF instantons which are not in the Euclidean Kerr family of solutions. It has been proven by Aksteiner and Andersson [1] that, as the Chen–Teo family consists of Hermitian and therefore one-sided Petrov–Penrose type D solutions, the Chen–Teo instantons do not arise as an analytic continuation of any Lorentzian black holes.

4.1. Explicit formulae

Let f be a quartic polynomial with four real roots. Set

$$\begin{aligned} f &= f(\xi) = a_4\xi^4 + a_3\xi^3 + a_2\xi^2 + a_1\xi + a_0\,, \\ F &= f(x)y^2 - f(y)x^2\,, \\ H &= (\nu x + y)\left[(\nu x - y)(a_1 - a_3 xy) - 2(1-\nu)\left(a_0 - a_4 x^2 y^2\right)\right]\,, \\ G &= f(x)\left[(2\nu-1)a_4 y^4 + 2\nu a_3 y^3 + a_0\nu^2\right] \\ &\quad -f(y)\left[\nu^2 a_4 x^4 + 2\nu a_1 x + (2\nu-1)a_0\right]\,. \end{aligned}$$

The family of metrics

$$g = \frac{kH}{(x-y)^3}\left(\frac{\mathrm{d}x^2}{f(x)} - \frac{\mathrm{d}y^2}{f(y)} - \frac{f(x)f(y)}{kF}\mathrm{d}\phi^2\right) + \frac{1}{FH(x-y)}(F\mathrm{d}\tau + G\mathrm{d}\phi)^2 \tag{4.1}$$

is Ricci-flat for any choice of the parameters $(a_0, \ldots, a_4, \nu, k)$. Two out of five parameters $(a_0, \ldots, a_4)$ can be fixed by scalings, so (4.1) is a five-parameter family. The Riemann curvature is regular if the range of (x, y) is restricted to the rectangle in Fig. 2, where $r_1 < r_2 < r_3 < r_4$ are the roots f.

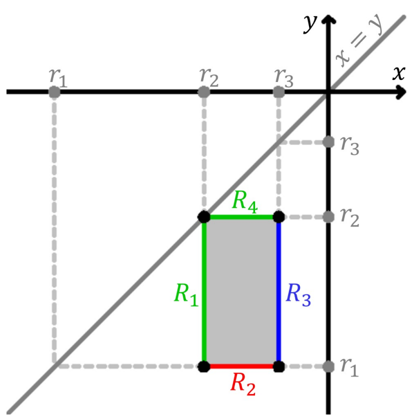

Fig. 2. The Chen-Teo domain.

To avoid the conical singularities and ensure the asymptotic flatness, one makes a choice

$$\begin{aligned} r_1 &= \frac{4s^2(1-s)}{1-2s+2s^2}\,, \qquad r_2 = -1\,, \qquad r_3 = \frac{1-2s}{s(1-2s+2s^2)}\,, \\ r_4 &= \infty\,, \qquad \nu = -2s^2\,, \qquad s \in (1/2, \sqrt{2}/2)\,. \end{aligned} \tag{4.2}$$

This leads to a two-parameter family of AF instantons on $M = \mathbb{CP}^2 \setminus S^1$.

4.2. The rod structure

The Chen–Teo metrics (4.1) admit two commuting Killing vectors $K_i = \partial/\partial\phi^i$, where $\phi^i = (\phi, \tau)$. Any metric with two commuting Killing vectors can locally be put in the form

$$g = \Omega^2 \left(\mathrm{d}r^2 + \mathrm{d}z^2\right) + J_{ij}\mathrm{d}\phi^i\mathrm{d}\phi^j\,, \qquad i,j = 1,2\,, \tag{4.3}$$

where $J = J(r,z)$ is a 2 by 2 symmetric matrix, and the (r,z) coordinates are defined by

$$r^2 = \det(J)\,, \qquad \star_2\mathrm{d}z = \mathrm{d}r\,.$$

The space of orbits of the T^2 action is the upper half-plane $\mathbb{H} = \{(r,z), r>0\}$ with the boundary $\partial\mathbb{H}$, where $\mathrm{rank}(J(0,z)) < 2$. Generically, this rank is equal to 1. It vanishes at the turning points $z_1, z_2, \ldots, z_N$, where both Killing vectors vanish. These turning points divide the z-axis into $(N+1)$ rods [29]

$$I_1 = (-\infty, z_1)\,, \quad I_2 = (z_1, z_2)\,, \ldots, I_N = (z_{N-1}, z_N)\,, \quad I_\infty = (z_N, \infty)\,.$$

In the Lorentzian case, these rods correspond to horizons or axes of rotation, and in the Riemannian case, they are axes. The rod data associated to (4.3) consists of a collection of $(N+1)$ rods, together with the lengths $(z_k - z_{k-1}), k = 2, \ldots, N$ of the finite rods, and the constant rod vectors $V_2, \ldots, V_N$ such that V_k vanishes on the rod I_k. Each of these vectors can be expanded as $V_k = V_k^1 K_1 + V_k^2 K_2$, and then the admissibility condition [30] is

$$\det \begin{pmatrix} V_k^1 & V_k^2 \\ V_{k+1}^1 & V_{k+1}^2 \end{pmatrix} = \pm 1\,.$$

While the rod structure does not uniquely determine the metric of the instanton, it specifies the topology of the underlying four-manifold [43]. The number of turning points is equal to the Euler signature. In the Chen–Teo case, there exist three turning points, so that $\chi(M) = 3$ for the Chen–Teo instanton. Closing up the semi-infinite rods gives the triangular rod structure of $\mathbb{CP}^2$ with three turning points as the triangle vertices, and three rods as sides. Joining the rods adds $S^1 \times \mathbb{R}^3$ to M, and so $M = \mathbb{CP}^1 \setminus S^1 \times \mathbb{R}^3 \cong \mathbb{CP}^1 \setminus S^1$. The signature of the Chen–Teo family is 1.

4.3. The Yang equation and ASDYM

The Ricci-flat condition on (4.3) reduces to the Yang equation

$$r^{-1}\partial_r\left(rJ^{-1}\partial_r J\right) + \partial_z\left(J^{-1}\partial_z J\right) = 0\,. \tag{4.4}$$

Once a solution to this equation has been found, the conformal factor Ω can be found by a single integration.

The Yang equation (4.4) also arises as a reduction of anti-self-dual Yang–Mills equations [53, 55]. To see it, let us consider the complexified Minkowski space $M_{\mathbb{C}} = \mathbb{C}^4$, with coordinates $(W, Z, \widetilde{W}, \widetilde{Z})$ such that the metric and the volume form are

$$\mathrm{d}s^2 = 2\left(\mathrm{d}Z\mathrm{d}\widetilde{Z} - \mathrm{d}W\mathrm{d}\widetilde{W}\right), \qquad \mathrm{vol} = \mathrm{d}W \wedge \mathrm{d}\widetilde{W} \wedge \mathrm{d}Z \wedge \mathrm{d}\widetilde{Z}\,.$$

Let $\Phi \in \Lambda^1(M_{\mathbb{C}}) \otimes \mathfrak{sl}(2)$ and $F = \mathrm{d}\Phi + \Phi \wedge \Phi$. The anti-self-dual Yang–Mills (ASDYM) equations are $F = -\star_4 F$ (now $\star_4$ is taken w.r.t. the flat metric on $\mathbb{C}^4$), or

$$F_{WZ} = 0\,, \qquad F_{\widetilde{W}\widetilde{Z}} = 0\,, \qquad F_{W\widetilde{W}} - F_{Z\widetilde{Z}} = \,. \tag{4.5}$$

The first two equations imply the existence of a gauge choice such that

$$\Phi = J^{-1}\partial_{\widetilde{W}} J\,\mathrm{d}\widetilde{W} + J^{-1}\partial_{\widetilde{Z}} J\,\mathrm{d}\widetilde{Z}\,, \qquad J = J\left(W, Z, \widetilde{W}, \widetilde{Z}\right) \in SL(2, \mathbb{C})\,. \tag{4.6}$$

The final equation in (4.5) holds iff

$$\partial_Z\left(J^{-1}\partial_{\widetilde{Z}} J\right) - \partial_W\left(J^{-1}\partial_{\widetilde{W}} J\right) = 0\,. \tag{4.7}$$

Setting

$$Z = t + z\,, \qquad \widetilde{Z} = t - z\,, \qquad W = r\,\mathrm{e}^{i\theta}\,, \qquad \widetilde{W} = r\,\mathrm{e}^{-i\theta}\,,$$

and performing a symmetry reduction $J = J(r, z)$ reduces (4.7) to (4.4).

4.4. Twistor construction

The twistor correspondence for ASDYM is based on an observation that ASDYM condition is equivalent to the flatness of a connection Φ on α-planes in $M_{\mathbb{C}}$

$$\mu = W + \lambda\widetilde{Z}\,, \qquad \nu = Z + \lambda\widetilde{W}\,. \tag{4.8}$$

The twistor space $PT \equiv \mathbb{CP}^3 \setminus \mathbb{CP}^1$ is the space of all such planes. It can be covered by two open sets, with affine coordinates (μ, ν, λ) in an open set, where $\lambda \neq \infty$. Points in $M_{\mathbb{C}}$ correspond to rational curves (twistor lines) in PT, and points in PT correspond to α-planes in $M_{\mathbb{C}}$. The conformal structure on $M_{\mathbb{C}}$ is encoded in the algebraic geometry of curves in PT: p_1, p_2 are null separated iff L_1, L_2 intersect.

The connection between twistor theory and ASDYM is provided by the following:

Theorem 4.1 (Ward [52]) *There exists a 1–1 correspondence between gauge equivalence classes of ASDYM connections Φ, and holomorphic vector bundles $E \to PT$ trivial on twistor lines.*

To read off the solution (4.7) from this theorem cover PT with two open sets: U, where $\lambda \neq \infty$ and $\widetilde{U}$, where $\lambda \neq 0$. The bundle E is then characterised by its patching matrix: $P = P(\mu, \nu, \lambda)$. The triviality on twistor lines implies that there exists a splitting $P = P_U {P_{\widetilde{U}}}^{-1}$, where P_U and $P_{\widetilde{U}}$ are holomorphic and invertible matrices on U and $\widetilde{U}$, respectively. The incidence relation (4.8) implies that P is constant along the vector fields $\{\partial_{\widetilde{Z}} - \lambda\partial_W, \partial_{\widetilde{W}} - \lambda\partial_Z\}$. Applying this to the splitting relation and using the Liouville theorem implies the existence of $\Phi \in \Lambda^1(M_{\mathbb{C}}) \otimes \mathfrak{sl}(2)$ such that

$$\Phi = \widetilde{H}^{-1}\partial_Z\widetilde{H}\,\mathrm{d}Z + \widetilde{H}^{-1}\partial_W\widetilde{H}\,\mathrm{d}W + H^{-1}\partial_{\widetilde{Z}}H\,\mathrm{d}\widetilde{Z} + H^{-1}\partial_{\widetilde{W}}H\,\mathrm{d}\widetilde{W}\,,$$

where $H = P_U(\lambda = 0)$, $\widetilde{H} = P_{\widetilde{U}}(\lambda = \infty)$. This is gauge equivalent to (4.6) with

$$J = H\widetilde{H}^{-1}\,. \tag{4.9}$$

4.5. Twistor bundle for toric Ricci flat metrics

Let us go back to the toric Ricci-flat metrics. For any of the Killing vectors K, we can find its twist potential: a function ψ such that

$$\mathrm{d}\psi = *(K \wedge \mathrm{d}K)\,.$$

Another solution to the Yang equation (4.4) then arises from a Bäcklund transformation

$$J' = \frac{1}{V}\begin{pmatrix} 1 & -\psi \\ -\psi & \psi^2 - V^2 \end{pmatrix}\,, \qquad V \equiv g(K, K)\,.$$

Let us pick a rod on which K is not identically zero. The following has been established in [21, 41, 56]: The patching matrix for the bundle E from Theorem 4.1 is an analytic continuation of $P(z) \equiv J'(0, z)$

$$P(\gamma), \quad \text{where} \quad \gamma = z + \frac{1}{2}r\left(\lambda - \frac{1}{\lambda}\right)\,.$$

The splitting procedure leads, via (4.9), to $J'(r, z)$ from which $J(r, z)$ can be recovered.

This patching matrix can be found for the Chen–Teo family [18]. It is given by

$$P(z) = \begin{pmatrix} C_1/C & Q/C \\ Q/C & C_2/C \end{pmatrix}\,, \tag{4.10}$$

where C_1, C_2, C are monic cubics, Q quadratic, with coefficients depending on the Chen–Teo parameters. Examining the outer rod and the asymptotics near $z=\infty$ gives

$$P \cong \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 2m & 2n \\ 2n & 2m \end{pmatrix} + O\left(\frac{1}{z^2}\right),$$

where m, n are mass and nut parameters. For Chen–Teo instanton with (4.2), we find

$$m = \sqrt{k}\,\frac{\left(1+2s^2\right)^2}{2\sqrt{1-4s^4}}, \qquad n = 0$$

in agreement with [37]. In general, the patching matrix P of the form of (4.10) where C, C_1, C_2 are monic polynomials of degree N and Q is a polynomial of degree $N-1$ subject to $\det(P)=-1$ leads to Ricci-flat ALF metrics with $N+1$ rods and N turning points. The ALE metrics with $N+1$ rods can also be constructed, but from a different ansatz [15, 50].

5. Other developments

5.1. ALF, ALE, ALG, ALH, and more

The ALE and ALF classes of gravitational instantons have been defined in (2.2) and (2.3) in terms of the asymptotic quotients of $\mathbb{R}^4$ and asymptotic S^2 fibrations, respectively. There is an alternative and unifying definition in terms of the volume growth of a ball of large radius R. It is of orders R^4 and R^3 for, respectively, ALE and ALF. This classification gives rise to more families of instantons: ALG and ALG* with the volume growth R^2, ALH with the volume growth R, and ALH* with the volume growth $R^{4/3}$ [5, 6, 32]. Unlike the ALE and ALF, these new families do not contain any examples which are known analytically in closed form. It is however the case that all classes are asymptotically described by the Gibbons–Hawking form (3.2) with the harmonic function given by

$$\begin{aligned} V &\sim \frac{N}{|\boldsymbol{x}|} && \text{for ALE}, \\ V &\sim 1 + \frac{N}{|\boldsymbol{x}|} && \text{for ALF}, \\ V &\sim 1 + \frac{N}{2\pi}\ln\left({x_1}^2 + {x_2}^2\right) && \text{for ALG and ALG*}, \\ V &\sim 1 + N x_3 && \text{for ALH and ALH*}. \end{aligned}$$

Therefore, the metrics are locally asymptotic to $\mathbb{R}^k \times T^{4-k}$ with $k=4$ for ALE, $k=3$ for ALF, $k=2$ for ALG, and $k=1$ for ALH. Let us focus on the ALH* case, and perform an affine transformation of x_3, such that $V = x_3$ in the Gibbons–Hawking ansatz (3.2). The coordinate x_3 is on the base $\mathbb{R}$ of the fibration $M \to \mathbb{R}$. The fibres are Nil 3-manifolds fibering over T^2 with periodic coordinates (x_1, x_2) with the fibre coordinate τ. The one-form A in the ansatz (3.2) is such that $\mathrm{d}A = \mathrm{d}x_1 \wedge \mathrm{d}x_2$ is the volume form on T^2. Setting $x_3 = r^{2/3}$ and rescalling (x_1, x_2, τ) by constants yields

$$g = \mathrm{d}r^2 + r^{2/3}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2\right) + r^{-2/3}\left(\mathrm{d}\tau + A\right)^2 .$$

The volume form is $\mathrm{vol} = r^{1/3}\mathrm{d}r \wedge \mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \mathrm{d}\tau$, so the volume growth is indeed $\int_M \mathrm{vol} \sim R^{4/3}$ if the range of r is bounded by R.

5.2. Einstein–Maxwell instantons

The gravitational instantons exist in the Einstein–Maxwell theory. Unlike the pure Einstein case, there exist many asymptotically flat solutions in the multi-centred class. These solutions arise as analytic continuations of the Israel–Wilson and Majumdar–Papapetrou black holes (see [12, 54, 58]), and are given by

$$g = V\widetilde{V}\left(\mathrm{d}x_1^2 + \mathrm{d}x_2^2 + \mathrm{d}x_3^2\right) + \frac{1}{V\widetilde{V}}(\mathrm{d}\tau + A)^2 , \tag{5.1}$$

where V and $\widetilde{V}$ are harmonic functions on $\mathbb{R}^3$, and the one-form A satisfies

$$\star_3\left(\widetilde{V}\mathrm{d}V - V\mathrm{d}\widetilde{V}\right) = \mathrm{d}A . \tag{5.2}$$

The Maxwell field is given by

$$F = \partial_i\left(V^{-1} - \widetilde{V}^{-1}\right)(\mathrm{d}\tau + A) \wedge \mathrm{d}x^i + \epsilon_{ijk}\partial_k\left(V^{-1} + \widetilde{V}^{-1}\right)V\widetilde{V}\mathrm{d}x^i \wedge \mathrm{d}x^j .$$

If $\tilde{V} = 1$ then (5.2) reduces to the monopole equation (3.3) and the metrics (5.1) are Ricci flat, and coincide with the Gibbons–Hawking ansatz (3.2). If

$$V = V_0 + \sum_{m=1}^{N} \frac{a_m}{|\boldsymbol{x} - \boldsymbol{x}_m|} , \qquad \widetilde{V} = \widetilde{V}_0 + \sum_{m=1}^{N} \frac{\tilde{a}_m}{|\boldsymbol{x} - \tilde{\boldsymbol{x}}_m|}$$

with $V_0, \widetilde{V}_0, a_m, \tilde{a}_m, \boldsymbol{x}_m, \tilde{\boldsymbol{x}}_m$ constant and $N, \widetilde{N}$ integers. In particular, if $V_0 = \widetilde{V}_0 \neq 0$, $N = \widetilde{N}$, and $\sum a_m = \sum \tilde{a}_m$, then the metrics (5.1) are AF. The Riemannian Majumdar–Papapetrou metrics have $V = \widetilde{V}$ and purely magnetic Maxwell field $F = -2 \star_3 \mathrm{d}V$. See [12] for other choices which lead to AE, ALE, and ALF solutions.

There also exist Einstein–Maxwell instantons with no Lorentzian counterpart and anti-self-dual Weyl curvature [39, 40]. An example is the Burns metric

$$g_{\text{Burns}} = \mathrm{d}r^2 + \frac{1}{4}r^2\left({\sigma_1}^2 + {\sigma_2}^2 + {\sigma_3}^2\right) + \frac{m}{4}\left({\sigma_1}^2 + {\sigma_2}^2\right) . \tag{5.3}$$

It is the unique scalar-flat Kähler metric on the total space of the line bundle $\mathcal{O}(-1) \to \mathbb{CP}^1$. It is also an AE Einstein–Maxwell gravitational instanton, with the self-dual part of the Maxwell field strength given by the Kähler form, and its anti-self-dual part given by the Ricci form. It is one of few gravitational instantons where the isometric embedding class is known: It has been shown in [16] that (5.3) can be isometrically embedded in $\mathbb{R}^7$, but not in $\mathbb{R}^6$.

5.3. Twistor theory and non-linear graviton

The twistor non-linear graviton approach of Penrose [46] parametrises holomorphic anti-self-dual Ricci flat metrics in terms of complex three-folds with 4-parameter family of rational curves and some additional structures. The Riemannian version of this correspondence has been given by Atiyah, Hitchin, and Singer [2], where the twistor space is the six-dimensional manifold arising as an S^2-bundle over a Riemannian manifold (M, g). Each fiber of the S^2-fibration parametrises the almost-complex structures in M. The twistor space is itself an almost-complex manifold, and its almost-complex structure is integrable iff (with respect to a chosen orientation on M) the Weyl tensor of g is ASD.

Theorem 5.1 ([46], [2]) *Hyper-Kähler four-manifolds (ASD Ricci flat metrics) are in one-to-one correspondence with three-dimensional complex manifolds (twistor spaces) admitting 4-parameter families of rational curves with some additional structure.*

This formulation is well suited to the study of gravitational instantons. In particular, the ALE class can be fully characterised twistorially [33–36]. In this case, there exists a holomorphic fibration $PT \to \mathcal{O}(k)$ for some integer k. If $k = 2$, then the associated instanton admits a tri-holomorphic Killing vector and belongs to the A_N Gibbons–Hawking class (3.2), [49]. If $k > 2$, then in general (M, g) does not admit a Killing vector, but it admits a tri-holomorphic Killing spinor which leads to a hidden symmetry of the associated heavenly equations [13, 14].

5.4. Euclidean quantum gravity

Euclidean quantum gravity which gave rise to the initial interest in gravitational instantons in the late 1970 does not any more aspire to the status of a fundamental theory of quantum gravity. According to Gibbons's interesting account [27], it never did. And yet, it is the only theory of quantum gravity with experimental predictions, including the black hole thermodynamics. In this theory, the gravitational instantons dominate the Euclidean path integral. So if a quantum gravity theory exists, and if it reduces to Einstein's general relativity in the classical limit, then Euclidean quantum gravity is here to stay, and will occupy a place similar to that the WKB approximation has in the quasi-classical limit relating the quantum mechanics to Newtonian physics. This short and subjective review has focused on recent, and not so recent, mathematical development. It remains to be seen what role will the gravitational instantons play in physics in the years to come.

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