

AXIAL MOMENTUM AND QUANTIZATION
OF THE MAJORANA FIELD*

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A new approach to quantization of the relativistic Majorana field is presented. It is based on the expansion of the field into eigenfunctions of the axial momentum — a novel observable introduced recently. Relativistic invariance is used as the main guiding principle instead of canonical formalism. The hidden structure of the quantized Majorana field in the form of real Clifford algebra of Hermitian fermionic operators is unveiled. All generators of the Poincaré transformations are found as solutions of certain operator equations, without invoking the principle of correspondence with classical conserved quantities. Also, operators of parity \hat{P} and time-reversal \hat{T} are constructed.

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1. Introduction

The Majorana field is a very interesting object of theoretical studies for several reasons. First, it is the most fundamental fermionic field. The more popular Dirac field is, in fact, composed of two Majorana fields [1]. Furthermore, it is still not excluded by experiments that neutrinos are quanta of the Majorana field, see, *e.g.*, [2]. In various attempts to go ‘beyond the Standard Model’ the Majorana neutrinos play an important role. Recently, Majorana fermions have been widely discussed also in condensed matter physics, see, *e.g.*, [3, 4].

Quantized Majorana field was introduced already in the pioneering work [1], straightaway in a remarkably modern form, after a noteworthy discussion of pseudoclassical Majorana field in terms of bispinors with anticommuting components, including the Hamiltonian formulation. As regards recent theoretical works involving the quantized Majorana field, we would like to

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mention works on description of neutrino flavor oscillations, see, *e.g.*, [5] and references therein, and works on the discrete symmetries P and PC, see, *e.g.*, [6] and references therein.

Our paper is devoted to quantization of a classical Majorana field. Let us stress that this classical field should not be confused with the more popular pseudoclassical Majorana field which has anticommuting components and is considered already in [1]. In our paper, the pseudoclassical field is not considered. The classical Majorana field has commuting components. It can be considered in three equivalent forms: as a four-component complex Dirac bispinor subject to the condition of invariance under the charge conjugation; as a complex two-component spinor which appears in the general solution of that condition; or as a four-component real bispinor obtained by taking the real and imaginary parts of the complex spinor, see, *e.g.*, [7, 8]. We prefer the latter form because it is in accordance with the fact that the set of real numbers is the proper algebraic number field for linear space of the Majorana bispinors¹.

There are several reasons for our interest in quantization of the classical Majorana field. Quantization of classical fields usually begins with a Lagrangian, canonical momentum and Hamiltonian formulation, but in the case of classical Majorana field, there is a problem. Our preliminary investigation shows that the classical massive Majorana field has a nonstandard, nonlocal Lagrangian, as opposed to the pseudoclassical Majorana field for which one can use essentially the Dirac Lagrangian. Quantization of this latter field is rather straightforward because one can use the canonical quantization. Quantization of the classical Majorana field is less obvious, and this fact makes it interesting. In the quantization presented below, we do not need the classical Lagrangian at all.

The main feature of the quantum theory of the Majorana field is well-known: it predicts a relativistic, spin-1/2 particle, without an antiparticle [1]. We would like to check whether one can arrive at this theory starting from an expansion of the classical Majorana field into eigenfunctions of the so-called axial momentum, a novel observable which has been introduced and discussed in our earlier papers [9, 10]. This expansion is distinguished by the fact that both the eigenfunctions and all coefficients are real, like the field. This is the second part of our motivation.

There is also a third reason. In the literature, the quantum Majorana field is often obtained by quantizing the complex Dirac field first, and next imposing the condition of invariance under the charge conjugation, see, *e.g.*, Section 6.2.3 in [2]. Such a shortcut procedure, however, requires a certain

¹ We prefer the convenient term ‘bispinor’ because there are four components, not to suggest that the Majorana bispinor is composed of two spinors. A more precise term is ‘real spinor’.

amount of care, as pointed out in [11] and [12]. Therefore, we would like to have a complete in itself quantization in which only the Majorana field is considered. Such a self-contained approach seems to be missing in the literature.

We consider only the massive Majorana field. When the field is massless, a local gauge symmetry is present, see Section 5.2 in [7]. In consequence, the structure of the quantum theory is very different from the massive case. We shall not study it here in order to keep the size of the paper within reasonable limits.

It turns out that the presented below new path to the old result offers interesting insights. As the most important one, we would consider the appearance of the real Clifford algebra generated by the so-called Majorana basis of Hermitian fermionic operators. This algebra seems to be the right mathematical structure for the quantized Majorana field. Another virtue of our approach is that it shows how one can construct the quantum theory essentially by maintaining the relativistic invariance. Last but not least, we see that eigenfunctions of the axial momentum can successfully replace eigenfunctions of the ordinary momentum (*i.e.*, the plane waves) in mode expansion for the Majorana field, and perhaps also for the Dirac field.

The plan of our paper is as follows. In Section 2, we briefly recall the expansion of the classical Majorana field into eigenfunctions of the axial momentum [10] for convenience of the reader and in order to fix notation. Section 3 is devoted to the quantization. The Majorana basis of operators is introduced and generators of the Poincaré transformations are found. The Fock space and particle interpretation are constructed in Section 4. In Section 5 we investigate the Majorana field operator in the Fock space. Section 6 contains explicit construction of operators representing in the Fock space the space inversion (unitary \hat{P}) and the time-reversal (antiunitary \hat{T}). Summary and remarks are presented in Section 7. In Appendix A, we have collected useful formulas pertinent to the Lorentz boosts and the Wigner rotations. Appendix B contains formulas helpful in computations of commutators in Sections 3 and 4.

We use the Minkowski metrics $\eta = \text{diag}(1, -1, -1, -1)$. Throughout the paper, summation over repeated indices is understood.

2. The classical Majorana field in the axial momentum basis

In this section, we fix the notation for the classical Majorana field and remind the axial momentum operator with its eigenfunctions called the axial plane waves. Next, we recall the expansion of the field in the basis of the axial plane waves presented in [10] and adjust it for the present goal. The amplitudes $b^\alpha(\mathbf{p})$ introduced in this expansion are to be replaced by Hermitian operators in the process of quantization.

Let us begin with the Dirac equation for the classical Majorana field $\psi(x)$

$$i\gamma^\mu \partial_\mu \psi(x) - m\psi(x) = 0, \quad (1)$$

where $m > 0$ and all matrices γ^μ are purely imaginary. We use the following matrices:

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, & \gamma^1 &= i \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}, & \gamma^2 &= i \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\ \gamma^3 &= -i \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, & \gamma_5 &= i\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}. \end{aligned}$$

Here, σ_k are the Pauli matrices and σ_0 is the two by two unit matrix. All four components ψ^α of the Majorana field are real-valued functions on the Minkowski spacetime M .

The space of solutions of Eq. (1) is invariant with respect to the Poincaré transformations

$$\psi'(x) = S(L)\psi(L^{-1}(x - a)), \quad (2)$$

where the four-vector a represents a translation in the Minkowski spacetime and L a proper orthochronous Lorentz transformation. The four by four real matrix $S(L)$ obeys the condition

$$S(L)^{-1}\gamma^\mu S(L) = L^\mu_\nu \gamma^\nu.$$

We shall consider also the discrete symmetries P and T

$$\psi_P(t, \mathbf{x}) = \eta_P i\gamma^0 \psi(t, -\mathbf{x}), \quad \psi_T(t, \mathbf{x}) = \eta_T \gamma^0 \gamma_5 \psi(-t, \mathbf{x}), \quad (3)$$

where the numerical factors η_P, η_T are equal to 1 or -1 .

The axial momentum operator has the form of [9]

$$\hat{\mathbf{p}}_5 = -i\gamma_5 \nabla.$$

Its normalized eigenfunctions, called the axial plane waves, read

$$\psi_{\mathbf{p}}(\mathbf{x}) = (2\pi)^{-3/2} \exp(i\gamma_5 \mathbf{p}\mathbf{x}) v,$$

where v an arbitrary real, constant and normalized bispinor, $v^T v = 1$, T denotes the matrix transposition. The matrix $\exp(i\gamma_5 \mathbf{p}\mathbf{x})$ is real and orthogonal, because $\gamma_5^* = -\gamma_5$ and $\gamma_5^T = -\gamma_5$. Thus,

$$\hat{\mathbf{p}}_5 \psi_{\mathbf{p}}(\mathbf{x}) = \mathbf{p} \psi_{\mathbf{p}}(\mathbf{x}), \quad \int d^3x \psi_{\mathbf{p}}^T(\mathbf{x}) \psi_{\mathbf{q}}(\mathbf{x}) = \delta(\mathbf{p} - \mathbf{q}).$$

Note that $\exp(i\gamma_5 \mathbf{p}\mathbf{x}) = \cos(\mathbf{p}\mathbf{x})I_4 + i\gamma_5 \sin(\mathbf{p}\mathbf{x})$. A detailed discussion of rather interesting properties of the axial momentum operator is given in [9] and [10].

The general solution of the Dirac equation (1) can be written as time-dependent superposition of the axial plane waves, see Section 4 in [10],

$$\psi(\mathbf{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3p}{E_p} \left(e^{i\gamma_5(\mathbf{p}\mathbf{x} - E_p t)} v_+(\mathbf{p}) + e^{-i\gamma_5(\mathbf{p}\mathbf{x} - E_p t)} v_-(\mathbf{p}) \right), \quad (4)$$

where $E_p = \sqrt{\mathbf{p}^2 + m^2} > 0$, and $v_{\pm}(\mathbf{p})$ are real bispinors which obey the following relations:

$$\left(E_p \gamma^0 - p^k \gamma^k \right) \gamma_5 v_{\pm}(\mathbf{p}) = \pm m v_{\mp}(\mathbf{p}).$$

Note that p denotes eigenvalues of the axial momentum, not of the standard momentum.

Next, we express the bispinor $v_-(\mathbf{p})$ by $v_+(\mathbf{p})$ and introduce a basis $e_{\alpha}(\mathbf{p})$, $\alpha = 1, 2, 3, 4$, in the linear space of real bispinors $v_+(\mathbf{p})$,

$$v_+(\mathbf{p}) = m \sqrt{E_p} e_{\alpha}(\mathbf{p}) b^{\alpha}(\mathbf{p}).$$

The coefficient $m \sqrt{E_p}$ is introduced in order to give the functions $b^{\alpha}(\mathbf{p})$ the dimension $\text{cm}^{3/2}$ consistent with the postulated anticommutators (8) below. We obtain

$$\psi(\mathbf{x}, t) = \frac{m}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{E_p}} \left[e^{-i\gamma_5 p x} + \frac{1}{m} e^{i\gamma_5 p x} p_{\mu} \gamma^{\mu} \gamma_5 \right] e_{\alpha}(\mathbf{p}) b^{\alpha}(\mathbf{p}), \quad (5)$$

where $p = (p^{\mu}) = (E_p, p^1, p^2, p^3)^{\text{T}}$, $p x = E_p t - \mathbf{p}\mathbf{x}$, and $p_{\mu} \gamma^{\mu} = E_p \gamma^0 - p^k \gamma^k$. Note that because γ_5 anticommutes with γ^{μ} , the order of matrices matters, *e.g.*, $\exp(i\gamma_5 p x) p_{\mu} \gamma^{\mu} = p_{\mu} \gamma^{\mu} \exp(-i\gamma_5 p x)$.

The basis $e_{\alpha}(\mathbf{p})$ is obtained by applying Lorentz boosts $H(\mathbf{p})$ to the rescaled Cartesian basis at $\mathbf{p} = 0$

$$e_{\alpha}^{\eta}(\mathbf{0}) = \frac{\delta_{\alpha\eta}}{\sqrt{m}}; \quad e_{\alpha}(\mathbf{p}) = S(H(\mathbf{p})) e_{\alpha}(\mathbf{0}); \quad H(\mathbf{p}) \overset{(0)}{p} = p,$$

where $\overset{(0)}{p} = (m, 0, 0, 0)^{\text{T}}$. The index $\eta = 1, 2, 3, 4$ enumerates components of the bispinor e_{α} . Bispinors are represented by matrices with one column and four rows. The explicit form of $H(\mathbf{p})$ can be found in Appendix A. The coefficient $1/\sqrt{m}$ is introduced in order to ensure that the basis bispinors e_{α} have the same dimensionality as v_+ , which is $\text{cm}^{1/2}$.

The choice of the basis made above implies the following Lorentz transformations of the functions b^{α} :

$$b'^{\alpha}(\mathbf{p}) = \sqrt{\frac{E_{L^{-1}\mathbf{p}}}{E_p}} S(\mathcal{R}(L, \mathbf{p}))_{\alpha\beta} b^{\beta}(l^{-1}(\mathbf{p})), \quad (6)$$

where the matrix $\mathcal{R}(L, \mathbf{p}) = H^{-1}(\mathbf{p}) L H(l^{-1}(\mathbf{p}))$ leaves $\overset{(0)}{p}$ invariant. Therefore, it represents a rotation, known as the Wigner rotation. The matrix $S(\mathcal{R}(L, \mathbf{p}))$ is real and orthogonal, see Appendix A. The argument $l^{-1}(\mathbf{p})$ of b^β on the r.h.s. of formula (6) is the spatial part of the four-vector $L^{-1}\mathbf{p}$: the i^{th} component of the three-vector $l^{-1}(\mathbf{p})$ is equal to $(L^{-1})^i_k p^k + (L^{-1})^i_0 E_p$.

In the case of translations, $\psi'(x) = \psi(x - a)$. In consequence,

$$b'^\alpha(\mathbf{p}) e_\alpha(\mathbf{p}) = b^\beta(\mathbf{p}) \exp(i\gamma_5 p a) e_\beta(\mathbf{p})$$

and

$$b'^\alpha(\mathbf{p}) = \exp(i\gamma_5 p a)_{\alpha\beta} b^\beta(\mathbf{p}) \quad (7)$$

because

$$\exp(i\gamma_5 p a) e_\beta(\mathbf{p}) = \exp(i\gamma_5 p a)_{\alpha\beta} e_\alpha(\mathbf{p}).$$

The matrix $\exp(i\gamma_5 p a)$ in (7) is real and orthogonal.

Real-valued functions $b^\alpha(\mathbf{p})$ parameterize the space of real solutions of the Dirac equation (1) for the classical Majorana field. They are not restricted by any constraints.

3. Quantization: the algebraic part

Here, we consider algebraic aspects of the would-be quantum operators. The actual realization of them as Hermitian operators is possible only when we introduce a Hilbert space, *e.g.*, the Fock space defined in the next section. For convenience, we shall use the term ‘operator’ already in the present section.

The quantized Majorana field $\hat{\psi}(x)$ is obtained by replacing the real-valued functions $b^\alpha(\mathbf{p})$ with Hermitian operator valued functions $\hat{b}^\alpha(\mathbf{p})^2$, which obey the anticommutation constraints³

$$\left[\hat{b}^\alpha(\mathbf{p}), \hat{b}^\beta(\mathbf{q}) \right]_+ = \kappa_{\alpha\beta}(\mathbf{p}) I \delta(\mathbf{p} - \mathbf{q}),$$

where $\kappa_{\alpha\beta}(\mathbf{p})$ are real-valued, dimensionless functions of \mathbf{p} , $\kappa_{\alpha\beta}(\mathbf{p}) = \kappa_{\beta\alpha}(\mathbf{p})$, and I is the identity operator. We demand that these constraints are invariant with respect to the Lorentz and translation transformations (6) and (7). Simple calculations show that in this case, $\kappa_{\alpha\beta}(\mathbf{p}) = \kappa_0 \delta_{\alpha\beta}$, where κ_0 is a real, dimensionless, positive constant. Rescaling the operators \hat{b}^α , we may put $\kappa_0 = 1$. Thus,

$$\left[\hat{b}^\alpha(\mathbf{p}), \hat{b}^\beta(\mathbf{q}) \right]_+ = \delta_{\alpha\beta} I \delta(\mathbf{p} - \mathbf{q}). \quad (8)$$

Such a set of Hermitian fermionic operators is known in condensed matter physics as the Majorana basis of operators.

² In fact, they turn out to be operator-valued generalized functions of \mathbf{p} .

³ We anticipate the fermionic character of the field.

The crucial point to be addressed is the relativistic invariance of the quantum theory. In such a theory, transformations (6) and (7) should be implemented by unitary operators $U(\Lambda, a)$, where $\Lambda \in \text{SL}(2, \mathbb{C})$. $\text{SL}(2, \mathbb{C})$ is the universal covering group of the proper orthochronous Lorentz group. The two groups are isomorphic in a vicinity of the unit element. Thus, Λ can be regarded as a function of the Lorentz transformation L . For a detailed discussion of the relation between these two groups see, *e.g.*, [14]. Our task is to find the operators $U(\Lambda, a)$. To this end, it suffices to consider infinitesimal transformations, *i.e.*, from a small vicinity of the unit element. Then

$$U(\Lambda(L), a) \cong I + ia^\mu \hat{P}_\mu + \frac{i}{2} \omega^{\mu\nu} \hat{M}_{\mu\nu},$$

where a^μ parameterize translations in the spacetime and $\omega_{\mu\nu} = -\omega_{\nu\mu}$ parameterize the proper orthochronous Lorentz group around I_4 , namely $L = \exp \omega$, where $\omega = (\omega^\mu{}_\nu)$. Since $\hat{M}_{\mu\nu} = -\hat{M}_{\nu\mu}$, there are six independent generators of Lorentz transformations.

The Hermitian operators

$$\hat{P}_\mu = -i \left. \frac{\partial U(\sigma_0, a)}{\partial a^\mu} \right|_{a=0}, \quad \hat{M}_{\mu\nu} = -i \left. \frac{\partial U(\Lambda(L), a=0)}{\partial \omega^{\mu\nu}} \right|_{\omega=0} \quad (9)$$

are identified with, respectively, the total four-momentum and the total angular momentum of the field — the most important observables for the quantized field. The matrix σ_0 appears because it is the unit element in the $\text{SL}(2, \mathbb{C})$ group. In the second formula (9) the matrix Λ is regarded as a function of the Lorentz matrix L . The operators \hat{P}_μ , $\hat{M}_{\mu\nu}$ are called the generators of the representation because they essentially determine the representation $U(\Lambda(L), a)$ [13].

In the case of translations, the postulated quantum version of condition (7) reads

$$U^{-1}(\sigma_0, a) \hat{b}^\alpha(\mathbf{p}) U(\sigma_0, a) = \exp(i\gamma_5 p a)_{\alpha\beta} \hat{b}^\beta(\mathbf{p}).$$

Differentiating both sides of this formula with respect to a^μ and putting $a = 0$, we obtain the condition for \hat{P}_μ

$$\left[\hat{P}_\mu, \hat{b}^\alpha(\mathbf{p}) \right] = -p_\mu (\gamma_5)_{\alpha\beta} \hat{b}^\beta(\mathbf{p}), \quad (10)$$

where $p_0 = E_p = \sqrt{\mathbf{p}^2 + m^2}$. As shown in Appendix B, there is a general formula for operators satisfying such conditions. Condition (10) has the form of formula (B.1) with \hat{P}_μ and $r_{\alpha\beta}(\mathbf{p}, \mathbf{q}) = ip_\mu (\gamma_5)_{\alpha\beta} \delta(\mathbf{p} - \mathbf{q})$ in the place of \hat{X} and $x_{\alpha\beta}(\mathbf{p}, \mathbf{q})$, respectively. Formula (B.2) gives the following Hermitian operators:

$$\hat{P}_\mu = \frac{1}{2} \int d^3p p_\mu \hat{b}^\alpha(\mathbf{p}) (\gamma_5)_{\alpha\beta} \hat{b}^\beta(\mathbf{p}) + d_\mu I, \quad (11)$$

where d_μ are arbitrary real constants. These operators commute with each other

$$[\hat{P}_\mu, \hat{P}_\nu] = 0$$

as expected for the generators of translations. This can be checked with the help of formulas (B.3), (B.4).

Similar, but more tedious calculations give the generators $\hat{M}_{\mu\nu}$. In order to obtain conditions for $\hat{M}_{\mu\nu}$ analogous to (10), we postulate the quantum version of the Lorentz transformation (6)

$$U^{-1}(\Lambda(L), 0) \hat{b}^\alpha(\mathbf{p}) U(\Lambda(L), 0) = \sqrt{\frac{E_{L^{-1}p}}{E_p}} S(\mathcal{R}(L, \mathbf{p}))_{\alpha\beta} \hat{b}^\beta(l^{-1}(\mathbf{p})). \quad (12)$$

We differentiate both sides of formula (12) with respect to ω_{jk} and next we put $\omega = 0$. This gives the following condition for \hat{M}_{jk} :

$$[\hat{M}_{jk}, \hat{b}^\alpha(\mathbf{p})] = \frac{i}{4} [\gamma^j, \gamma^k]_{\alpha\beta} \hat{b}^\beta(\mathbf{p}) - i \left(p^j \frac{\partial}{\partial p^k} - p^k \frac{\partial}{\partial p^j} \right) \hat{b}^\alpha(\mathbf{p}). \quad (13)$$

Calculations giving the r.h.s. are explained in Appendix A. Formula (13) has the form as in (B.1) with

$$m_{\alpha\beta}^{jk}(\mathbf{p}, \mathbf{q}) = \frac{1}{4} [\gamma^j, \gamma^k]_{\alpha\beta} \delta(\mathbf{p} - \mathbf{q}) - \left(p^j \frac{\partial}{\partial p^k} - p^k \frac{\partial}{\partial p^j} \right) \delta_{\alpha\beta} \delta(\mathbf{p} - \mathbf{q})$$

in place of $x_{\alpha\beta}(\mathbf{p}, \mathbf{q})$. Formula (B.2) gives

$$\begin{aligned} \hat{M}_{jk} &= -\frac{i}{8} \int d^3p \hat{b}^\alpha(\mathbf{p}) [\gamma^j, \gamma^k]_{\alpha\beta} \hat{b}^\beta(\mathbf{p}) \\ &\quad + \frac{i}{2} \int d^3p \hat{b}^\alpha(\mathbf{p}) \left(p^j \frac{\partial}{\partial p^k} - p^k \frac{\partial}{\partial p^j} \right) \hat{b}^\alpha(\mathbf{p}) + d_{jk} I, \end{aligned} \quad (14)$$

where $d_{jk} = -d_{kj}$ are arbitrary real constants.

Similarly, differentiation of formula (12) with respect to ω_{0k} at $\omega = 0$ gives the condition (see Appendix A for details of the calculation of the r.h.s.)

$$\begin{aligned} [\hat{M}_{0j}, \hat{b}^\alpha(\mathbf{p})] &= -\frac{i}{4} \frac{1}{E_p + m} (p^r \delta_{sj} - p^s \delta_{rj}) (\gamma^r \gamma^s)_{\alpha\beta} \hat{b}^\beta(\mathbf{p}) \\ &\quad + \frac{i}{2} \int d^3q \left(\left(E_p \frac{\partial}{\partial p^j} - E_q \frac{\partial}{\partial q^j} \right) \delta(\mathbf{p} - \mathbf{q}) \right) \hat{b}^\alpha(\mathbf{q}), \end{aligned} \quad (15)$$

which is satisfied by

$$\begin{aligned} \hat{M}_{0j} = & \frac{i}{8} \int d^3p \frac{1}{E_p + m} (p^r \delta_{sj} - p^s \delta_{rj}) \hat{b}^\alpha(\mathbf{p}) (\gamma^r \gamma^s)_{\alpha\beta} \hat{b}^\beta(\mathbf{p}) \\ & - \frac{i}{4} \int d^3p d^3q \hat{b}^\alpha(\mathbf{p}) \hat{b}^\alpha(\mathbf{q}) \left(E_p \frac{\partial}{\partial p^j} - E_q \frac{\partial}{\partial q^j} \right) \delta(\mathbf{p} - \mathbf{q}) + d_{0j} I, \end{aligned} \quad (16)$$

where the real constants d_{0j} are arbitrary.

The generators should obey certain commutator relations which follow directly from the properties of the Poincaré group, see, *e.g.*, [13]. In the parameterization of the Poincaré group introduced above formula (9), they have the form of

$$\left[\hat{P}_\mu, \hat{P}_\nu \right] = 0, \quad (17)$$

$$\left[\hat{M}_{\rho\lambda}, \hat{P}_\mu \right] = i \left(\eta_{\mu\rho} \hat{P}_\lambda - \eta_{\mu\lambda} \hat{P}_\rho \right), \quad (18)$$

$$\left[\hat{M}_{\alpha\beta}, \hat{M}_{\mu\nu} \right] = i \left(\eta_{\alpha\mu} \hat{M}_{\beta\nu} - \eta_{\alpha\nu} \hat{M}_{\beta\mu} - \eta_{\beta\mu} \hat{M}_{\alpha\nu} + \eta_{\beta\nu} \hat{M}_{\alpha\mu} \right), \quad (19)$$

where $\eta_{\mu\rho}$ are components of the Minkowski metric tensor.

Our operators (11), (14), and (16) do not obey the commutation relations (18), (19) unless the constants d_μ , $d_{\mu\nu}$ vanish. To show this, we use formulas (B.3), (B.4) from Appendix B for the terms with the \hat{b}^α operators. It turns out that already these terms alone satisfy commutators (18), (19). The terms with the identity operator of course give vanishing contributions to the l.h.s. of commutators (18) and (19), but they explicitly appear on the r.h.s.'s. In this way, we obtain from (18), (19) the following conditions:

$$0 = \eta_{\mu\rho} d_\nu - \eta_{\mu\nu} d_\rho, \quad 0 = \eta_{\alpha\mu} d_{\beta\nu} - \eta_{\alpha\nu} d_{\beta\mu} + \eta_{\beta\nu} d_{\alpha\mu} - \eta_{\beta\mu} d_{\alpha\nu},$$

which imply that $d_\lambda = 0$ and $d_{\mu\nu} = 0$. Nevertheless, we shall keep these constants nonvanishing. The reason is that we prefer the normal ordered generators in the Fock space, discussed in the next section, because their eigenvalues are consistent with the particle interpretation. Indeed, the normal ordered generators are regarded as physical observables for the quantized field, in particular as its total four-momentum and total angular momentum. The normal ordering can be interpreted as a special choice of the constants d_μ , $d_{\mu\nu}$ ⁴. Thus, because we insist on having the particle interpretation, the relativistic invariance of the model critically depends on whether the normal ordered generators, and not the ones above, obey the commutator relations (17), (18), and (19).

⁴ In this case, the constants are given by integrals over \mathbf{p} which are divergent unless there is a cutoff.

Note that the Poincaré generators contain the imaginary unit i as an overall coefficient. Therefore, the operator $U(L(\Lambda), a)$ contains products of the operators $\hat{b}^\alpha(\mathbf{p})$ with only real coefficients. Thus, the algebraic structure described above is a real, infinite-dimensional Clifford algebra with the operators $\hat{b}^\alpha(\mathbf{p})$ as its Hermitian generating elements.

4. The complex Fock space and particle interpretation

First, let us introduce two annihilation and two creation operators $\hat{a}_\lambda(\mathbf{p})$, $\hat{a}_\lambda^\dagger(\mathbf{p})$, where $\lambda = 1, 2$

$$\hat{a}_\lambda(\mathbf{p}) = c_{\lambda\alpha} \hat{b}^\alpha(\mathbf{p}), \quad \hat{a}_\lambda^\dagger(\mathbf{p}) = c_{\lambda\alpha}^* \hat{b}^\alpha(\mathbf{p}), \quad (20)$$

where $c_{\lambda\alpha}$ are constants, $*$ denotes the complex conjugation, $\alpha = 1, 2, 3, 4$. By assumption, these operators have the following anticommutators:

$$\left[\hat{a}_\lambda^\dagger(\mathbf{p}), \hat{a}_\sigma(\mathbf{q}) \right]_+ = \delta_{\lambda\sigma} \delta(\mathbf{p} - \mathbf{q}) I, \quad (21)$$

$$\left[\hat{a}_\lambda^\dagger(\mathbf{p}), \hat{a}_\sigma^\dagger(\mathbf{q}) \right]_+ = 0, \quad [\hat{a}_\lambda(\mathbf{p}), \hat{a}_\sigma(\mathbf{q})]_+ = 0. \quad (22)$$

Inserting formulas (20) and using anticommutators (8), we obtain from of (21), (22) constraints for the constants $c_{\lambda\alpha}$, namely

$$c_{\lambda\alpha} c_{\sigma\alpha} = 0, \quad c_{\lambda\alpha} c_{\sigma\alpha}^* = \delta_{\lambda\sigma}. \quad (23)$$

These constraints do not have a unique solution because their l.h.s.'s are invariant with respect to arbitrary transformations of the form of $c'_{\lambda\alpha} = c_{\lambda\beta} \mathcal{O}_{\beta\alpha}$, where $\mathcal{O}_{\beta\alpha}$ form a real, orthogonal, four by four matrix.

With the Fock space and particle interpretation as the goal, we would like to obtain the four-momentum operator in the form typical for the quantum theory of free fields. In our case, it reads

$$:\hat{P}_\mu: = \int d^3p p_\mu \hat{a}_\lambda^\dagger(\mathbf{p}) \hat{a}_\lambda(\mathbf{p}), \quad (24)$$

where $p_0 = \sqrt{\mathbf{p}^2 + m^2}$, $\lambda = 1, 2$, and $: :$ denotes the normal ordering of products of the operators $\hat{a}_\lambda, \hat{a}_\lambda^\dagger$. We start from formula (11) for the operator \hat{P}_μ , in which we substitute

$$\hat{b}^\alpha(\mathbf{p}) = c_{\lambda\alpha}^* \hat{a}_\lambda(\mathbf{p}) + c_{\lambda\alpha} \hat{a}_\lambda^\dagger(\mathbf{p}). \quad (25)$$

This formula is inverse to (20). The constants $c_{\lambda\alpha}, c_{\lambda\alpha}^*$ obey conditions (23). In the resulting expression for \hat{P}_μ , there are unwanted terms with the products $\hat{a}_\lambda \hat{a}_\sigma$ and $\hat{a}_\lambda^\dagger \hat{a}_\sigma^\dagger$. These terms vanish when the constants $c_{\lambda\alpha}$ obey the following conditions:

$$c_{\lambda\alpha} (\gamma_5)_{\alpha\beta} c_{\sigma\beta} = 0. \quad (26)$$

If we add yet another condition, namely,

$$c_{\lambda\alpha}(\gamma_5)_{\alpha\beta}c_{\sigma\beta}^* = \delta_{\lambda\sigma}, \quad (27)$$

the four-momentum operator acquires the form

$$\hat{P}_\mu = \frac{1}{2} \int d^3p p_\mu \left(\hat{a}_\lambda^\dagger(\mathbf{p}) \hat{a}_\lambda(\mathbf{p}) - \hat{a}_\lambda(\mathbf{p}) \hat{a}_\lambda^\dagger(\mathbf{p}) \right) + d_\mu I.$$

In the last step, we apply the normal ordering and drop the term $d_\mu I$. On a heuristic level, one can say that the term generated by the normal ordering, which is proportional to I , is canceled by the term $d_\mu I$ with appropriately chosen constant d_μ . The final form of the four-momentum operator is given by formula (24).

At this point, it is clear that we may use the standard Fock space as the complex Hilbert space for the quantized Majorana field. In particular, the Fock vacuum state $|0\rangle$ is defined by the conditions

$$\hat{a}_1(\mathbf{p})|0\rangle = 0, \quad \hat{a}_2(\mathbf{p})|0\rangle = 0. \quad (28)$$

The Fock basis of quantum states of the field is created by the operators $\hat{a}_\lambda^\dagger(\mathbf{p})$ acting on the vacuum state. Such states are eigenvectors of the $:\hat{P}_\mu:$ operators. Note that the Fock space is linear over the set of complex numbers \mathbb{C} , while the classical Majorana field is real-valued.

We see that the energy operator $:\hat{P}_0:$ is non-negative. Let us recall that when quantizing the Dirac field at a certain stage, one has to redefine the vacuum state: an empty vacuum defined by conditions analogous to (28) is abandoned in favor of the state known as the Dirac sea, otherwise, the energy operator is not bounded from below. In the Majorana case, such redefinition is not needed. In this sense, the Dirac sea is absent here.

Condition (27) is satisfied if $c_{\sigma\beta}^*$ are components of two orthonormal eigenspinors of the matrix γ_5 corresponding to eigenvalue $+1$. Thus,

$$(\gamma_5)_{\alpha\beta}c_{\sigma\beta}^* = c_{\sigma\alpha}^*$$

($\sigma = 1, 2$ enumerates the eigenvectors). Since γ_5 is imaginary, $c_{\sigma\beta}$ give other two eigenspinors which correspond to the eigenvalue -1 . Condition (26) is then reduced to $c_{\lambda\alpha} c_{\sigma\alpha} = 0$, which coincides with the first condition in (23). Also, the second condition in (23) is satisfied if the eigenspinors are orthonormal. The eigenvalues of γ_5 are double degenerate, hence the orthonormal eigenspinors are not fixed uniquely. We take

$$(c_{1\alpha}^*) = \frac{1}{\sqrt{2}} \begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad (c_{2\alpha}^*) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix}.$$

The resulting relation between the operators \hat{b}^α and \hat{a}_λ , \hat{a}_λ^\dagger can be summarized in the matrix form

$$\begin{pmatrix} \hat{b}^1(\mathbf{p}) \\ \hat{b}^2(\mathbf{p}) \\ \hat{b}^3(\mathbf{p}) \\ \hat{b}^4(\mathbf{p}) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 0 & -i & 0 \\ 0 & i & 0 & -i \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{a}_1(\mathbf{p}) \\ \hat{a}_2(\mathbf{p}) \\ \hat{a}_1^\dagger(\mathbf{p}) \\ \hat{a}_2^\dagger(\mathbf{p}) \end{pmatrix}. \quad (29)$$

The matrix on the r.h.s. of formula (29) (with the factor $1/\sqrt{2}$ included) is unitary.

The three generators of spatial rotations, $:\hat{M}_{12}:$, $:\hat{M}_{23}:$, and $:\hat{M}_{31}:$, are obtained from formula (14). We use formula (29) for $\hat{b}^\alpha(\mathbf{p})$, next we apply the normal ordering, and remove by hand the terms proportional to the identity operator I . It turns out that the spin part comes out in a nonstandard form, namely $\hat{M}_{12} \supset \sigma_1$, $\hat{M}_{23} \supset \sigma_2$, $\hat{M}_{31} \supset \sigma_3$. Therefore, we apply additional unitary transformation

$$\begin{pmatrix} \hat{a}_1(\mathbf{p}) \\ \hat{a}_2(\mathbf{p}) \end{pmatrix} = Q \begin{pmatrix} \hat{d}_1(\mathbf{p}) \\ \hat{d}_2(\mathbf{p}) \end{pmatrix}, \quad (30)$$

with the matrix Q

$$Q = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

This transformation cyclically permutes the Pauli matrices⁵

$$Q^\dagger \sigma_1 Q = \sigma_3, \quad Q^\dagger \sigma_2 Q = \sigma_1, \quad Q^\dagger \sigma_3 Q = \sigma_2.$$

Anticommutators of the operators $\hat{d}_\lambda(\mathbf{p})$, $\hat{d}_\lambda^\dagger(\mathbf{p})$ have the form of (21), (22) of course,

$$\begin{aligned} \left[\hat{d}_\lambda^\dagger(\mathbf{p}), \hat{d}_\sigma(\mathbf{q}) \right]_+ &= \delta_{\lambda\sigma} \delta(\mathbf{p} - \mathbf{q}) I, \\ \left[\hat{d}_\lambda^\dagger(\mathbf{p}), \hat{d}_\sigma^\dagger(\mathbf{q}) \right]_+ &= 0, \quad \left[\hat{d}_\lambda(\mathbf{p}), \hat{d}_\sigma(\mathbf{q}) \right]_+ = 0. \end{aligned}$$

Definition (28) of the vacuum state can be equivalently written as

$$\hat{d}_\lambda(\mathbf{p})|0\rangle = 0.$$

Transformations (29) and (30) together give the following unitary transformation:

$$\begin{pmatrix} \hat{b}^1(\mathbf{p}) \\ \hat{b}^2(\mathbf{p}) \\ \hat{b}^3(\mathbf{p}) \\ \hat{b}^4(\mathbf{p}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} i & 1 & -i & 1 \\ i & -1 & -i & -1 \\ 1 & -i & 1 & i \\ 1 & i & 1 & -i \end{pmatrix} \begin{pmatrix} \hat{d}_1(\mathbf{p}) \\ \hat{d}_2(\mathbf{p}) \\ \hat{d}_1^\dagger(\mathbf{p}) \\ \hat{d}_2^\dagger(\mathbf{p}) \end{pmatrix}. \quad (31)$$

⁵ It follows that Q^3 commutes with the Pauli matrices. Indeed, $Q^3 = \exp(i\pi/4)\sigma_0$.

Transformation (30) applied in the four-momentum operator gives

$$:\hat{P}_\mu: = \int d^3p p_\mu \hat{d}_\lambda^\dagger(\mathbf{p}) \hat{d}_\lambda(\mathbf{p}).$$

The final result for the generators \hat{M}_{ik} reads

$$:\hat{M}_{ik} := \epsilon_{ikj} \int d^3p \hat{d}_\lambda^\dagger(\mathbf{p}) \left(-\frac{1}{2}(\sigma_j)_{\lambda\eta} + i\delta_{\lambda\eta} \epsilon_{jmn} p^m \frac{\partial}{\partial p^n} \right) \hat{d}_\eta(\mathbf{p}). \quad (32)$$

Here, $(\sigma_j)_{\lambda\eta}$ denotes matrix elements of the Pauli matrices and ϵ_{ikl} is the totally antisymmetric symbol, $\epsilon_{123} = +1$. We notice the spin part with the Pauli matrices and the angular momentum part with the derivatives $\partial/\partial p^n$.

The generators of boosts $:\hat{M}_{0k}:$ are obtained from formula (16) in a similar manner

$$:\hat{M}_{0k} := \int d^3p \hat{d}_\lambda^\dagger(\mathbf{p}) \left(\frac{\epsilon_{kjl} p^l}{2(E_p + m)} (\sigma_j)_{\lambda\eta} - i \left(E_p \frac{\partial}{\partial p^k} + \frac{p^k}{2E_p} \right) \delta_{\lambda\eta} \right) \hat{d}_\eta(\mathbf{p}). \quad (33)$$

The normal ordered generators satisfy commutator relations (17), (18), and (19). This can be checked with the help of formula (B.5) from Appendix B. We conclude that the constructed quantum model in the Fock space is relativistically invariant.

The index $\lambda = 1, 2$ can be related to the eigenvalues of the operator $\hat{\Sigma}^3$, which is the 3rd component of the spin operator

$$\hat{\Sigma}^i = \frac{1}{2} \int d^3p \hat{d}_\kappa^\dagger(\mathbf{p}) (\sigma_i)_{\kappa\eta} \hat{d}_\eta(\mathbf{p})$$

in the single-particle subspace of the Fock space. Such a subspace is spanned on the basis states

$$|\mathbf{p} \lambda\rangle = \hat{d}_\lambda^\dagger(\mathbf{p}) |0\rangle,$$

which are eigenstates of $:\hat{P}^\mu:$ and $\hat{\Sigma}^3$

$$:\hat{P}^\mu: |\mathbf{p} \lambda\rangle = p^\mu |\mathbf{p} \lambda\rangle, \quad \hat{\Sigma}^3 |\mathbf{p} 1\rangle = \frac{1}{2} |\mathbf{p} 1\rangle, \quad \hat{\Sigma}^3 |\mathbf{p} 2\rangle = -\frac{1}{2} |\mathbf{p} 2\rangle,$$

where $p^0 = \sqrt{\mathbf{p}^2 + m^2}$.

The general single-particle state has the form of $|\psi\rangle = \int d^3p \psi_\lambda(\mathbf{p}) |\mathbf{p} \lambda\rangle$. Such states form a subspace of the Fock space which is invariant under the Poincaré transformations generated by the operators $:\hat{P}_\mu:$, $:\hat{M}_{ik}:$, and $:\hat{M}_{0k}:$. In fact, we have obtained the unitary irreducible representation of the Poincaré group in the single-particle subspace of the Fock space. It is characterised by spin-1/2, the time-like four momentum $p_\mu p^\mu = m^2 > 0$, and the positive energy $E_p > 0$.

5. The field operator in the Fock space

The quantum field operator in the Fock space is obtained from formula (5) by replacing the classical variables $b^\alpha(\mathbf{p})$ with the operators $\hat{b}^\alpha(\mathbf{p})$

$$\hat{\psi}(\mathbf{x}, t) = \frac{m}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{E_p}} \left[e^{-i\gamma_5 p x} + \frac{1}{m} e^{i\gamma_5 p x} p_\mu \gamma^\mu \gamma_5 \right] e_\alpha(\mathbf{p}) \hat{b}^\alpha(\mathbf{p}), \quad (34)$$

where $\hat{b}^\alpha(\mathbf{p})$ are to be eliminated with the help of relation (31).

This form of the field operator can be significantly modified. First, using the definition of the basis $e_\alpha(\mathbf{p})$ and formula $S(H(\mathbf{p}))^{-1} p_\mu \gamma^\mu S(H(\mathbf{p})) = m\gamma^0$, we may rewrite it as

$$\begin{aligned} \hat{\psi}(\mathbf{x}, t) = & \frac{m}{(2\pi)^{3/2}} \int \frac{d^3p}{\sqrt{E_p}} S(H(\mathbf{p})) \left[e^{i\gamma_5(\mathbf{p}\mathbf{x} - E_p t)} \right. \\ & \left. + e^{-i\gamma_5(\mathbf{p}\mathbf{x} - E_p t)} \gamma^0 \gamma_5 \right] e_\alpha(\mathbf{0}) \hat{b}^\alpha(\mathbf{p}). \end{aligned}$$

Next, we substitute $\exp(\pm i\gamma_5 p x) = \cos(px)I_4 \pm i\gamma_5 \sin(px)$ and notice that

the matrices $i\gamma_5$, $i\gamma^0$, and $\gamma^0\gamma_5 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$ merely permute

the basis bispinors $e_\alpha(\mathbf{0})$ and, in some cases, change their sign. The sin and cos functions are expressed by $\exp(\pm ipa)$ functions. Finally, we introduce the creation and annihilation operators using formula (31). After all these steps, we obtain the field operator in a more transparent form

$$\hat{\psi}(\mathbf{x}, t) = \sqrt{m} \int \frac{d^3p}{\sqrt{(2\pi)^3 E_p}} \left[e^{-ipx} v_\lambda(\mathbf{p}) \hat{d}_\lambda(\mathbf{p}) + e^{ipx} v_\lambda^*(\mathbf{p}) \hat{d}_\lambda^\dagger(\mathbf{p}) \right], \quad (35)$$

where $v_1(\mathbf{p}) = S(H(\mathbf{p}))v_1$, $v_2(\mathbf{p}) = S(H(\mathbf{p}))v_2$, and

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \quad v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}.$$

The constant bispinors v_λ, v_λ^* are normalized eigenvectors of the matrix γ^0

$$\gamma^0 v_\lambda = v_\lambda, \quad \gamma^0 v_\lambda^* = -v_\lambda^*. \quad (36)$$

The definition of the boosted bispinors $v_1(\mathbf{p}), v_2(\mathbf{p})$ implies that

$$(m\gamma^0 + p^i \gamma^0 \gamma^i) v_\lambda(\mathbf{p}) = E_p v_\lambda(\mathbf{p}), \quad \overline{v_\lambda(\mathbf{p})} v_\sigma(\mathbf{p}) = \delta_{\lambda\sigma},$$

where $\overline{v_\lambda(\mathbf{p})} = v_\lambda^\dagger(\mathbf{p})\gamma^0$. The first equation shows that they are eigenvectors of the Hermitian matrix $m\gamma^0 + p^i\gamma^0\gamma^i$ (which coincides with the Dirac Hamiltonian), while the second one gives the normalization of the boosted bispinors. For the boosted complex conjugate bispinors, $v_1^*(\mathbf{p}) = S(H(\mathbf{p}))v_1^*$ and $v_2^*(\mathbf{p}) = S(H(\mathbf{p}))v_2^*$, we have

$$(m\gamma^0 + p^i\gamma^0\gamma^i)v_\lambda^*(-\mathbf{p}) = -E_p v_\lambda^*(-\mathbf{p}), \quad \overline{v_\lambda^*(\mathbf{p})}v_\sigma^*(\mathbf{p}) = -\delta_{\lambda\sigma}.$$

The quantized Majorana field obeys the Dirac equation. First, using formulas (35) and (36), we write

$$m\hat{\psi}(\mathbf{x}, t) = \int \frac{d^3p \sqrt{m}}{\sqrt{(2\pi)^3 E_p}} S(H(\mathbf{p})) \left[e^{-ipx} m\gamma^0 v_\lambda \hat{d}_\lambda(\mathbf{p}) - e^{ipx} m\gamma^0 v_\lambda^* \hat{d}_\lambda^\dagger(\mathbf{p}) \right].$$

Next, we apply on the r.h.s. the formula $S(H(\mathbf{p}))m\gamma^0 S(H(\mathbf{p}))^{-1} = p_\mu\gamma^\mu$

$$\begin{aligned} m\hat{\psi}(\mathbf{x}, t) &= \int \frac{d^3p \sqrt{m}}{\sqrt{(2\pi)^3 E_p}} \left[e^{-ipx} p_\mu\gamma^\mu S(H(\mathbf{p}))v_\lambda \hat{d}_\lambda(\mathbf{p}) \right. \\ &\quad \left. - e^{ipx} p_\mu\gamma^\mu S(H(\mathbf{p}))v_\lambda^* \hat{d}_\lambda^\dagger(\mathbf{p}) \right] = i\gamma^\mu \partial_\mu \hat{\psi}(x). \end{aligned}$$

The field $\hat{\psi}(\mathbf{x}, t)$ obeys also the following equation:

$$\partial_t \hat{\psi}(\mathbf{x}, t) = i \left[: \hat{P}_0 : , \hat{\psi}(\mathbf{x}, t) \right]. \quad (37)$$

It has the form of the evolution equation for quantum operators in the Heisenberg picture. For this reason, we may consider $: \hat{P}_0 :$ as the Hamiltonian of the quantized Majorana field, and the time-dependent field $\hat{\psi}(\mathbf{x}, t)$ as the field operator in the Heisenberg picture. It is an important conceptual step. The point is that the presented above quantization is not based on canonical formalism, in particular, we have not considered any classical Hamiltonian. The operator $: \hat{P}_0 :$ has been introduced in Section 4 as the generator of time translations in the context of representations of the Poincaré group, not as a quantum version of certain classical Hamiltonian.

The field operator (35) is local in the sense proper for fermionic fields: bilinear local operators of the form $\hat{\psi}^\alpha(x)A_{\alpha\beta}\hat{\psi}^\beta(x)$, $\hat{\psi}^\alpha(y)B_{\alpha\beta}\hat{\psi}^\beta(y)$, where $A_{\alpha\beta}, B_{\alpha\beta}$ are complex numbers, commute if $(x-y)^2 < 0$. The commutator of such operators is proportional to the anticommutator $[\hat{\psi}^\alpha(x), \hat{\psi}^\beta(y)]_+$ which can be calculated easily

$$\left[\hat{\psi}^\alpha(x), \hat{\psi}^\beta(y) \right]_+ = \left(\left(imI_4 - \gamma^\mu \frac{\partial}{\partial x^\mu} \right) \gamma^0 \right)_{\alpha\beta} \Delta(x-y), \quad (38)$$

where $\Delta(x-y)$ is the Jordan–Pauli function, $\Delta(x-y) = 0$ if $(x-y)^2 < 0$.

Let us close this section with a remark on the well-known current density $j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x)$, where ψ is the classical Majorana field, $\bar{\psi}(x) = \psi(x)^T\gamma^0$. Its existence poses a puzzle. Because the current is conserved, $\partial_\mu j^\mu = 0$ if ψ is a solution of the Dirac equation (1), there exists the conserved charge $Q = \int d^3x j^0(\mathbf{x}, t)$. Usually, such a conserved charge is associated with a U(1) symmetry, or rather SO(2) because the field is real — we invoke here the inverse Noether's theorem. However, in the Majorana case, it is hard to point out such a symmetry. Analyzing this problem, the first question is about the existence of Lagrangian for the classical real-valued Majorana field, because a Lagrangian is needed in the Noether theorem. Rather surprisingly, it turns out that the answer is not quite trivial. Since this topic clearly lies far outside the scope of the present paper, we will not pursue it here. (We plan a separate manuscript devoted to it.) Instead, let us consider the quantum counterpart of the current density $j^\mu(x)$. In order to write it, one has to regularize the product of field operators preserving the Hermiticity of the current. We choose the point-splitting regularization,

$$\hat{j}_\epsilon^\mu(x) = \frac{1}{2} \left(\hat{\psi}(x)^T \gamma^0 \gamma^\mu \hat{\psi}(x + \epsilon) + \hat{\psi}(x + \epsilon)^T \gamma^0 \gamma^\mu \hat{\psi}(x) \right),$$

where the four-vector ϵ is constant and non-vanishing. Since all matrices $\gamma^0\gamma^\mu$ are symmetric,

$$\begin{aligned} \hat{j}_\epsilon^\mu(x) &= \frac{1}{2} (\gamma^0\gamma^\mu)_{\beta\alpha} \left[\psi^\alpha(x), \psi^\beta(x + \epsilon) \right]_+ \\ &= \frac{1}{2} \text{Tr} \left(\gamma^0\gamma^\mu \left(imI_4 - \gamma^\nu \frac{\partial}{\partial x^\nu} \right) \gamma^0 \right) \Delta(-\epsilon) = 0. \end{aligned}$$

The last equality follows from $\text{Tr} \gamma^\mu = 0$ and $\partial\Delta(-\epsilon)/\partial x^\nu = 0$. This result for the regularized current suggests that the current and the related charge may not exist in the quantum theory of the Majorana field.

6. The discrete symmetries P and T

6.1. The space inversion P

As mentioned in Section 2, the space of solutions of the Dirac equation (1) is invariant with respect to the transformation

$$\psi_P(\mathbf{x}, t) = \eta_P i \gamma^0 \psi(-\mathbf{x}, t), \quad (39)$$

which represents the space inversion $P : \mathbf{x} \rightarrow -\mathbf{x}$. The coefficient η_P is real. Note that $(\psi_P)_P(\mathbf{x}, t) = -\eta_P^2 \psi(\mathbf{x}, t)$. Since $P^2 = I$, we expect that $(\psi_P)_P$ is physically equivalent to ψ . This is the case when $\eta_P^2 = 1$ ⁶. Inserting

⁶ It is a well-known fact that the overall sign of fermionic fields is physically irrelevant.

formula (5) for the Majorana field in (39) and using the definition of the basis bispinors $e_\alpha(\mathbf{p})$, we obtain the corresponding transformation of the functions $b^\alpha(\mathbf{p})$

$$b_P^\alpha(\mathbf{p}) = \eta_P i(\gamma_5)_{\alpha\beta} b^\beta(-\mathbf{p}). \quad (40)$$

The space inversion can be implemented in the quantum theory by constructing a unitary operator \hat{P} in the Fock space such that

$$\hat{P} \hat{b}^\alpha(\mathbf{p}) \hat{P}^{-1} = \eta_P i(\gamma_5)_{\alpha\beta} \hat{b}^\beta(-\mathbf{p}). \quad (41)$$

Formula (41) and transformation (31) give

$$\hat{P} \hat{d}_\lambda(\mathbf{p}) \hat{P}^{-1} = \eta_P i \hat{d}_\lambda(-\mathbf{p}), \quad \hat{P} \hat{d}_\lambda^\dagger(\mathbf{p}) \hat{P}^{-1} = -\eta_P i \hat{d}_\lambda^\dagger(-\mathbf{p}). \quad (42)$$

The second formula in (42) determines the linear operator \hat{P} provided that we know how it acts on the vacuum state. We assume that the vacuum state is invariant⁷

$$\hat{P}|0\rangle = |0\rangle. \quad (43)$$

This assumption is consistent with the first formula in (42). (It would be inconsistent if, for example, on the r.h.s. of that formula there was the operator \hat{d}_λ^\dagger .) The basis in the n -particle sector of the Fock space is formed by vectors

$$|\mathbf{p}_1 \lambda_1, \mathbf{p}_2 \lambda_2, \dots, \mathbf{p}_n \lambda_n\rangle = \frac{1}{\sqrt{n!}} \hat{d}_{\lambda_1}^\dagger(\mathbf{p}_1) \hat{d}_{\lambda_2}^\dagger(\mathbf{p}_2), \dots, \hat{d}_{\lambda_n}^\dagger(\mathbf{p}_n) |0\rangle. \quad (44)$$

It is clear that

$$\hat{P} |\mathbf{p}_1 \lambda_1, \mathbf{p}_2 \lambda_2, \dots, \mathbf{p}_n \lambda_n\rangle = (-\eta_P i)^n |-\mathbf{p}_1 \lambda_1, -\mathbf{p}_2 \lambda_2, \dots, -\mathbf{p}_n \lambda_n\rangle. \quad (45)$$

Note that the $|(-\eta_P i)^n| = 1$, therefore, transformation (45) does not change the norm of the basis vectors. Formulas (43) and (45) are taken for the definition of the operator \hat{P} in the Fock space. In order to calculate its action on an arbitrary state, it is sufficient to expand that state in the (44) basis. Formulas (42), as well as the unitarity of \hat{P} , now reappear as easy to prove theorems.

The four-momentum operator $:\hat{P}_\mu:$ has the standard four-vector transformation law with respect to the space inversion, namely,

$$\hat{P} : \hat{P}^0 : \hat{P}^{-1} = : \hat{P}^0 :, \quad \hat{P} : \hat{P}^i : \hat{P}^{-1} = - : \hat{P}^i : .$$

⁷ Discussion of this point within the general framework of the relativistic quantum theory of fields can be found in Section 3.4 of Ref. [14].

Transformation law of the field operator $\hat{\psi}$ mimics formula (39) for the classical field

$$\hat{P}\hat{\psi}(\mathbf{x}, t)\hat{P}^{-1} = \eta_P i \gamma^0 \hat{\psi}(-\mathbf{x}, t).$$

To see this, compute $\hat{P}\hat{\psi}(\mathbf{x}, t)\hat{P}^{-1}$ using formulas (35) and (42). Next, recover the matrix γ^0 with the help of Eqs. (36), and move it to the left using formula $S(H(\mathbf{p}))\gamma^0 = \gamma^0 S(H(-\mathbf{p}))$. In the last step, change the integration variable from \mathbf{p} to $-\mathbf{p}$.

6.2. The time reversal T

Time reversal T acts on the classical field as follows:

$$\psi_T(\mathbf{x}, t) = \eta_T \gamma^0 \gamma_5 \psi(\mathbf{x}, -t). \quad (46)$$

It is a symmetry of the Dirac equation (1) in the sense that if $\psi(\mathbf{x}, t)$ is a solution of it, so is $\psi_T(\mathbf{x}, t)$. The coefficient η_T is real. Similarly as in the case of space inversion, $\eta_T^2 = 1$ and $(\psi_T)_T(\mathbf{x}, t) = -\psi_T(\mathbf{x}, t)$. Calculations analogous the ones leading to formula (40) give

$$b_T^\alpha(\mathbf{p}) = \eta_T (\gamma^0 \gamma_5)_{\alpha\beta} b^\beta(-\mathbf{p}).$$

In order to implement this transformation in the Fock space, we seek a unitary, or antiunitary, operator \hat{T} such that

$$\hat{T} \hat{b}^\alpha(\mathbf{p}) \hat{T}^{-1} = \eta_T i (\gamma^0 \gamma_5)_{\alpha\beta} \hat{b}^\beta(-\mathbf{p}). \quad (47)$$

Moreover, it should leave the vacuum state unchanged

$$\hat{T} |0\rangle = |0\rangle. \quad (48)$$

It turns out that the unitary option has to be abandoned, as expected on the basis of general experience with other models. To see this, let us assume that \hat{T} is unitary, hence linear. Operators $\hat{b}^\alpha(\mathbf{p})$ in (47) are expressed by $\hat{d}^\alpha(\mathbf{p})$ according to formula (31). Simple algebraic calculations show that

$$\hat{T} \hat{d}_1(\mathbf{p}) \hat{T}^{-1} = -\eta_T i \hat{d}_2^\dagger(-\mathbf{p}), \quad \hat{T} \hat{d}_2(\mathbf{p}) \hat{T}^{-1} = \eta_T i \hat{d}_1^\dagger(-\mathbf{p}).$$

Here, the linearity of the operator \hat{T} has been used in order to move this operator close to $\hat{d}_\lambda(\mathbf{p})$, to the position as on the l.h.s.'s of the formulas right above. We see that condition (48) is not consistent with these formulas, *e.g.*, $\hat{T} \hat{d}_1(\mathbf{p}) \hat{T}^{-1} |0\rangle = 0$, while $\hat{d}_2^\dagger(-\mathbf{p}) |0\rangle = |-\mathbf{p} 2\rangle \neq 0$.

On the other hand, antiunitary \hat{T} is antilinear, hence, when moving this operator close to $\hat{d}_\lambda(\mathbf{p})$, one has to complex conjugate coefficients encountered on the way. In this case, we obtain

$$\hat{T} \hat{d}_1(\mathbf{p}) \hat{T}^{-1} = \eta_T i \hat{d}_2(-\mathbf{p}), \quad \hat{T} \hat{d}_2(\mathbf{p}) \hat{T}^{-1} = -\eta_T i \hat{d}_1(-\mathbf{p}). \quad (49)$$

These formulas are consistent with (48) and we may proceed with the definition of the operator \hat{T} as in the case of space inversion \hat{P} .

Now, equipped with the definition of the operator \hat{T} , we return to formulas (49) which change their status: from a conjecture to an easy to prove theorem. We can derive also the transformation law of the field operator $\hat{\psi}$. It resembles formula (46) for the classical field

$$\hat{T} \hat{\psi}(\mathbf{x}, t) \hat{T}^{-1} = \eta_T \gamma^0 \gamma_5 \hat{\psi}(\mathbf{x}, -t). \quad (50)$$

The field operator is given by formula (35). When computing the l.h.s. in (50), one has to remember about the complex conjugation due to the antiunitarity of \hat{T} . The matrix $\gamma^0 \gamma_5$ is recovered with the help of the following formulas:

$$v_1 = i\gamma^0 \gamma_5 v_2^*, \quad v_2 = -i\gamma^0 \gamma_5 v_1^*, \quad v_1^* = -i\gamma^0 \gamma_5 v_2, \quad v_2^* = i\gamma^0 \gamma_5 v_1.$$

7. Summary and remarks

We have shown how to quantize the classical Majorana field starting from its expansion into the eigenfunctions of the axial momentum. The model has the particle interpretation with a single spin-1/2 fermion. There is no antiparticle, as expected in the case of Majorana field. The constructed quantum model essentially coincides with the results of other approaches to quantization of the field, which is the desired outcome. Also, the discrete symmetries: unitary \hat{P} and antiunitary \hat{T} , have been implemented in the model (the charge conjugation is trivial, $\hat{C} = I$). It is clear that the same eigenfunction expansion can be used also in the quantum theory of the Dirac field.

As for differences with other approaches to quantization of the Majorana field: (a) Our approach is self-contained — we do not refer to the quantized Dirac field. (b) The mode expansion (5) is novel. (c) On a more technical level, we have made the specific choice of the basis bispinors $e_\alpha(\mathbf{p})$, as described in above formula (6). Due to it, the coefficient functions $b^\alpha(\mathbf{p})$ in formula (5) have clear relativistic transformation laws (6), (7). The concrete form of basis bispinors v_λ in formula (35) is a consequence of that choice.

The presented quantization is not based on canonical formalism. Instead, we have used as the guiding principle the relativistic invariance. In particular, the operator $:\hat{P}_0:$ has been introduced as a generator of time translations, without reference to a correspondence with a classical Hamiltonian. Noteworthy is also the fact that there is no need to consider the Dirac sea when searching for the vacuum state.

The fact that the expansion (5) has led to the expected quantum model confirms that the axial momentum is a useful quantum mechanical observable, in spite of its apparent peculiarities discussed in [9, 10].

On a more general ground, we think that probably the most interesting aspect of our work is the appearance of the Majorana basis of operators, $\hat{b}^\alpha(\mathbf{p})$, in the context of quantized relativistic fields. These operators were replaced by the annihilation and creation operators $\hat{a}_\lambda(\mathbf{p})$, $\hat{a}_\lambda^\dagger(\mathbf{p})$ in Section 4 with the goal of obtaining the standard Fock space and the particle interpretation. That construction provides a particular complex representation of the Majorana basis of operators. A very interesting question arises about the existence and features of other representations.

Appendix A

The Lorentz boosts and the Wigner rotations

Calculations in Sections 3 and 5 require detailed knowledge of the Lorentz boosts and of the Wigner rotations. For convenience of the reader, we have collected relevant formulas in this appendix. Most of them are well-known, perhaps except formula (A.2) which is found in the monograph [14].

The proper orthochronous Lorentz matrices in a vicinity of the unit matrix I_4 can be written in the exponential form, $L = \exp(\omega)$, where ω is four by four real matrix. Its elements $\omega^\mu{}_\nu$ obey the condition $\omega_{\mu\nu} = -\omega_{\nu\mu}$, where $\omega_{\mu\nu} = \eta_{\mu\kappa}\omega^\kappa{}_\nu$, $\eta_{\mu\kappa}$ are components of the Minkowski metric. With this parametrization, $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]/8)$. In the Majorana representation, the matrices γ^μ are purely imaginary, $S(L)$ are real, and $S(L)^T \gamma^0 = \gamma^0 S(L)^{-1}$. As the independent parameters on the Lorentz group, we take $\omega^{23}, \omega^{31}, \omega^{12}$, and ω^{0i} with $i = 1, 2, 3$.

Let us stress that we raise or lower indices using the Minkowski metric, for example, $\omega^0{}_i = -\omega^{0i}$. Moreover, the δ_{ik} denotes the Kronecker symbol (not tensor), which always takes values 0 or +1. In this notation, the trivial Lorentz transformation $L = I_4$ has matrix elements denoted as $(I_4)^\mu{}_\nu$, where $(I_4)^0{}_0 = 1$, $(I_4)^i{}_k = \delta_{ik}$, $(I_4)^i{}_0 = 0$, $(I_4)^0{}_k = 0$. All this may sound trivial, but the reality is that in the calculations reported below, it is very easy to make a sign mistake related to the level of indices.

The matrix elements $H^\mu{}_\nu(\mathbf{p})$ of the boost $H(\mathbf{p})$ have the following form:

$$H^0{}_0(\mathbf{p}) = \frac{E_p}{m}, \quad H^0{}_i(\mathbf{p}) = H^i{}_0(\mathbf{p}) = \frac{p^i}{m}, \quad H^i{}_j(\mathbf{p}) = \delta_{ij} + \frac{p^i p^j}{m(m + E_p)}. \quad (\text{A.1})$$

The matrix $H(\mathbf{p})$ is symmetric. It turns out that $(H(\mathbf{p}))^{-1} = H(-\mathbf{p})$. In the case of this boost,

$$S(H(\mathbf{p})) = n_\mu(\mathbf{p})\gamma^\mu\gamma^0, \quad (\text{A.2})$$

where

$$n_0(\mathbf{p}) = \frac{m + E_p}{\sqrt{2m(m + E_p)}}, \quad n_i(\mathbf{p}) = \frac{p_i}{\sqrt{2m(m + E_p)}}, \quad (\text{A.3})$$

see Exercise 2.4.9 in [14]. Note that $p_i = -p^i$, $\mathbf{p} = (p^i)$.

In Section 3, we use infinitesimal form of the Wigner rotations

$$\mathcal{R}(L, \mathbf{p}) = H^{-1}(\mathbf{p}) L H(l^{-1}(\mathbf{p})).$$

In the case of rotations, the Lorentz matrix has the block diagonal form

$$L_R = \left(\begin{array}{c|ccc} 1 & 0 & 0 & 0 \\ \hline 0 & & & \\ 0 & & R & \\ 0 & & & \end{array} \right).$$

It turns out that

$$\mathcal{R}(L_R, \mathbf{p}) = L_R.$$

In the linear approximation around I_4 , which is sufficient for our purposes, $L_R \cong I_4 + \omega$, where $\omega = (\omega^\mu{}_\nu)$ with $\omega^0{}_i = \omega^i{}_0 = 0$, and

$$S(L_R) \cong I_4 + \sum_{i < k} \omega^{ik} [\gamma^i, \gamma^k] / 4, \quad l^{-1}(\mathbf{p})^i \cong p^i + \omega^{ik} p^k.$$

These formulas are used on the r.h.s. of formula (12) in the derivation of condition (13).

In the case of infinitesimal boosts, $L \cong I_4 + \delta L$, where

$$\delta L = \left(\begin{array}{c|ccc} 0 & \omega^0{}_1 & \omega^0{}_2 & \omega^0{}_3 \\ \hline \omega^1{}_0 & & & \\ \omega^2{}_0 & & 0 & \\ \omega^3{}_0 & & & \end{array} \right).$$

Furthermore, $l^{-1}(\mathbf{p})^i \cong p^i - \omega^i{}_0 E_p$, and $\mathcal{R}(L, \mathbf{p}) \cong I_4 + \delta\mathcal{R}$, where

$$\delta\mathcal{R}^0{}_\mu = 0, \quad \delta\mathcal{R}^\mu{}_0 = 0, \quad \delta\mathcal{R}^i{}_k \cong \frac{1}{E_p + m} \left(\omega^{i0} p^k - \omega^{k0} p^i \right).$$

The first term on the r.h.s. of formula (15) comes from

$$S(\mathcal{R}(L, \mathbf{p})) \cong I_4 + \frac{1}{8} \delta\mathcal{R}_{ik} \left[\gamma^i, \gamma^k \right],$$

and the second term from $\sqrt{E_{L^{-1}p}/E_p} b^\alpha(l^{-1}(\mathbf{p}))$.

Appendix B

The commutator equations

Formulas (B.2)–(B.5) shown below facilitate calculations in Sections 3 and 4. We seek a Hermitian operator \hat{X} which obeys the condition

$$\left[\hat{X}, \hat{b}^\alpha(\mathbf{p}) \right] = i \int d^3\mathbf{q} x_{\alpha\beta}(\mathbf{p}, \mathbf{q}) \hat{b}^\beta(\mathbf{q}), \quad (\text{B.1})$$

where the real-valued functions $x_{\alpha\beta}$ are antisymmetric in the following sense:

$$x_{\alpha\beta}(\mathbf{p}, \mathbf{q}) = -x_{\beta\alpha}(\mathbf{q}, \mathbf{p}).$$

Condition (B.1) is satisfied by the Hermitian operator

$$\hat{X} = -\frac{i}{2} \int d^3\mathbf{p} d^3\mathbf{q} \hat{b}^\alpha(\mathbf{p}) x_{\alpha\beta}(\mathbf{p}, \mathbf{q}) \hat{b}^\beta(\mathbf{q}) + c_0 I, \quad (\text{B.2})$$

where c_0 is an arbitrary real constant. This can be checked with the help of anticommutators (8) and the formula $[\hat{A}\hat{B}, \hat{C}] = \hat{A}[\hat{B}, \hat{C}]_+ - [\hat{A}, \hat{C}]_+ \hat{B}$.

Let

$$\hat{Y} = -\frac{i}{2} \int d^3\mathbf{p} d^3\mathbf{q} \hat{b}^\alpha(\mathbf{p}) y_{\alpha\beta}(\mathbf{p}, \mathbf{q}) \hat{b}^\beta(\mathbf{q}) + d_0 I,$$

where d_0 is an arbitrary real constant, and the real-valued functions $y_{\alpha\beta}(\mathbf{p}, \mathbf{q})$ are antisymmetric in the above sense. The commutator of the Hermitian operators \hat{X} and \hat{Y} is given by the formula

$$\left[\hat{X}, \hat{Y} \right] = -\frac{1}{2} \int d^3\mathbf{p} d^3\mathbf{q} \hat{b}^\alpha(\mathbf{p}) w_{\alpha\beta}(\mathbf{p}, \mathbf{q}) \hat{b}^\beta(\mathbf{q}), \quad (\text{B.3})$$

where

$$w_{\alpha\beta}(\mathbf{p}, \mathbf{q}) = \int d^3\mathbf{s} \left(x_{\alpha\eta}(\mathbf{p}, \mathbf{s}) y_{\eta\beta}(\mathbf{s}, \mathbf{q}) - y_{\alpha\eta}(\mathbf{p}, \mathbf{s}) x_{\eta\beta}(\mathbf{s}, \mathbf{q}) \right). \quad (\text{B.4})$$

Thus, the ‘matrix’ w for the commutator is given by the commutator of the ‘matrices’ x and y .

Commutators of the Poincaré generators discussed in Section 4 can be checked with the help of formula (B.5) below. We consider operators \hat{W} and \hat{Z}

$$\hat{W} = \int d^3p d^3q \hat{d}_i^\dagger(\mathbf{p}) w_{ik}(\mathbf{p}, \mathbf{q}) \hat{d}_i(\mathbf{q}), \quad \hat{Z} = \int d^3p d^3q \hat{d}_i^\dagger(\mathbf{p}) z_{ik}(\mathbf{p}, \mathbf{q}) \hat{d}_i(\mathbf{q}),$$

where $i, k = 1, 2$ and w_{ik}, z_{ik} are (generalized) functions of \mathbf{p}, \mathbf{q} . Their commutator has the following form:

$$[\hat{W}, \hat{Z}] = \int d^3p d^3q \hat{d}_i^\dagger(\mathbf{p}) r_{ik}(\mathbf{p}, \mathbf{q}) \hat{d}_i(\mathbf{q}), \quad (\text{B.5})$$

where

$$r_{ik}(\mathbf{p}, \mathbf{q}) = \int d^3s (w_{ij}(\mathbf{p}, \mathbf{s}) z_{jk}(\mathbf{s}, \mathbf{q}) - z_{ij}(\mathbf{p}, \mathbf{s}) w_{jk}(\mathbf{s}, \mathbf{q})).$$

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