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Statistical Interpretation of the Klein-Gordon Equation *)

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It is possible to find a statistical interpretation of the equations containing time derivatives of second and higher orders by a method similar to that used for the Schrödinger equation. This enables us to calculate cross sections for scattering processes involving waves obeying various types of fields equations without help of the quantum theory of fields. We shall explain the procedure on the special case of the Klein-Gordon equation with a given perturbation

$$(\Box - m^2)\,\psi = g\,\varphi\psi, \tag{1}$$

where $\psi(x)$ is a complec field, $\varphi(x)$ a given external potential, g a small coupling constant and x (x_μ) a point of the space-time. To distinguish points we shall often

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write 1,2,... instead of $x', x'', \ldots$ It is most convenient to use the Green-function treatment of field equations introduced by Feynman[1] (cf. also Rzewuski[2]). We first express the solution of (1) by means of the boundary values on a hypersurface consisting of two hyperplanes $t = t_1'$ and $t = t_1''$ and an arbitrary timelike hypersurface connecting those planes at infinity. The contribution from the latter vanishes and we get

$$\psi(2) = \int \psi(1')\, X_4(1')\, G(1'2)\, d^3x_1' - \int \psi(1'')\, X_4(1'')\, G(1''2)\, d^3x_1''. \tag{2}$$

Here $t_1'' > t_2 > t_1'$, $d^3x = dx\,dy\,dz$,

$$X_4(x) = \frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t} \quad (3), \qquad G(12) = \sum_{n=0}^{\infty} G^{(n)}(12), \tag{4}$$

$$G^{(n)}(12) = g \int G^{(0)}(13)\, \varphi(3)\, G^{(n-1)}(32)\, d^4x_3 \tag{5}$$

and

$$G^{(0)}(12) = \frac{1}{(2\pi)^4} \int \frac{e^{-ikx_{12}}}{k^2 - m^2}\, d^4k, \; kx_{12} = k_\mu(x_{1\mu} - x_{2\mu}). \tag{6}$$

The integrand in (6) has two poles at $k_4 = \pm\sqrt{\vec{k}^2 + m^2}$. To give (6) an unambiguous meaning we fix the path of integration so as to make it pass the negative pole on an infinitesimal circle in the lower half of the complex k_4-plane and the positive pole in the upper half of this plane. The consequence of this prescription is that the operator $X_4(1)G^{(0)}(12)$ propagates plane waves

$$\psi^0(x) = \psi^0(p)\, e^{-ipx}, \quad p^2 = m^2 \tag{7}$$

with positive energy ($p_4 > 0$) forwards in time and the complex conjugate of (7) backwards in time:

$$\int \psi^0(1)\, X_4(1)\, G^{(0)}(12)\, d^3x_1 = \begin{cases} \psi^0(2) & \text{for } t_2 > t_1 \\ 0 & \;\;\text{,,} \quad t_2 < t_1 \end{cases} \tag{8}$$

$$\int \psi^{0*}(1)\, X_4(1)\, G^{(0)}(12)\, d^3x_1 = \begin{cases} 0 & \;\;\text{,,} \quad t_2 > t_1 \\ -\psi^0(2) & \;\;\text{,,} \quad t_2 < t_1. \end{cases} \tag{9}$$

With help of (2), (4), (5), (8) and (9) we may express the solution of (1) by means of $G^{(0)}$, φ and the incoming plane wave $\psi_1^0(x)$

$$\psi_1(2) = \psi_1^0(2) + g \int G(23)\, \varphi(3)\, \psi_1^0(3)\, d^4x_3, \tag{10}$$

and similarly the solution of the complex conjugate equation to (1) if ψ_2^0 denotes the outgoing plane wave

$$\psi_2^*(2) = \psi_2^{0*}(2) + g \int d^4x_3\, \psi_2^{0*}(3)\, \varphi(3)\, G(32), \tag{11}$$

the only assumption being $t_1' \to -\infty$, $t_1'' \to +\infty$.

[1] Feynman R. P., Phys. Rev., **76**, 749 (1949).

[2] Rzewuski J., Stud. Soc. Sc. Torunensis (in press); Acta Phys. Polonica, preceding letter.

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Now we ask for the probability amplitude for scattering in the external field φ of the plane wave ψ_1^0 at $t_1'=-\infty$ into the state ψ_2^0 at $t_1''=+\infty$. In the case of the Schrödinger equation we argue as follows: At $t_1'=-\infty$ we have an incoming plane wave ψ_1^0 which is the solution of the unperturbed Klein-Gordon equation. The perturbation φ changes ψ_1^0 into say ψ_1 which is the solution of the full equation. On a hypersurface say $t=t_1''$ this ψ_1 may be considered as a superposition of plane waves like (7). The probability amplitude that at a measurement we find the state ψ_2^0 is simply the corresponding expansion coefficient

$$\int \psi_2^{0*}(1'')\,\psi_1(1'')\,d^3x_1'' \tag{12}$$

if the ψ^0 are normalized to unity. This coefficient is independent of the time t_1 in the limit $t_1 \to +\infty$.

We generalize this procedure for the Klein-Gordon equation and take

$$A(\psi_1^0 \to \psi_2^0) = \int \psi_2^{0*}(1'')\,Y(1'')\,\psi_1(1'')\,d^3x_1'' \tag{13}$$

as the transition amplitude $\psi_1^0 \to \psi_2^0$. Here $\psi_1(1'')$ depends linearly on the initial state $\psi_1^0(1')$ according to (10) and $Y(1'')$ is an unknown operator. The linear dependence on the initial and final state is necessary in view of the linearity of quantum theory. The operator $Y(x)$ is easily determined by a symmetry argument. Indeed, by means of (11), we can express $\psi_2^*(1')$ by the outgoing wave $\psi_2^{0*}(1'')$ at $+\infty$. Thus the same transition amplitude may be written

$$A(\psi_1^0 \to \psi_2^0) = \int \psi_2^*(1')\,Y'(1')\,\psi_1^0(1')\,d^3x_1', \tag{14}$$

where Y' is another unknown operator. It is understood that in (13) the limes $t_1'' \to \infty$ and in (14) $t_1' \to -\infty$ is taken. From the identity of (13) and (14) for all possible ψ_1^0 and φ_2^0 it follows that

$$Y(x) = Y'(x) = X_4(x) = \frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial t}. \tag{15}$$

Normalizing the plane waves in such a way that the unperturbed transition $\psi_1^0 \to \psi_1^0$ has an amplitude of modulus one, we get

$$\psi_1^0(p) = \frac{1}{\sqrt{2p_4 V}}, \tag{16}$$

where V is the considered volume of space. Now we can finally write (13) or (14) in either of the forms

$$\begin{aligned} A(\psi_1^0 \to \psi_2^0) &= -i\,\delta_{\psi_1^0\psi_2^0} - g\int d^4x\,\psi_2^{0*}(x)\,\varphi(x)\,\psi_1(x) = \\ &= -i\,\delta_{\psi_1^0\psi_2^0} - g\int d^4x\,\psi_2^*(x)\,\varphi(x)\,\psi_1^0(x). \end{aligned} \tag{17}$$

Here $\delta_{\psi_1^0\psi_2^0} = \begin{cases} 1 \text{ if } p_1 = p_2 \\ 0 \text{ if } p_1 \neq p_2 \end{cases}$. (We consider a finite V and therefore a discrete spectrum of the p's. In the case of continous spectrum the δ-function should be used). We

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may note that the final result (17) has not the covariant form of (10) or (11). This is caused by the non-invariance of the definition of the transition amplitude and the resulting non-invariance of the normalization procedure. However, the physically important second terms of both forms of (17) become relativistically invariant after multiplication with $\sqrt{(p_1)_4 (p_2)_4}$, e. g.

$$-g \sqrt{(p_2)_4} \int d^4x \, \psi_2^{0*}(x) \, \varphi(x) \, \psi_1(x) \cdot \sqrt{(p_1)_4}. \tag{18}$$

An analogous procedure may be applied to relativistic equations of higher order. A full account of this work will be published shortly in this journal.

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