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THE PHYSICS OF THE CHIRAL FERMIONS*

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We review those aspects of chiral gauge theories which are related to the violation of the decoupling property. The case of the top quark is worked out in detail. The mechanism of anomaly cancellation in the low-energy effective theory is illustrated in a simple model.

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1. Introduction

These lectures review some low-energy features of gauge theories with massive chiral fermions. The standard model (SM), present theory of electroweak interactions, describes three generations of fermions transforming in chiral representations of the gauge group $SU(2)_L \otimes U(1)_Y$. Compared to the electroweak scale defined by the Fermi constant G_F, all fermions are essentially massless, with the exception of the top quark, whose mass is even larger than the vector boson masses. This remarkable hierarchy, totally mysterious at the present time, is accounted for in the theory by a corresponding hierarchy of coupling constants, which singles out the top Yukawa coupling as the largest.

Aim of these lectures is to describe the consequences of this basic fact. To start with, we review the decoupling theorem of Appelquist and Carazzone [1]. We show how the decoupling property is violated in the SM with an heavy top quark just because of the assumed relation between masses and couplings.

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Such a violation is not academic. Indeed it controls the pattern of the potentially largest electroweak radiative corrections and it shows up in the real world with specific signals. Apart from the existence of lower bounds on the top mass and the probably imminent (maybe recent, for the reader) discovery of the top at the Tevatron collider, it is astonishing how strong and real is the indirect evidence for the top quark in $B\bar{B}$ oscillations. The often underestimated agreement of the large set of electroweak precision measurements at LEP and SLC with the SM expectations is perhaps less striking but it represents a highly non-trivial fact.

Instead of discussing the full one-loop radiative corrections, necessary to perform a complete analysis of the LEP/SLC data [2], we illustrate the pattern of the leading corrections in the framework of an effective Lagrangian. More than a device used to simplify the discussion of the quantum theory, the effective Lagrangian approach reproduces automatically the infinite set of Ward identities of $SU(2)_L \otimes U(1)_Y$ [3], some of which has revealed so useful in dealing with leading higher-order computations.

A final lecture is devoted to the mechanism of anomaly cancellation in the low-energy theory. The gauge invariance of the effective action, an indispensable requirement, is apparently broken by the anomalous fermion content of the low-energy spectrum. This breaking is however repaired by a Wess–Zumino term whose gauge variation exactly compensates the gauge variation coming from the classical action. The independence of the physical amplitudes from the gauge parameter is thus guaranteed.

2. The decoupling theorem

In this section we briefly review the decoupling theorem [1]. Consider a field theory with particles of mass M. If the energy at which we perform measurements is much smaller than M, these particles will affect the predictions of the theory only through their virtual effects.

The decoupling theorem states that, in the limit $M \to \infty$, the above mentioned effects are unobservable. More precisely, the effects from heavy particles are either suppressed by inverse powers of M, or they renormalize parameters of the low-energy theory, that is they can be absorbed into renormalizations of couplings, masses, wave functions of the theory obtained by removing the heavy particles.

Examples of theories enjoying the decoupling property are theories with an exact gauge symmetry, like, for instance, QED or QCD. The U(1) gauge invariant Lagrangian of QED is:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu \psi - M\bar{\psi}\psi \,, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength of the photon field A_μ and the covariant derivative $D_\mu\psi$ is given by:

$$D_\mu\psi = (\partial_\mu - ieA_\mu)\psi\,. \tag{2.2}$$

Suppose that we are interested in the behaviour of the electromagnetic field A_μ for energies much smaller than the electron mass M. The first effects potentially affected by M will show up at one-loop order. Consider, as the simplest case, the one-loop contribution $-i\Pi_{\mu\nu}(p)$ to the photon self-energy. Using dimensional regularization, one obtains:

$$-i\Pi_{\mu\nu}(p) = -e^2(\mu^2)^{\frac{4-d}{2}} \int \frac{d^dk}{(2\pi)^d}\,\mathrm{Tr}\left(\gamma_\mu \frac{1}{\not k - \not p - M}\gamma_\nu \frac{1}{\not k - M}\right)\,. \tag{2.3}$$

By introducing the Feynman parametrization, by evaluating the trace and performing the usual shift in the integration variable, one has:

$$\begin{aligned}-i\Pi_{\mu\nu}(p) = &- 4e^2(\mu^2)^{\frac{4-d}{2}} \int \frac{d^dk}{(2\pi)^d} \int_0^1 dt \frac{1}{(k^2-\Omega)^2}\left[\left(\frac{2}{d}-1\right)k^2 g_{\mu\nu}\right.\\ &\left. + (p^2 g_{\mu\nu} - 2p_\mu p_\nu)t(1-t) + M^2 g_{\mu\nu}\right],\end{aligned} \tag{2.4}$$

where

$$\Omega = \Omega(t) = M^2 - p^2 t(1-t)\,. \tag{2.5}$$

After the Wick rotation, the integration over the loop variable gives:

$$\begin{aligned}-i\Pi_{\mu\nu}(p) &= -i\frac{8e^2}{(4\pi)^{d/2}}(\mu^2)^{\frac{4-d}{2}}(p^2 g_{\mu\nu} - p_\mu p_\nu)\int_0^1 dt\; t(1-t)\Omega^{\frac{d-4}{2}}\Gamma\left(\frac{4-d}{2}\right)\\ &= i\frac{4}{3}\frac{e^2}{(4\pi)^2}(p^2 g_{\mu\nu} - p_\mu p_\nu)\left[A + 6\int_0^1 dt\; t(1-t)\ln\frac{\Omega(t)}{\mu^2}\right] + \ldots\,,\end{aligned} \tag{2.6}$$

where

$$\begin{aligned}A &= -\frac{2}{4-d} + \gamma_E - \ln 4\pi\,,\\ \gamma_E &\simeq 0.577\end{aligned} \tag{2.7}$$

and dots in Eq. (2.6) stand for terms which vanish in the limit $d \to 4$. Since we are considering external momenta much smaller than the electron mass

M, we can expand the function $\Omega(t)$ in powers of p^2/M^2 and we perform the (convergent) integration over the Feynman parameter t term by term. The result is:

$$-i\Pi_{\mu\nu}(p) = i\frac{4}{3}\frac{e^2}{(4\pi)^2}(p^2 g_{\mu\nu} - p_\mu p_\nu)\left[A + ln\frac{M^2}{\mu^2} - \frac{1}{5}\frac{p^2}{M^2} + \ldots\right] . \quad (2.8)$$

The previous equation provides a simple example of how the decoupling property for QED works at one-loop order. The one-loop self-energy correction (2.8) can be represented by a set of local terms in an effective low-energy QED Lagrangian:

$$\mathcal{L}_{\text{eff}} = -\frac{1}{4}(1 + \delta Z)F_{\mu\nu}F^{\mu\nu} + c_1 F_{\mu\nu}\Box F^{\mu\nu} + \ldots , \quad (2.9)$$

where dots stand for higher dimensional terms. The coefficients δZ and c_1 are fixed to reproduce the result given in Eq. (2.8):

$$\delta Z = \frac{4}{3}\frac{e^2}{(4\pi)^2}\left[A + \ln\frac{M^2}{\mu^2}\right] ,$$
$$c_1 = \frac{4}{3}\frac{e^2}{(4\pi)^2}\left[-\frac{1}{5M^2}\right] . \quad (2.10)$$

We see that the potentially dangerous logarithmic dependence on M occurs in the term proportional to $(p^2 g_{\mu\nu} - p_\mu p_\nu)$ and, it is thus absorbed by the wave- function renormalization of the photon field - δZ - leading to no observable effect. The next term of the expansion (2.8) cannot be absorbed in a renormalization of parameters and is related to an independent operator in the low-energy effective theory. However, since the coefficient c_1 is inversely proportional to M^2, it vanishes in the limit $M \to \infty$, giving again, in this limit, no observable effect. This is obviously true also for the remaining terms in the expansion (2.8).

One can proceed in a completely analogous way with other Green functions, for instance the four-point photon Green function. In this way it is easy to check, at one-loop order, the validity of the decoupling property for QED or QCD. Appelquist and Carazzone [1], extended the proof to all orders in perturbation theory.

Different from theories possessing an exact gauge symmetry, theories with spontaneously broken gauge symmetries can be shown not to necessarily satisfy the decoupling property. The point is that, whereas in the case of an exact gauge symmetry mass terms are gauge invariant, in the spontaneously broken case masses are generated from interaction terms in the process of symmetry breaking. The typical mass is of the kind:

$$M = \lambda \langle\varphi\rangle , \quad (2.11)$$

where $\langle\varphi\rangle$ is the vacuum expectation value (VEV) of a scalar field φ and λ is a dimensionless coupling. It is clear that, in such a case, the large mass limit can be achieved in two different ways:

(A) λ fixed , $\langle\varphi\rangle$ large;
(B) λ large, $\langle\varphi\rangle$ fixed.

The first alternative is commonly considered in discussing grand unified theories (GUTs). In GUTs this choice is suggested by the physical hierarchy between the two widely separated VEVs associated to the GUT scale and to the electroweak one. Other physical situations can however be described more efficiently by adopting the point of view (B). Consider for instance the effective Lagrangian for low-energy charged current electroweak processes:

$$\mathcal{L}_{CC} = 2\sqrt{2}G_F J_\mu^+ J^{-\mu} , \tag{2.12}$$

where

$$J_\mu^- = \bar{u}\gamma_\mu(1-\gamma_5)d + \dots . \tag{2.13}$$

is the total charged current. In the standard model (SM) of electroweak interactions, the Fermi constant, $G_{\rm F}$, is given by:

$$G_{\rm F} = \frac{g^2}{8M_W^2} = \frac{1}{\sqrt{2}v^2} , \tag{2.14}$$

where g is the $SU(2)_{\rm L}$ coupling constant, $v = 246$ GeV is the VEV of the neutral component of the Higgs doublet. The last expression of the previous equality represents G_F as a function of g and v. From Eqs (2.12) and (2.14), by considering the large M_W limit according to (A), one obtains

$$\mathcal{L}_{CC} \to 0 \tag{2.15}$$

in agreement with the decoupling theorem. However this is not really the case we are interested in. Following (A) we are considering all SM particles infinitely heavy at the same time. On the contrary, we have in mind a situation where the external momenta, of the order of the fermion masses, are much smaller than the W mass:

$$p^2 \simeq m_f^2 \ll M_W^2 , \tag{2.16}$$

where m_f is a generic fermion mass. By denoting with y_f the corresponding Yukawa coupling, the previous relation implies:

$$y_f \ll g . \tag{2.17}$$

In this case, the large mass limit only reflects the fact that the (light) fermion Yukawa coupling is much smaller than the $SU(2)_L$ gauge coupling constant, and therefore it is better represented by the option (B). As we can see from Eq. (2.14), "sending g to infinity" and keeping v fixed leaves $\mathcal{L}_{CC}$ invariant and non-vanishing. The decoupling property is violated.

Notice that, when we speak of large g, in the case (B), we are not saying that g must be much larger than one. Indeed, the ideal case occurs when g, while satisfying the relation (2.17), still remains smaller than one and the usual perturbative analysis applies. This is what happens in the previous example.

A further freedom we have in theories with a spontaneously broken symmetry is that we can allow a single member in a particle multiplet to become heavy with respect to the rest of the spectrum (which is forbidden in the exact case). For instance, in the SM, we can consider the case of an heavy top quark whose left-handed component transforms, together with that of the bottom quark, in an $SU(2)_L$ doublet.

In this case it may seem that the large mass limit is not compatible with the gauge symmetry one starts with, since one is removing a member of a representation. Indeed, to maintain the gauge symmetry in the light sector, one must embed the light degrees of freedom into nonlinear multiplets, and the symmetry becomes non-linearly realized [4]. The theory containing the light particles is now non-renormalizable from the beginning and this, as we shall see in a moment, can be regarded as a failure of the decoupling property.

To illustrate this point, we consider the Higgs sector of the SM, described by the $SU(2)_L \otimes U(1)_Y$ invariant Lagrangian:

$$\mathcal{L}_H = \frac{1}{4}\,\mathrm{Tr}(D_\mu H^\dagger D^\mu H) - V(\mathrm{Tr}(H^\dagger H)) \,. \qquad (2.18)$$

The 2 by 2 matrix H contains the usual $SU(2)_L$ doublet of complex scalar fields:

$$H = \sqrt{2}\begin{pmatrix} \varphi^0 & -\varphi^+ \\ \varphi^- & (\varphi^0)^* \end{pmatrix} \,. \qquad (2.19)$$

The covariant derivative $D_\mu H$ is defined by:

$$D_\mu H = \partial_\mu H - g\hat{W}_\mu H + g' H \hat{B}_\mu \qquad (2.20)$$

with the $SU(2)_L \otimes U(1)_Y$ gauge fields, $\vec{W}_\mu$ and B_μ, embedded in matrices:

$$\hat{W}_\mu = \frac{1}{2i}\vec{W}_\mu \cdot \vec{\tau} \,,$$
$$\hat{B}_\mu = \frac{1}{2i}B_\mu \tau^3 \,. \qquad (2.21)$$

The scalar potential V is given by:

$$V = \frac{M^2}{8v^2}\left(\frac{\mathrm{Tr}(H^\dagger H)}{2} - v^2\right)^2 , \tag{2.22}$$

and it depends on v, the VEV of $\sqrt{2}\varphi^0$, and M, the mass of the Higgs. By shifting the neutral component according to:

$$\varphi^0 = \frac{h + v + i\chi}{\sqrt{2}} \tag{2.23}$$

one can give the scalar potential the form:

$$V = \frac{M^2}{8v^2}(h^2 + \chi^2 + 2\varphi^+\varphi^- + 2hv)^2 . \tag{2.24}$$

Suppose now that the Higgs is much heavier than the particles we can excite in a set of physical measurements [5]. We would like to know if the virtual effects on measurable quantities due to the heavy Higgs decouple or not, in the sense specified above. For the moment, we restrict the analysis to the tree-level approximation. If we imagine a process with a certain number of external light particles, it is not so evident, even in the tree-level approximation, whether the Higgs exchange will produce negligible effects or not. Indeed, as we can see from Eq. (2.24), there are interaction terms among the Higgs and the other unphysical scalars which grow as M^2, allowing in principle an overcompensation of the negative powers of M contained in the Higgs propagator.

To fix the ideas, we consider the scattering $\varphi^+\varphi^- \to \varphi^+\varphi^-$ among unphysical scalars. (As guaranteed by the so-called equivalence theorem [6], this scattering amplitude is the high-energy approximation to the scattering amplitude among longitudinally polarized charged vector bosons, and, to be consistent, we will work in the energy interval $M_W \ll E \ll M$.) At tree level, the amplitude is the sum of a contact term, plus s and t channel Higgs exchanges. One obtains:

$$A(\varphi^+\varphi^- \to \varphi^+\varphi^-) = -2i\frac{M^2}{v^2} - \frac{iM^4}{v^2}\left(\frac{1}{s-M^2} + \frac{1}{t-M^2}\right) , \tag{2.25}$$

where we have separately listed the three contributions. While in the large M limit (at fixed scattering angle) the leading term, of order M^2, cancels, the amplitude, given by:

$$A(\varphi^+\varphi^- \to \varphi^+\varphi^-) = -i\frac{u}{v^2} + O\left(\frac{1}{M^2}\right) , \tag{2.26}$$

is still different from zero.

The general case can be analyzed similarly. We recall that, summing over all tree-level amplitudes with internal Higgs lines amounts to:

1. Solve the classical equation of motion for the Higgs field;
2. Substitute back the solution in the original action.

The first step is usually hard to accomplish, because of the non-linearity of the field equations. However we are not interested in the full solution of the equations of motion, but rather in their limit when M is much larger than the energy and/or other mass parameters. To this end we parametrize the scalar multiplet H as follows:

$$H = \sigma U = \sigma \exp\left(i\frac{\vec{\xi}\cdot\vec{\tau}}{v}\right) . \tag{2.27}$$

The Lagrangian for the scalar sector reads:

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\sigma\partial^\mu\sigma - \frac{M^2}{8v^2}(\sigma^2 - v^2)^2 + \frac{\sigma^2}{4}\,\mathrm{Tr}(D_\mu U^\dagger D^\mu U) . \tag{2.28}$$

The Higgs degree of freedom is now described by the field σ. In the large M limit, the solution of the equation of motion for σ is simply:

$$\sigma = v , \tag{2.29}$$

since for large M the action is dominated by the scalar potential. Plugging back this solution in the Lagrangian of Eq. (2.28), one finds [7]:

$$\mathcal{L}(M = \infty) = \frac{v^2}{4}\,\mathrm{Tr}(D_\mu U^\dagger D^\mu U) . \tag{2.30}$$

This is the Lagrangian for a (gauged) nonlinear σ-model [8]. It contains an infinite set of operators depending on the would-be Goldstone fields $\vec{\xi}$, which makes it a non-renormalizable theory. No field redefinition can turn it into a renormalizable Lagrangian. It represents the sum of all one-particle irreducible tree diagrams with infinitely heavy Higgs internal lines. It is easy to check that the amplitude for $\xi^+\xi^- \to \xi^+\xi^-$ derived from it coincides with that evaluated before in Eq. (2.26).

The Higgs particle was originally a member of an $\mathrm{SU(2)_L}$ doublet (see Eq. (2.19)). To separate it from the rest of the doublet consistently with the gauge invariance, we have performed the field transformation in Eq. (2.27). This leads to a low-energy theory for the would-be Goldstone modes $\vec{\xi}$ with the electroweak symmetry realized non-linearly [7, 9–11].The occurrence of infinitely many higher-dimensional operators with coefficients not suppressed by inverse power of M signals the failure of the decoupling property.

This failure persists at the quantum level. However, owing to a remarkable property of the SM, at one-loop order the dependence of the generic physical observable upon the Higgs mass is only logarithmic (screening theorem) [5]. Power-like effects are possible, but only at higher orders. In physical terms this means that detection of the Higgs through its virtual effects will not be easy. This screening effect is strictly related to the minimal structure of the SM and power-like dependencies can be generated in modest extensions of the SM as, for instance, in models containing two scalar doublets [12].

Chiral fermions, that is fermions whose left and right-handed components transform according to inequivalent representations of the gauge group (contrary to vector-like fermions), provide other examples of violation of the decoupling property [13]. Chiral fermions do not admit gauge invariant mass terms and their masses are generated via the spontaneous breaking of the gauge symmetry, from Yukawa interactions. When a chiral fermion is made heavier than the other matter fields by a relatively large Yukawa coupling, its effects at low energies do not decouple. The mass suppression associated to the propagator can be compensated by the mass enhancement provided by vertices with an overall non-vanishing effect. This mechanism is well exemplified in the SM by the top quark. The top is by far the heaviest of the known fermions. The top Yukawa coupling - y_t - is of order one (about 0.6 for $m_t = 150$ GeV), much larger than the other Yukawa couplings and comparable with the $SU(2)_L$ gauge coupling g ($g \simeq 0.65$). In the ideal case where we could neglect g and g', the top quark would provide extremely clean signals of breakdown of the decoupling property, ordered only by powers of y_t and, as we shall see, easy to compute. In the real world y_t and the gauge couplings g and g' are of the same order and, depending on the physical observable considered, we expect significant corrections to the previous, ideal case.

For the time being we consider this ideal, gaugeless limit of the SM, and first we look for the low-energy effective action for an heavy top quark, in the tree-level approximation. As we have seen previously, we have to solve the classical equation of motion for the top quark, in the limit $E \ll m_t$. The Lagrangian for the quarks reads:

$$\mathcal{L}_q = i\bar{q}_L\gamma^\mu D_\mu q_L + i\bar{q}_R\gamma^\mu D_\mu q_R - \left[\bar{q}_L H \frac{m_q}{v} q_R + \text{h.c.}\right] . \qquad (2.31)$$

We have put both the up and the down type quarks in a single multiplet q and indices in the generation space are understood. The covariant derivatives acting on the left and right-handed quarks are defined below:

$$D_\mu q_L = \left(\partial_\mu - g\hat{W}_\mu - g'\hat{B}_\mu^{(L)}\right) q_L , \qquad (2.32)$$

$$D_\mu q_{\rm R} = \left(\partial_\mu - g'\hat{B}_\mu^{(\rm R)}\right) q_{\rm R} \,. \tag{2.33}$$

The combinations $\hat{B}_\mu^{(\rm L,R)}$ are given by:

$$\hat{B}_\mu^{(\rm L)} = \frac{1}{6i} B_\mu \,, \tag{2.34}$$

$$\hat{B}_\mu^{(\rm R)} = \frac{1}{2i}\left(\tau^3 + \frac{1}{3}\right) B_\mu \,. \tag{2.35}$$

In the large m_t limit, the only relevant term of the action is the Yukawa coupling of Eq. (2.31), which, more explicitly, reads:

$$\mathcal{L}_Y = -\frac{\sigma}{v}\left(\bar{u}_{\rm L}^0 \bar{d}_{\rm L}^0\right) U \begin{pmatrix} m_u & 0 \\ 0 & m_d \end{pmatrix} \begin{pmatrix} u_{\rm R}^0 \\ d_{\rm R}^0 \end{pmatrix} + \text{h.c.} \,, \tag{2.36}$$

where m_u and m_d are the 3 by 3 quark mass matrices in the up and down sectors, respectively. Diagonal mass matrices $m_u^{\rm D}$, $m_d^{\rm D}$ and mass eigenstates $u_{\rm L,R}$, $d_{\rm L,R}$ are introduced via a bi-unitary transformation:

$$\begin{aligned} m_u^{\rm D} &= \mathcal{V}_{\rm L}^u m_u \mathcal{V}_{\rm R}^{u\,\dagger} \,, \\ m_d^{\rm D} &= \mathcal{V}_{\rm L}^d m_d \mathcal{V}_{\rm R}^{d\,\dagger} \,, \end{aligned} \tag{2.37}$$

$$\begin{aligned} u_{\rm L,R} &= \mathcal{V}_{\rm L,R}^u u_{\rm L,R}^0 \,, \\ d_{\rm L,R} &= \mathcal{V}_{\rm L,R}^d d_{\rm L,R}^0 \,. \end{aligned} \tag{2.38}$$

The Yukawa Lagrangian, in terms of mass eigenvalues and eigenstates, reads:

$$\mathcal{L}_Y = -\frac{\sigma}{v}\left(\bar{u}_{\rm L} \bar{d}_{\rm L}^W\right) U \begin{pmatrix} m_u^D & 0 \\ 0 & V_{\rm CKM} m_d^D V_{\rm CKM}{}^\dagger \end{pmatrix} \begin{pmatrix} u_{\rm R} \\ d_{\rm R}^W \end{pmatrix} + \text{h.c.} \,, \tag{2.39}$$

where $V_{CKM} = \mathcal{V}_{\rm L}^d \mathcal{V}_{\rm L}^{u\,\dagger}$ is the Cabibbo–Kobayashi–Maskawa (CKM) mixing matrix and $d_{\rm L,R}^W = V_{\rm CKM} d_{\rm L,R}$. If we assume that all fermion masses but the top one are zero, we can isolate from the previous equation the top sector:

$$\mathcal{L}_Y = -\frac{\sigma}{v}\left(\bar{t}_{\rm L} \bar{b}_{\rm L}^W\right) U \begin{pmatrix} m_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_{\rm R} \\ b_{\rm R}^W \end{pmatrix} + \text{h.c.} \,. \tag{2.40}$$

Notice that the top couples to the "weak" bottom combination:

$$b^W = V_{td} d + V_{ts} s + V_{tb} b \,, \tag{2.41}$$

mainly made of the physical bottom quark.

The equations of motion for the top quark, in the large m_t limit, decouple into two separate equations for the left and for the right-handed components:

$$U\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}t_{\rm R}\\ b^W_{\rm R}\end{pmatrix} = 0\,, \tag{2.42}$$

$$\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}U^\dagger\begin{pmatrix}t_{\rm L}\\ b^W_{\rm L}\end{pmatrix} = 0\,, \tag{2.43}$$

whose solutions are:

$$t_{\rm R} = 0\,, \tag{2.44}$$

$$\begin{pmatrix}t_{\rm L}\\ b^W_{\rm L}\end{pmatrix} = U\begin{pmatrix}0\\ b^{W\prime}_{\rm L}\end{pmatrix}\,. \tag{2.45}$$

Inserting back these solutions in the Lagrangian $\mathcal{L}_q$ of Eq. (2.31), one obtains [14]:

$$\mathcal{L}_q = i\bar{b}\gamma^\mu\partial_\mu b + \frac{g'}{3}B_\mu\bar{b}\gamma^\mu b - \frac{i}{2}\bar{b}^W_{\rm L}\gamma^\mu b^W_{\rm L}\,{\rm Tr}(TV_\mu) + \ldots\,, \tag{2.46}$$

where

$$T = U\tau^3U^\dagger\,, \tag{2.47}$$

$$V_\mu = D_\mu U\cdot U^\dagger\,, \tag{2.48}$$

and dots stand for the remaining light quarks. The gauge invariant combination:

$$-\frac{i}{2}\bar{b}^W_{\rm L}\gamma^\mu b^W_{\rm L}\,{\rm Tr}(TV_\mu) = \bar{b}^W_{\rm L}\gamma^\mu b^W_{\rm L}\left[\frac{g}{2\cos\theta}Z_\mu + \frac{\partial_\mu\xi^3}{v} + \ldots\right]\,, \tag{2.49}$$

contains infinitely many terms representing interactions of the $V - A$ bottom current with gauge and would-be Goldstone bosons. This is precisely the term which represents the non-vanishing sum of all tree diagrams with internal, heavy top lines. (The apparent flavour changing neutral current contained in the first term of the expansion $Z_\mu\bar{b}^W_{\rm L}\gamma^\mu b^W_{\rm L}$ is cancelled by the contributions from the light fermions.)

As we shall see in the next Sections, at one-loop measurable quantities start depending upon the square of the top mass, leading to new physical effects and making non-trivial the agreement between the SM predictions and the available data from precision experiments.

3. Non-decoupling in the real world

In this section we summarize the phenomenological relevance of the large one-loop top quark electroweak corrections.

An important development took place in 1987, with the discovery of $B^0 - \bar{B}^0$ oscillations. It became then clear that the top quark was much heavier than previously expected.

The first evidence of $B^0 - \bar{B}^0$ oscillations was found by the UA1 collaboration [15]. The theoretical interpretation of their data was however difficult, since B_d^0 and B_s^0 were produced in an essentially unknown mixture. Later on the signal was confirmed by the ARGUS [16] and CLEO [17] collaborations, who found evidence for equal sign dileptons in the decay of $\Upsilon(4s)$. The $\Upsilon(4s)$ resonance, through its decay into a $B_d^0 - \bar{B}_d^0$ pair, gives rise to a final state containing two charged leptons: l^+, coming from B_d^0 and l^-, coming from $\bar{B}_d^0$. If a $B_d^0 - \bar{B}_d^0$ mixing is allowed, then, sometimes the B_d^0 decay produces l^-, the $\bar{B}_d^0$ gives rise to l^+ and, in a fraction of the events, one will find equal sign dileptons. The relevant parameter is the ratio of equal sign to opposite sign leptons:

$$r_d = \left. \frac{N(l^+l^+) + N(l^-l^-)}{N(l^+l^-)} \right|_{\Upsilon(4s)}, \tag{3.1}$$

which experimentally is given by:

$$r_d = 0.17 \pm 0.10 . \tag{3.2}$$

More recently also the LEP collaborations have found an excess of like sign lepton pairs in $e^+e^- \to l^\pm l^\pm + X$, coming both from B_d^0 and B_s^0. Moreover they were able to detect the predicted time dependence of the B_d^0 oscillations [18].

The $B_d^0 - \bar{B}_d^0$ system is described by the two- dimensional effective Hamiltonian:

$$H = \begin{pmatrix} M & M_{12} \\ M_{12}^* & M \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma \end{pmatrix}, \tag{3.3}$$

written in the base $(B_d^0 \bar{B}_d^0)$. In the B system Γ_{12} is approximately zero and the two eigenstates of the hamiltonian have essentially the same width Γ, which can be extracted directly from the measured lifetime τ_B. On the other hand, the mass difference ΔM between the two eigenvalues is given by:

$$\Delta M = 2|M_{12}| . \tag{3.4}$$

Introducing the two parameters:

$$x = \frac{\Delta M}{\Gamma}, \qquad y = \frac{\Delta \Gamma}{2\Gamma}, \tag{3.5}$$

one obtains:

$$r_d = \frac{x^2 + y^2}{2 + x^2 - y^2} \simeq \frac{x^2}{2 + x^2}. \tag{3.6}$$

The x parameter can be estimated from the $\Delta B = 2$ non-leptonic effective hamiltonian, which in the SM arises as a result of a second order weak interaction:

$$H(\Delta B = 2) = \frac{G_F}{16\pi^2}(V_{tb}^* V_{td})^2 m_t^2 f\left(\frac{m_t^2}{M_W^2}\right) \eta \bar{b}\gamma^\mu(1-\gamma_5)d\bar{b}\gamma_\mu(1-\gamma_5)d, \tag{3.7}$$

where η is a factor of order one which accounts for the QCD corrections and $f(x)$ is a slowly varying function of x:

$$f(x) = \frac{1}{4} + \frac{9}{4(1-x)} - \frac{3}{2(1-x)^2} + \frac{3}{2}\frac{x^2 \ln x}{(x-1)^3}, \tag{3.8}$$

with $f(1) = {}^3\!/_4$ and $f(\infty) = {}^1\!/_4$. From $H(\Delta B = 2)$ and Eq. (3.4) one can derive ΔM:

$$\Delta M = 2|\langle \bar{B}_d^0 | H(\Delta B = 2) | B_d^0 \rangle| \,. \tag{3.9}$$

This requires the computation of the hadronic matrix element:

$$\langle \bar{B}_d^0 | \bar{b}\gamma^\mu(1-\gamma_5)d\bar{b}\gamma_\mu(1-\gamma_5)d | B_d^0 \rangle = \tfrac{4}{3} B_B f_B^2 m_B \,, \tag{3.10}$$

where m_B is the B meson mass, f_B its decay constant and B_B parametrizes a possible departure from the so-called vacuum saturation approximation in which $B_B = 1$. From equations (3.5), (3.7), (3.9) and (3.10) one obtains:

$$x = \frac{G_F^2}{6\pi^2} m_t^2 f\left(\frac{m_t^2}{M_W^2}\right) B_B f_B^2 m_B \tau_B |V_{tb}^* V_{td}|^2 \eta \,. \tag{3.11}$$

Notice the leading quadratic dependence of x on the top mass, coming from the box diagram which, in the SM, gives rise to $H(\Delta B = 2)$. We shall come back to this point in section 4. Apart from the top mass, x also depends on the hadronic parameter $B_B f_B$ and the CKM combination $|V_{tb}^* V_{td}|$. On the former quantity, we have estimates from the QCD sum rules approach and from the lattice, from which we expect $B_B f_B$ in the range 100 – 300 MeV [19]. The CKM angles involving the top quark are presently unknown, but restrictions on them can be derived from the unitarity of the CKM matrix V assuming that only three generations are present. One finds [20]:

$$0.003 \leq |V_{tb}^* V_{td}| \leq 0.019 \,. \tag{3.12}$$

Finally $\eta = 0.78 - 0.85$ [21]. Putting everything together [22], one realizes that, even pushing all the unknown quantities to the extreme upper limit compatible with the present bounds or our theoretical understanding, a large value for m_t ($m_t \geq 50$ GeV) is required to have consistence with the experimental value of r_d.

Other observables affected by potentially large m_t corrections are those related to the electroweak precision measurements done at LEP/SLC. With the exception of the partial width of the Z into $\bar{b}b$, which we will discussed at the end of this section, the leading top quark effects, at one-loop level, are dominated by the gauge bosons self-energy corrections. To count the number of independent parameters occurring in this sector [23], we start from the usual definition:

$$-i\Pi_{ij}^{\mu\nu}(p) = -i\left[\Pi_{ij}(p^2)g^{\mu\nu} + (p^\mu p^\nu \text{ terms})\right] \,, \tag{3.13}$$

where $-i\Pi_{ij}^{\mu\nu}(p)$ denote the set of self-energies for the gauge boson fields. The indices i, j can take the values 0 (for the field B) and $1, 2, 3$ (for the fields W^i), or, alternatively, the values γ, Z, W. From now on we will discard the irrelevant terms proportional to $p^\mu p^\nu$. Furthermore, we make a Taylor expansion of the top contribution to the scalar function $\Pi_{ij}(p^2)$, around the point $p^2 = 0$:

$$\Pi_{ij}(p^2) = A_{ij} + p^2 F_{ij} + \ldots \,. \tag{3.14}$$

This expansion, meaningful for $p^2 \ll m_t^2$ contains real coefficients A_{ij}, F_{ij}, *etc.* Moreover, since $\Pi_{ij}(p^2)$ has dimension two in units of mass, it is reasonable to neglect the dots in Eq. (3.14), representing terms suppressed by positive powers of (p^2/m_t^2).

As a consequence of the exact electromagnetic gauge invariance, we have $A_{\gamma\gamma} = A_{\gamma Z} = 0$. (More precisely, the fermionic contribution to $A_{\gamma Z}$ vanishes, and the bosonic one is zero in the unitary gauge.) Then we are left with the six independent coefficients A_{ZZ}, A_{WW}, $F_{\gamma\gamma}$, $F_{\gamma Z}$, F_{ZZ}, F_{WW}, carrying the main dependence on m_t. Three combinations of them are however unobservable, being related to the fundamental constants of the electroweak theory: the electromagnetic fine structure constant α, the Fermi constant $G_{\rm F}$ and the mass of the Z gauge vector boson M_Z. Indeed, the quantum corrections induced by the gauge vector boson self-energies provide the following shifts in the fundamental constants [1]:

$$\frac{\delta\alpha}{\alpha} = -F_{\gamma\gamma}\,, \tag{3.15}$$

$$\frac{\delta G_F}{G_F} = \frac{A_{WW}}{M_W^2}\,, \tag{3.16}$$

$$\frac{\delta M_Z^2}{M_Z^2} = -\left(\frac{A_{ZZ}}{M_Z^2} + F_{ZZ}\right)\,. \tag{3.17}$$

We conclude that, in our approximation, the parameters carrying in the SM the leading top quark dependence are three combinations among the six coefficients A_{ZZ}, A_{WW}, $F_{\gamma\gamma}$, $F_{\gamma Z}$, F_{ZZ}, F_{WW}. These combinations can be identified by looking at the radiative corrections for three independent physical observables, which we choose as the ratio of the gauge boson masses M_W/M_Z, the forward-backward asymmetry A^μ_{FB} in $e^+e^- \to \mu^+\mu^-$ at the Z peak and the partial width of the Z into charged leptons, Γ_l.

- $\dfrac{M_W}{M_Z}$

We trade M_W/M_Z for the observable Δr_W defined as follows:

$$\left(\frac{M_W}{M_Z}\right)^2 = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} - \frac{\mu^2}{M_Z^2(1-\Delta r_W)}}\,, \tag{3.18}$$

where

$$\mu^2 = \frac{\pi\alpha(M_Z^2)}{\sqrt{2}G_F} = (38.454\ \mathrm{GeV})^2\,. \tag{3.19}$$

One finds:

$$\Delta r_W = -\frac{\cos^2\theta}{\sin^2\theta}\left(\frac{A_{ZZ}}{M_Z^2} - \frac{A_{WW}}{M_W^2}\right) + \frac{\cos 2\theta}{\sin^2\theta}(F_{WW} - F_{33}) + 2\frac{\cos\theta}{\sin\theta}F_{30}\,. \tag{3.20}$$

- A^μ_{FB}, Γ_l

By forward-backward asymmetry at the peak we mean the quantity quoted by the LEP experiments, which is corrected for all QED effects,

[1] In a general analysis of the one-loop corrections one should also include in δG_F contributions coming from boxes, vertices and fermion self-energies. Similarly, the right-hand side of Eq. (3.17) would read $-\Pi_{ZZ}(M_Z^2)/M_Z^2$ [24, 25]. However, since here we are only interested in the dependence upon the top quark mass, the additional contributions can be neglected.

including initial and final state radiation and also for the effect of the imaginary part of the photon vacuum polarization diagram. The partial width of Z into charged leptons is inclusive of photon emissions: $\Gamma_l = \Gamma(Z -> l\bar{l} + \text{photons})$.

Also in this case we proceed through a series of definitions inspired to the lowest order relations:

$$A^{\mu}_{FB}(p^2 = M_Z^2) = 3\left(\frac{g_V g_A}{g_V^2 + g_A^2}\right)^2 , \tag{3.21}$$

$$\Gamma_l = \frac{G_F M_Z^3}{6\pi\sqrt{2}} (g_V^2 + g_A^2)\left(1 + \frac{3\alpha}{4\pi}\right) , \tag{3.22}$$

$$g_A = \tfrac{1}{2}\left(1 + \frac{\Delta\rho}{2}\right) , \tag{3.23}$$

$$\frac{g_V}{g_A} = -1 + 4\sin^2\hat{\theta} , \tag{3.24}$$

$$\sin^2\hat{\theta} = (1 + \Delta k)\sin^2\tilde{\theta} , \tag{3.25}$$

$$\sin^2\tilde{\theta} = \tfrac{1}{2} - \sqrt{\tfrac{1}{4} - \frac{\mu^2}{M_Z^2}} = 0.23118 \ (\text{for} \ \ M_Z = 91.187 \text{ GeV}) . \tag{3.26}$$

With these definitions, the knowledge of A^{μ}_{FB}, which depends only on the ratio g_V/g_A is equivalent to that of the parameter Δk, given in Eq. (3.25). On the other hand the parameter $\Delta\rho$, entering the definition of the Z coupling to charged leptons as an overall factor, is fixed by Γ_l. One finds:

$$\Delta k = -\frac{\cos^2\theta}{\cos 2\theta}\left(\frac{A_{ZZ}}{M_Z^2} - \frac{A_{WW}}{M_W^2}\right) + \frac{1}{\cos 2\theta}\frac{\cos\theta}{\sin\theta} F_{30} , \tag{3.27}$$

$$\Delta\rho = \frac{A_{ZZ}}{M_Z^2} - \frac{A_{WW}}{M_W^2} . \tag{3.28}$$

Indeed the whole set of self-energy corrections can be accounted for by an effective neutral current Hamiltonian given by:

$$H_{NC} = \left(4\sqrt{2} G_F M_Z^2\right)^{1/2}\left(1 + \frac{\Delta\rho}{2}\right)\left[J^{\mu}_{3L} - (1 + \Delta k)\sin^2\tilde{\theta} J^{\mu}_{\text{em}}\right] Z_{\mu} . \tag{3.29}$$

So far we have selected three physical quantities in order to isolate their leading dependence upon the top quark mass. By looking at the expressions obtained for the quantities Δr_W, Δk and $\Delta\rho$, we recognize that they are functions of the following three combinations of self-energy corrections [23, 26, 27]:

$$\begin{aligned}\epsilon_1 &= \frac{A_{ZZ}}{M_Z^2} - \frac{A_{WW}}{M_W^2}\,,\\ \epsilon_2 &= F_{WW} - F_{33}\,,\\ \epsilon_3 &= \frac{\cos\theta}{\sin\theta}F_{30}\,.\end{aligned} \tag{3.30}$$

We can summarize the results as follows:

$$\begin{aligned}\Delta r_W &= -\frac{\cos^2\theta}{\sin^2\theta}\epsilon_1 + \frac{\cos 2\theta}{\sin^2\theta}\epsilon_2 + 2\epsilon_3\,,\\ \Delta k &= -\frac{\cos^2\theta}{\cos 2\theta}\epsilon_1 + \frac{1}{\cos 2\theta}\epsilon_3\,,\\ \Delta\rho &= \epsilon_1\,.\end{aligned} \tag{3.31}$$

We remind that the relationship exhibited by Eqs (3.30)–(3.31) reflects the fact that in the SM most of the top quark contribution to the considered observables is contained in the vacuum polarization functions of the vector gauge bosons, suitably expanded as in Eq. (3.14). The more general dependence of Δr_W, Δk and $\Delta\rho$ on the SM radiative corrections can be easily derived along lines similar to those followed here, and it would include vertex, box and fermion self-energy corrections as well. The latter do not contain any further significant dependence on m_t.

Within the SM, the combinations in Eq. (3.30) have the following asymptotic dependence on m_t:

$$\epsilon_1 = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}} + \dots\,, \tag{3.32}$$

$$\epsilon_2 = -\frac{G_F M_W^2}{2\pi^2\sqrt{2}}\ln\left(\frac{m_t}{M_Z}\right) + \dots\,, \tag{3.33}$$

$$\epsilon_3 = -\frac{G_F M_W^2}{6\pi^2\sqrt{2}}\ln\left(\frac{m_t}{M_Z}\right) + \dots\,. \tag{3.34}$$

Notice that the potentially largest top quark correction, namely the one quadratic in m_t, appears only in ϵ_1, while in ϵ_2 and ϵ_3 the dependence on m_t is only logarithmic.

Eq. (3.31) is the starting point of the so-called non-standard analysis of the electroweak data [27, 28]. Indeed, forgetting about the way Eq. (3.31) was derived, one can take it as the *definition* of the ϵ parameters, which become true physical observables, with the advantage that the strongest dependence on m_t has been confined in ϵ_1. The inclusion of a larger set of experimental data, to provide further information on the ϵ parameters, demands some further assumptions, which can be ordered according to an increasing amount of model dependence. This offers a common ground to compare various theoretical frameworks (SM [27, 28], minimal supersymmetric standard model [29], extended gauge models [30], ...). From the experimental values $M_W/M_Z = 0.8798 \pm 0.0028$, $A^l_{FB} = 0.0170 \pm 0.0016$ and $\Gamma_l = 83.975 \pm 0.20$ MeV, one finds [31]:

$$\begin{aligned}\epsilon_1 &= (0.42 \pm 0.24) \cdot 10^{-2}\,,\\ \epsilon_2 &= (-0.25 \pm 0.56) \cdot 10^{-2}\,,\\ \epsilon_3 &= (0.35 \pm 0.31) \cdot 10^{-2}\,.\end{aligned} \tag{3.35}$$

If vacuum polarization corrections were always dominating, at least for the part concerning the top dependence, then, from the effective hamiltonian H_{NC} in Eq. (3.29), one would conclude that, for all flavours f, the partial width Γ_f of the Z boson into $f\bar{f}$ is given by[2]:

$$\Gamma_f = N_C^f \frac{G_F M_Z^3}{6\pi\sqrt{2}} \left[(g_V^f)^2 + (g_A^f)^2\right]\,, \tag{3.36}$$

with g_V^f and g_A^f given by:

$$g_V^f = \left(1 + \frac{\Delta\rho}{2}\right)\left(T_{3L}^f - 2Q_{\rm em}^f \sin^2\hat{\theta}\right)\,, \tag{3.37}$$

$$g_A^f = -\left(1 + \frac{\Delta\rho}{2}\right) T_{3L}^f\,. \tag{3.38}$$

Then the effective fermionic couplings of the Z boson would be characterized by universal, flavour-independent, corrections: $\Delta\rho$ and Δk. However, because of the occurrence of vertex corrections, this conclusion is not true, not even for the top contribution, and the largest violation of Eqs (3.37)–(3.38) takes place for $f = b$. For Z decaying into $b\bar{b}$, besides the vacuum polarization effects one should also take into account the vertex corrections and the fermion self-energy corrections, where the exchange of charged unphysical scalars gives rise to additional terms quadratic in m_t [32]. In this case one has still the expression of Eq. (3.36) for the partial width Γ_b, but g_A^b and g_V^b are replaced by:

$$g_A^b = \tfrac{1}{2}\left(1 + \frac{\Delta\rho}{2}\right)(1 + \epsilon_b)\,, \tag{3.39}$$

$$\frac{g_V^b}{g_A^b} = \frac{-1 + \frac{4}{3}\sin^2\hat{\theta} - \epsilon_b}{1 + \epsilon_b}\,. \tag{3.40}$$

[2] Apart from QED and QCD corrections

The new corrections are isolated in the parameter ϵ_b, whose asymptotic dependence on the top quark mass reads:

$$\epsilon_b = -\frac{G_F m_t^2}{4\pi^2\sqrt{2}} + \dots . \tag{3.41}$$

The additional corrections can be derived from an effective Hamiltonian of the form:

$$H_b = -\tfrac{1}{2}\epsilon_b \left(4\sqrt{2}G_F M_Z^2\right)^{1/2} Z^\mu \bar{b}_L \gamma_\mu b_L . \tag{3.42}$$

The presence of the ϵ_b term in Γ_b, quadratic in m_t, singles out this partial width as particularly interesting quantity, whose peculiar dependence on m_t is potentially able to provide additional and independent information on the top quark.

From the present value $\Gamma_b = 385.3 \pm 3.9$ MeV and by removing from Γ_b the QCD correction, one obtains [31]:

$$\epsilon_b = (0.46 \pm 0.45) \cdot 10^{-2} . \tag{3.43}$$

4. An effective Lagrangian for the heavy top quark

As examples of violation of the decoupling property, we have seen that the heaviness of the top quark shows up in three independent effects: the $B^0 - \bar{B}^0$ oscillations; the vacuum polarization corrections (in particular the $\Delta\rho$ parameter) affecting all LEP/SLC observables and the M_W/M_Z mass ratio; the non-universal correction of the $Zb\bar{b}$ vertex detectable through the measure of the Γ_b partial width. These effects can be described by the effective hamiltonians given in Eqs (3.7) (3.29) and (3.42). In this section we will discuss the general structure of the effective Lagrangian which reproduces, at one-loop order, the above mentioned effects [14, 33].

To start with we observe that, one-loop results are not correctly reproduced by the effective Lagrangian we have derived in the large m_t limit in Section 2, namely $\mathcal{L}_{\rm cl}$ given by the sum of $\mathcal{L}_q$ of Eq. (2.46), $\mathcal{L}_H$ of Eq. (2.18) and the terms for the other light fermions and the gauge vector bosons. This has to do with the fact that $\mathcal{L}_{\rm cl}$ was obtained via a classical limit, corresponding to the sum of all tree-level diagrams containing heavy top quark lines. To deal correctly with the one-loop computation, we must first perform the (regularized) loop integration and subsequently take the large m_t limit. In general this leads to a result which is a divergent function of the ultraviolet cutoff. For this reason the opposite way, namely first taking the large m_t limit — which amounts to use $\mathcal{L}_{\rm cl}$ — and then performing the loop integration, generally leads to a different result. In formulae:

$$\lim_{m_t \to \infty} \int dk \; F_\Lambda(k,p) = \int dk \; \lim_{m_t \to \infty} F_\Lambda(k,p) + \Delta(p)\,, \qquad (4.1)$$

where p stands for a collection of external momenta, k is the loop variable, Λ an ultraviolet cutoff. The function $\Delta(p)$ represents the $O(\hbar)$ correction which we should add to the result obtained working with $\mathcal{L}_{\rm cl}$, to correctly reproduce the one-loop result in the large m_t limit.

The interesting fact is that $\Delta(p)$ can be represented as an effective Lagrangian. For external momenta lighter than m_t the integrals on both sides of the Eq. (4.1) have equal imaginary parts in all possible channels, and therefore the function $\Delta(p)$ is an analytic function of the variables p. If we consider its expansion in p^2, for dimensional reasons, there will be only a finite number of terms not vanishing in the large m_t limit. Moreover, for amplitudes with a sufficiently large number of external legs the loop integral is convergent, the $m_t \to \infty$ limit and the loop integral commute and $\Delta(p) = 0$. We conclude that at one-loop order the correct results of the large m_t limit are reproduced by adding to $\mathcal{L}_{\rm cl}$ a finite number of local terms, which we collectively denote by $\Delta\mathcal{L}$. The low-energy theory, in the $m_t \to \infty$ limit and to one-loop accuracy, is described by the effective Lagrangian:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_{\rm cl} + \Delta\mathcal{L}\,. \qquad (4.2)$$

The term $\Delta\mathcal{L}$ is further restricted by the symmetry of the low-energy theory. For the moment we require $\Delta\mathcal{L}$ to be $SU(2)_L \otimes U(1)_Y$ invariant (see however the next Section). In Section 2 we have already made use of nonlinear realizations of the $SU(2)_L \otimes U(1)_Y$ symmetry to describe $\mathcal{L}_{\rm cl}$ [7, 9–11]. The nonlinear realization naturally provides a low-energy expansion, ordered by the number of derivatives acting on the light fields. In our case such an expansion should contain at least the terms of order p^4. Indeed gauge invariance relates terms of order p^4 to terms of order p^2 containing two gauge vector bosons [34]. The latter is just what we have called F_{ij} in the analysis of vacuum polarization (see Eq. (3.14)). In addition to $SU(2)_L \otimes U(1)_Y$ gauge invariance we will also ask for CP invariance. We list below the invariant operators which are relevant to our discussion, containing up to four derivatives and built out the gauge vector bosons W, Z, A and the would be Goldstone bosons ξ[3]:

$$\mathcal{L}_0 = \frac{v^2}{4}[{\rm Tr}(TV_\mu)]^2\,,$$

[3] For a complete list, see, for instance, Ref.[35].

$$\mathcal{L}_1 = i\frac{gg'}{2}B_{\mu\nu}\,\mathrm{Tr}(T\hat{W}^{\mu\nu})\,,$$
$$\mathcal{L}_8 = \frac{g^2}{4}[\mathrm{Tr}(T\hat{W}_{\mu\nu})]^2\,. \tag{4.3}$$

These operators contribute to $\Delta\mathcal{L}$ in Eq. (4.3) through the term:

$$(\Delta\mathcal{L})_B = a_0\mathcal{L}_0 + a_1\mathcal{L}_1 + a_8\mathcal{L}_8\,. \tag{4.4}$$

In the fermionic sector, we consider the following invariant terms:

$$\mathcal{L}_1^b = \left(-\frac{i}{2}\right)\bar{b}_{\mathrm{L}}^W\gamma_\mu b_{\mathrm{L}}^W\,\mathrm{Tr}(TV_\mu)\,,$$
$$\mathcal{L}_2^b = \frac{G_F}{\sqrt{2}}(\bar{b}_{\mathrm{L}}^W\gamma_\mu b_{\mathrm{L}}^W)^2\,, \tag{4.5}$$

whose contribution to $\Delta\mathcal{L}$ is given by:

$$(\Delta\mathcal{L})_b = \beta_1\mathcal{L}_1^b + \beta_2\mathcal{L}_2^b\,. \tag{4.6}$$

The coefficients $a_0, a_1, a_8, \beta_1, \beta_2$ are easily found by comparing $(\Delta\mathcal{L})_B$, $(\Delta\mathcal{L})_b$ with the effective hamiltonians $H(\Delta B = 2)$ of Eq. (3.7), H_{NC} of Eq. (3.29) and H_b of Eq. (3.42)[4]. To do this one should expand the various combinations appearing in the expressions of the invariants $\mathcal{L}_i$. For instance:

$$\mathrm{Tr}(TV_\mu) = i\frac{g}{\cos\theta}Z_\mu + \frac{2i}{v}\partial_\mu\xi^3 + \ldots\,. \tag{4.7}$$

One has:

$$a_0\mathcal{L}_0 = -\tfrac{1}{4}a_0v^2(gW_\mu^3 - g'B_\mu)^2 + \ldots\,,$$
$$a_1\mathcal{L}_1 = \tfrac{1}{2}a_1gg'B_{\mu\nu}(\partial^\mu W^{3^\nu} - \partial^\nu W^{3^\mu}) + \ldots\,,$$
$$a_8\mathcal{L}_8 = -\tfrac{1}{4}a_8g^2(\partial_\mu W^3{}_\nu - \partial_\nu W^3{}_\mu)^2 + \ldots\,. \tag{4.8}$$

The dots stand for trilinear and quadrilinear terms in the gauge vector bosons and for terms containing the would-be Goldstone bosons, needed to

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[4] The contribution of $\mathcal{L}_{\mathrm{cl}}$ to the right-hand side of the Eq. (4.1) vanishes when we extract the leading top effects. Indeed $\mathcal{L}_{\mathrm{cl}}$ gives a divergent contribution to the relevant Green functions, which we choose to subtract at vanishing external momenta, as an additional prescription to deal with the new infinities of the effective non-renormalizable theory.

ensure the gauge invariance of each structure. By evaluating the contribution of the above terms to the vacuum polarization functions one can relate the a_i coefficients to the ϵ parameters as follows[5]:

$$\begin{aligned} \epsilon_1 &= 2a_0 , \\ \epsilon_2 &= -g^2 a_8 , \\ \epsilon_3 &= -g^2 a_1 . \end{aligned} \tag{4.9}$$

From the behaviour of the ϵ's in the large m_t limit, given in Eqs (3.32)–(3.34), we find:

$$\begin{aligned} a_0 &= \tfrac{1}{2}(\Delta\rho)_{\rm top} + \dots , \\ a_8 &= \frac{1}{16\pi^2}\left[\ln\left(\frac{m_t}{M_Z}\right)\right] + \dots , \\ a_1 &= \frac{1}{16\pi^2}\left[\tfrac{1}{3}\ln\left(\frac{m_t}{M_Z}\right)\right] + \dots , \end{aligned} \tag{4.10}$$

where we have defined:

$$(\Delta\rho)_{\rm top} = \frac{3G_F m_t^2}{8\pi^2\sqrt{2}} . \tag{4.11}$$

Similarly, in the fermionic sector one finds:

$$\begin{aligned} \beta_2 &= -\tfrac{1}{3}(\Delta\rho)_{\rm top}\left[4f\left(\frac{m_t^2}{M_W^2}\right)\eta\right] , \\ \beta_1 &= -\tfrac{2}{3}(\Delta\rho)_{\rm top} . \end{aligned} \tag{4.12}$$

The term in square brackets in the left-hand side of Eq. (1.12) is equal to one in the large m_t limit and for QCD interactions turned off.

So far we have just recast the content of the Section 3 into a more elegant form, which however does not seem to provide any additional information with respect to what already seen in the separate discussion of the various physical effects. To appreciate the usefulness of the point of view adopted here we will mention two facts.

As stressed in the second section, the violation of the decoupling property is related to a hierarchy of coupling constants which may arise by considering the low energy limit of a given fundamental theory. In the case

[5] ϵ_2 and ϵ_3 receive also an additional contribution from the operator $\mathcal{L}_{13}$ (see Ref. [36]), which however can be eliminated by using the equations of motion.

of an heavy top quark, such hierarchy, in the ideal case, is represented by the inequality:

$$y_l,\ g,\ g' \ll y_t\,, \tag{4.13}$$

where y_l and y_t are the Yukawa couplings for the light quarks and top quark, respectively. If we consider the extreme case when all the coupling constants but the top one are put to zero, we are lead to conclude that for the top quark the violation of the decoupling property is modeled by a pure Yukawa interaction. This point is particularly transparent in the effective Lagrangian we have obtained. In $\mathcal{L}_{\rm eff}$ the coefficients a_0, β_1 and β_2 represent the leading effects in the large m_t limit. If we turn the gauge interactions off, such effects do not collapse. Indeed the operators $\mathcal{L}_0$, $\mathcal{L}_1^b$ and $\mathcal{L}_2^b$ do not vanish, indicating the Yukawa origin in the SM of the largest corrections due to the top quark[6].

Second, on the practical side, the effective Lagrangian can be seen as the book-keeping of an infinite set of Ward identities which may be useful in actual computations. Rather than analyzing the general structure to these identities we will discuss their physical content on one example. Suppose we are interested in the evaluation of the coefficient β_1 of $\mathcal{L}_{\rm eff}$, which is related to the $Zb\bar{b}$ vertex correction. We should compute the contribution of the top quark to the operator $\mathcal{L}_1^b$. By expanding the exponential of the would-be Goldstone fields in the combination $\mathrm{Tr}(TV_\mu)$ (see Eq. (4.7)), one obtains:

$$\mathcal{L}_1^b = -\frac{i}{2}\bar{b}_{\rm L}^W \gamma_\mu b_{\rm L}^W \left[i\frac{g}{\cos\theta} Z_\mu + \frac{2i}{v}\partial_\mu \xi^3 + \ldots \right]. \tag{4.14}$$

This equation show that $\mathrm{SU(2)_L \otimes U(1)_Y}$ gauge invariance relates the $Zb\bar{b}$ function to the $\xi^3 b\bar{b}$ function in a well precise way and that, to compute β_1, we can in fact consider the latter, by retaining the term linear in the ξ^3 momentum.

It is clear that one does not need the effective Lagrangian $\mathcal{L}_{\rm eff}$ to derive the Ward Identities of $\mathrm{SU(2)_L \otimes U(1)_Y}$, which are implied just by the symmetry and the particle content. It is however true that many of these identities can be in practice read immediately from $\mathcal{L}_{\rm eff}$, with no further effort. In this sense, the situation closely resembles to what one had with the current algebra (whose analogue here is $\mathrm{SU(2)_L \otimes U(1)_Y}$) and the PCAC hypothesis (the spontaneous breaking of the symmetry) in the old times. Indeed they can be either analyzed in abstract or, as happened with the nonlinear σ-model, in the context of a specific field theoretical realization.

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[6] More generally, in the gaugeless limit we may regard the gauge vector bosons appearing in the various operators as classical external fields coupled to light fermions, Higgs and Goldstone bosons, which are the quantum degrees of freedom in the surviving Yukawa theory.

Each of the two possibilities has its own advantages and can be preferred depending on the specific problem at hand.

In the last part of this Section we will show how to compute the coefficients a_0, β_1 and β_2 exploiting the relevant Ward identities and the underlying Yukawa nature of the effects. We start from the definition of the gaugeless limit of the SM in the top-bottom sector, assuming a vanishing mass for the bottom:

$$\begin{aligned}\mathcal{L}_Y &= i\bar{b}\gamma^\mu\partial_\mu b + i\bar{t}\gamma^\mu\partial_\mu t + \frac{v^2}{4}\,\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) \\ &- \left(\bar{t}_{\rm L}\bar{b}_{\rm L}^W\right) U \begin{pmatrix} m_t & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} t_{\rm R} \\ b_{\rm R}^W \end{pmatrix} + \text{h.c.} + \dots \qquad (4.15)\end{aligned}$$

The dots stand for additional terms as, for instance, those depending on the Higgs field. The Feynman rules read:

$$\begin{aligned}\xi^+\bar{t}b &\leftrightarrow -\frac{\sqrt{2}}{v}m_t a_- \,, \\ \xi^-\bar{b}t &\leftrightarrow \frac{\sqrt{2}}{v}m_t a_+ \,, \\ \xi^3\bar{t}t &\leftrightarrow \frac{m_t}{v}(a_+ - a_-)\,, \\ \xi^+\bar{b}b &\leftrightarrow 0\,, \qquad (4.16)\end{aligned}$$

where

$$a_\pm = \frac{1\pm\gamma_5}{2}\,. \qquad (4.16)$$

- $B^0 - \bar{B}^0$ oscillations

To compute β_2 we consider the top quark contribution to the four-fermion operator $\mathcal{L}_2^b$. In the Yukawa theory defined above by $\mathcal{L}_Y$ such contribution is represented by two independent box diagrams, with top and charged Goldstone bosons circulating in the loop, which we will evaluate taking vanishing external momenta. The total amplitude $A_{\rm box}$, ultraviolet convergent, is given by:

$$\begin{aligned}A_{\rm box} &= \int \frac{d^4k}{(2\pi)^4}\bar{u}(0)\left(\frac{\sqrt{2}}{v}m_t a_+\right)\frac{i}{\not{k}-m_t}\left(-\frac{\sqrt{2}}{v}m_t a_-\right)u(0)\frac{i}{k^2} \\ &\cdot \bar{v}(0)\left(\frac{\sqrt{2}}{v}m_t a_+\right)\frac{i}{\not{k}-m_t}\left(-\frac{\sqrt{2}}{v}m_t a_-\right)v(0)\frac{i}{k^2} + \text{ crossed} \\ &= -\frac{i}{v^2}\frac{(\Delta\rho)_{\rm top}}{3}\left[\bar{u}(0)\gamma^\mu a_- u(0)\cdot\bar{v}(0)\gamma^\mu a_- v(0)\right. \\ &\left. + \bar{v}(0)\gamma^\mu a_- u(0)\cdot\bar{u}(0)\gamma^\mu a_- v(0)\right]. \qquad (4.18)\end{aligned}$$

From comparison with Eqs (4.5)–(4.6), one has:

$$\beta_2 = -\tfrac{1}{3}(\Delta\rho)_{\rm top} \tag{4.19}$$

which is the correct large m_t limit of β_2 in Eq. (4.12).

- $Z\bar{b}b$ vertex corrections [32]
  From Eqs (4.6) and (4.7) one has:

$$\beta_1 \mathcal{L}_1^b = \beta_1 \left[\frac{g}{2\cos\theta} Z_\mu + \frac{1}{2v}\partial_\mu \xi^3 + \dots\right] \bar{b}_{\rm L}^W \gamma_\mu b_{\rm L}^W \,. \tag{4.20}$$

We choose to compute the correction to the $\xi^3 \bar{b}b$ Green function. From the second term in the previous equation, we derive the amplitude:

$$A_{\xi^3\bar{b}b} = \beta_1 \bar{u}(p)\frac{\not{p}}{v} a_- v(0)\,, \tag{4.21}$$

where p_μ is the four-momentum of the incoming ξ^3. We should compare this result with the (finite) contribution due to the top-ξ loop, which reads:

$$\begin{aligned} A_{\xi^3\bar{b}b} = & \int \frac{d^4k}{(2\pi)^4}\bar{u}(p)\left(\frac{\sqrt{2}}{v}m_t a_+\right)\frac{i}{\not{k}+\not{p}-m_t}\frac{m_t}{v}(a_+ - a_-) \\ & \cdot \frac{i}{\not{k}-m_t}\left(-\frac{\sqrt{2}}{v}m_t a_-\right) v(0)\frac{i}{k^2} \\ = & -2i\frac{m_t^2}{v^2}\int \frac{d^4k}{(2\pi)^4}\frac{m_t^2}{k^2(k^2-m_t^2)^2}\cdot \bar{u}(p)\frac{\not{p}}{v}a_- v(0) + \dots \\ = & -\frac{2}{3}(\Delta\rho)_{\rm top}\cdot \bar{u}(p)\frac{\not{p}}{v}a_- v(0)\,, \end{aligned} \tag{4.22}$$

where dots in the second equality stand for higher-order terms in p. From Eqs (4.21) and (4.22) one finds:

$$\beta_1 = -\frac{2}{3}(\Delta\rho)_{\rm top} \tag{4.23}$$

in agreement with the result given in Eq. (4.12).

- $\Delta\rho$ parameter [37]
  To evaluate the a_0 coefficient, we look at the expansion of the operator $a_0\mathcal{L}_0$:

$$a_0\mathcal{L}_0 = a_0\frac{v^2}{4}[{\rm Tr}(TV_\mu)]^2 = a_0\left[\frac{1}{4}\frac{g^2v^2}{\cos^2\theta}Z^\mu Z_\mu + \dots + \partial^\mu\xi^3\partial_\mu\xi^3 + \dots\right]. \tag{4.24}$$

From this expression we notice that a_0 represents also a wave function renormalization of the ξ^3 field. Thus we are lead to consider the two-point function $-i\Pi_{\xi^3\xi^3}(p^2)$ for the field ξ^3. To match the results in the fundamental theory and in the effective one, we have to impose:

$$\Pi^{tt}_{\xi^3\xi^3}(p^2) + \Pi^{C}_{\xi^3\xi^3}(p^2) = \Pi^{0}_{\xi^3\xi^3}(p^2)\,, \tag{4.25}$$

where $\Pi^{tt,C,0}_{\xi^3\xi^3}(p^2)$ are, respectively, the contribution of the top quark loop, the contribution of the counter term and the contribution of $a_0\mathcal{L}_0$. The counter term needed to cancel the divergences of $\Pi^{tt}_{\xi^3\xi^3}(p^2)$ is given by:

$$\frac{\delta v^2}{4}\,\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) = \tfrac{1}{2}\frac{\delta v^2}{v^2}\partial^\mu\xi^i\partial_\mu\xi^i + \dots . \tag{4.26}$$

From this term we derive:

$$\Pi^{C}_{\xi^3\xi^3}(p^2) = \frac{\delta v^2}{v^2}p^2\,. \tag{4.27}$$

On the other hand from Eq. (4.24), one has:

$$\Pi^{0}_{\xi^3\xi^3}(p^2) = 2a_0p^2\,. \tag{4.28}$$

Before computing the top quark loop, we observe that we can get rid of the counter term contribution by writing the analogous matching condition for the $\xi^+\xi^+$ two-point function, $\Pi_{\xi^+\xi^+}(p^2)$:

$$\Pi^{tb}_{\xi^+\xi^+}(p^2) + \Pi^{C}_{\xi^+\xi^+}(p^2) = \Pi^{0}_{\xi^+\xi^+}(p^2)\,. \tag{4.29}$$

From Eqs (4.24) and (4.26), we obtain:

$$\Pi^{C}_{\xi^+\xi^+}(p^2) = \frac{\delta v^2}{v^2}p^2\,, \tag{4.30}$$

and

$$\Pi^{0}_{\xi^+\xi^+}(p^2) = 0\,. \tag{4.31}$$

By combining the conditions (4.25) and (4.29) and making use of Eqs (4.27) – (4.28) and (4.30) – (4.31), we find:

$$2a_0p^2 = \Pi^{tt}_{\xi^3\xi^3}(p^2) - \Pi^{tb}_{\xi^+\xi^+}(p^2)\,, \tag{4.32}$$

where it is understood that in the right-hand side we have to consider only the contribution proportional to p^2. By a direct evaluation of the Feynman amplitudes one obtains:

$$-i\Pi^{tt}_{\xi^3\xi^3}(p^2) =$$
$$12\left(\frac{m_t}{v}\right)^2 p^2 \int \frac{d^dk}{(2\pi)^4}\frac{[(1-\frac{2}{d})k^4 - 2m_t^2(1-\frac{1}{d})k^2 + m_t^4]}{(k^2-m_t^2)^4} + \ldots, \quad (4.33)$$

and

$$-i\Pi^{tb}_{\xi^+\xi^+}(p^2) = 12\left(\frac{m_t}{v}\right)^2 p^2 \int \frac{d^dk}{(2\pi)^4}\frac{(1-\frac{2}{d})}{k^2(k^2-m_t^2)} + \ldots, \quad (4.34)$$

where dots stand for higher terms in the p^2 expansion. The equation (4.32) now reads:

$$-2ia_0p^2 = -6\left(\frac{m_t}{v}\right)^2 p^2 \int \frac{d^4k}{(2\pi)^4}\frac{m_t^4}{k^2(k^2-m_t^2)^3} \quad (4.35)$$

from which we obtain:

$$2a_0(=\epsilon_1) = (\Delta\rho)_{\rm top} \quad (4.36)$$

in agreement with Eqs (4.10).

These examples show how the violation of the decoupling theory in the SM with an heavy top quark is related to the underlying Yukawa theory. From the practical point of view, one may have the impression of an unnecessary complication in dealing with a simple 1-loop computation. Moreover, to reach the accuracy required to compare the experimental prediction to the theoretical expectation, one should also include the corrections to the Yukawa limit taken in the fundamental theory.

Nevertheless the strategy followed above has been already useful in attacking more challenging computations as, for example, those concerning the leading two-loop effects in the pure electroweak theory [38] — $O(G_F^2 m_t^4)$ — or the mixed strong and electroweak corrections of $O(\alpha_s G_F m_t^2)$ [39].

We conclude this section with a comment concerning the size of the corrections to the results obtained in the gaugeless limit of the SM. These corrections are of order $(M_W/m_t)^2$ and, in principle, they can be large compared to the leading order results. At one-loop order one has two extreme cases. One-loop two-point functions are essentially untouched by the gaugeless limit, the only approximation coming from the subsequent expansion in p^2/m_t^2, which however works remarkably well already for $m_t^2 \simeq 2p^2$. On the other hand, vertex and box corrections may be largely modified in the full gauge theory. Consider for instance the function $f(m_t^2/M_W^2)$ appearing in the evaluation of the box diagram for $B^0 - \bar{B}^0$ oscillation (see Eq. (3.7)). The asymptotic value $f(\infty) = 1/4$ is not so close to the more realistic case $f(4) \simeq 0.57$. Moreover, also the first term in the expansion of $f(1/y)$ around $y = 0$ fails to provide the right correction (it does not even give the correct sign!):

$$f\left(\frac{1}{y}\right) = \frac{1}{4} - \left(\frac{9}{4} + \frac{3}{2}\ln y\right) y + \ldots . \tag{4.37}$$

By truncating the expression above at first order in y, one obtains $f(4) = 0.25 - 0.043 = 0.207$ rather far away from the physical value.

5. Heavy fermions and chiral anomalies

In the previous Section we have imposed the $SU(2)_L \otimes U(1)_Y$ gauge invariance on the low-energy Lagrangian $\mathcal{L}_{\rm eff}$. The physical basis of this requirement is the fact that the heaviness of the top quark is due to the magnitude of its Yukawa coupling, (much) bigger than the other coupling constants in the theory. This accidental hierarchy will always respect the gauge invariance of the theory which has to persist also in the low-energy theory. There is however a subtlety in the mechanism which maintains gauge invariance, due to the features of the classical term $\mathcal{L}_{\rm cl}$ in $\mathcal{L}_{\rm eff}$. The light matter fields entering $\mathcal{L}_{\rm cl}$ do not form an anomaly-free set of chiral fermions. This means that at one-loop order the gauge currents are not conserved and the $O(\hbar)$ contribution of $\mathcal{L}_{\rm cl}$ to the effective action is not gauge invariant. On the other hand, since the total effective action must be gauge invariant, the gauge variation of the terms induced by $\mathcal{L}_{\rm cl}$ has to be exactly compensated by the gauge variation of $\Delta\mathcal{L}$. So far we have only included gauge invariant operators in $\Delta\mathcal{L}$. In this section we will identify the additional, non-invariant contributions in $\Delta\mathcal{L}$ and we will detail the mechanism of anomaly cancellation.

We recall that anomalies may occur as violations of symmetry properties of a classical theory, in the regularization procedure which underlies the construction of the corresponding quantum theory [40]. The classical example is given by the axial current in QED (see Eq. (2.1)):

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma_5\psi . \tag{5.1}$$

At the classical level, the divergence of j_5^μ is proportional to the pseudoscalar density:

$$\partial_\mu j_5^\mu = 2iM\bar{\psi}\gamma_5\psi \tag{5.2}$$

so that, for $M = 0$, j_5^μ is conserved. Indeed, for $M = 0$ the QED Lagrangian of Eq. (2.1) possesses the chiral symmetry $U(1)_L \otimes U(1)_R$, which leads to the separate conservation of the vector current and of the axial-vector one. It is well known that this is no more true at one-loop order and the Ward identity (5.2) is replaced by:

$$\partial_\mu j_5^\mu = 2iM\bar{\psi}\gamma_5\psi + \frac{e^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}\,, \tag{5.3}$$

where $\tilde{F}^{\mu\nu} = 1/2\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. On the other hand, the vector current, associated to the U(1) local invariance, is still conserved.

In a chiral gauge theory, with left and right-handed fermions transforming according to inequivalent representations of the gauge group, there will be both vector and axial-vector gauge currents. In this case the problem of a possible (and unacceptable) breaking of the gauge symmetry via quantum effects immediately arises. It turns out that in this case it is the fermion content of the theory that decides if the gauge currents are anomalous or not, and a simple criterion can be formulated. It is convenient to define all the fermion fields to be left-handed. This is always possible: whenever a right-handed field $\psi_{\rm R}$ occurs, it may be always replaced by its charge-conjugate left-handed counterpart $(\psi^c)_{\rm L}$, with $\psi^c = C\bar{\psi}^T$. In this way all the gauge currents are of the kind:

$$j_\mu^A = \overline{\Psi_{\rm L}}\,\gamma_\mu T^A \Psi_{\rm L}\,, \tag{5.4}$$

where $\Psi_{\rm L}$ stands for the collection of fermion fields and T^A is the set of gauge generators for the representation $\Psi_{\rm L}$. An anomaly-free theory is characterized by the condition:

$$D^{ABC} = {\rm Tr}(T^A\{T^B, T^C\}) = 0\,, \tag{5.5}$$

corresponding to the vanishing of all possible quantum contributions to the anomalies via triangle diagrams. For instance, in the SM, the above condition is equivalent to the requirement:

$${\rm Tr}(Q_{\rm em}) = 0\,, \tag{5.6}$$

the trace being performed over all SU(2) doublets. A full fermion generation satisfies Eq. (5.6), since the quark contribution, $3 \times (^2/_3 - {}^1/_3) = +1$ exactly compensates the leptonic one, -1. An immediate consequence is that the removal of the top quark from the low-energy spectrum of $\mathcal{L}_{\rm cl}$ makes the theory anomalous.

Instead of investigating the problem in the SM, we consider here a simplified model, based on an abelian gauge symmetry U(1). The matter content of the theory consists of a "lepton" l and a "quark" q whose left-handed component transforms according opposite U(1) charges. The right-handed components are taken invariant:

$$\begin{aligned} l'_{\rm L} &= e^{i\alpha(x)} l_{\rm L}\,, \\ q'_{\rm L} &= e^{-i\alpha(x)} q_{\rm L}\,, \\ l'_{\rm R} &= 0\,, \\ q'_{\rm R} &= 0\,. \end{aligned} \tag{5.7}$$

To trigger the spontaneous breaking of the gauge symmetry we introduce also a complex scalar field φ transforming as follows:

$$\varphi' = e^{i\alpha(x)}\varphi\,. \tag{5.8}$$

Finally, we provide the equivalent of a lepton number L and a baryon number B, by requiring invariance under a global $\mathrm{U}(1)_{\rm L}\otimes\mathrm{U}(1)_B$ symmetry, with natural assignment:

$$\begin{aligned} L(q) = L(\varphi) = 0\,, &\qquad L(l) = 1\,, \\ B(l) = B(\varphi) = 0\,, &\qquad B(q) = 1\,. \end{aligned} \tag{5.9}$$

The Lagrangian for this model reads:

$$\begin{aligned} \mathcal{L} = &- \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} \\ &+ i\bar{l}_{\rm L}\gamma^\mu(\partial_\mu - igA_\mu)l_{\rm L} + i\bar{l}_{\rm R}\gamma^\mu\partial_\mu l_{\rm R} \\ &+ i\bar{q}_{\rm L}\gamma^\mu(\partial_\mu + igA_\mu)q_{\rm L} + i\bar{q}_{\rm R}\gamma^\mu\partial_\mu q_{\rm R} \\ &+ D_\mu\varphi^\dagger D^\mu\varphi - V(\varphi^\dagger\varphi) \\ &- y_l(\bar{l}_{\rm L}\varphi l_{\rm R} + \text{h.c.}) - y_q(\bar{q}_{\rm L}\varphi q_{\rm R} + \text{h.c.})\,. \end{aligned} \tag{5.10}$$

The potential $V(\varphi^\dagger\varphi)$ gives rise to the spontaneously broken phase, if we make the usual choice:

$$V(\varphi^\dagger\varphi) = \mu^2\varphi^\dagger\varphi + \lambda(\varphi^\dagger\varphi)^2\,, \tag{5.11}$$

with $\mu^2 < 0$ and $\lambda > 0$. The minimum is at:

$$\langle\varphi\rangle = \frac{v}{\sqrt{2}}\,, \tag{5.12}$$

with

$$v^2 = -\frac{\mu^2}{\lambda}\,. \tag{5.13}$$

We shift the scalar field as follows:

$$\varphi = \frac{(\sigma + v)}{\sqrt{2}} e^{i\xi/v} . \tag{5.14}$$

Notice that the would-be Goldstone boson ξ undergoes the following gauge transformation:

$$\xi' = \xi + \alpha(x)v . \tag{5.15}$$

The gauge symmetry is spontaneously broken and all the particles become massive via the Higgs mechanism. The mass spectrum is the following:

$$\begin{aligned} M_A^2 &= g^2 v^2 , \\ M_\sigma^2 &= 2\lambda v^2 , \\ m_l &= \frac{y_l v}{\sqrt{2}} , \\ m_q &= \frac{y_q v}{\sqrt{2}} . \end{aligned} \tag{5.16}$$

We are interested in the gauge current j^μ, given by:

$$\begin{aligned} j^\mu &= \bar{l_{\mathrm{L}}}\gamma^\mu l_{\mathrm{L}} - \bar{q_{\mathrm{L}}}\gamma^\mu q_{\mathrm{L}} + i(\varphi^\dagger D^\mu \varphi - (D^\mu \varphi^\dagger)\varphi) \\ &= \bar{l_{\mathrm{L}}}\gamma^\mu l_{\mathrm{L}} - \bar{q_{\mathrm{L}}}\gamma^\mu q_{\mathrm{L}} - v\partial^\mu \xi + \dots . \end{aligned} \tag{5.17}$$

In the second equality we have used the parametrization for the scalar field given in Eq. (5.14), writing down explicitly only the term linear in the field ξ. Dots denote terms with σ or more than one ξ, which, for the sake of simplicity, we will neglect from now on. The divergence of j_μ reads:

$$\partial^\mu j_\mu = -i m_l \bar{l}\gamma_5 l + i m_q \bar{q}\gamma_5 q - v\Box\xi + \dots . \tag{5.18}$$

By taking into account the equations of motion for the field ξ:

$$\Box\xi = -\frac{i}{v} m_l \bar{l}\gamma_5 l + \frac{i}{v} m_q \bar{q}\gamma_5 q + \dots . \tag{5.19}$$

we find that the gauge current is conserved also in the spontaneously broken phase, at least at the classical level. To see what happens with the quantum corrections, we write the generator T of the gauge transformations (5.7) in the base l_{L},q_{L},$(l^c)_{\mathrm{L}}$,$(q^c)_{\mathrm{L}}$:

$$T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{5.20}$$

The condition (5.5) for the absence of anomalies in the gauge currents now reads $\mathrm{Tr}(T^3) = 0$, which is clearly satisfied by T. In terms of triangle diagrams, the quark contribution is exactly cancelled by the lepton contribution, as in the SM. The model is anomaly-free.

To mimic the case of the SM, we now assume that the quark is much heavier than the other particles. As one can see from Eq. (5.16), this means that we are postulating the following hierarchy:

$$y_q \gg y_l, g, \lambda\,. \tag{5.21}$$

Such a choice does not interfere with the gauge invariance of the model, which we have checked above. As in the previous section, we introduce a low-energy effective action given by:

$$S_{\mathrm{eff}} = S_{\mathrm{cl}} + \Delta S\,. \tag{5.22}$$

Here S_{cl} is obtained from the original Lagrangian $\mathcal{L}$ of Eq. (5.10) simply by dropping the terms containing the heavy field q. The theory described by S_{cl} is still formally gauge invariant. The gauge current j^μ_{cl} is given by:

$$\begin{aligned} j^\mu_{\mathrm{cl}} &= \bar{l}_{\mathrm{L}}\gamma^\mu l_{\mathrm{L}} + i(\varphi^\dagger D^\mu\varphi - (D^\mu\varphi^\dagger)\varphi) \\ &= \bar{l}_{\mathrm{L}}\gamma^\mu l_{\mathrm{L}} - v\partial^\mu\xi + \ldots\,, \end{aligned} \tag{5.23}$$

with vanishing classical divergence. However, due to the anomalous fermion content of the theory, quantum corrections modify the classical Ward identity, and, as in the case of the axial vector current in QED, one obtains

$$\partial_\mu j^\mu_{\mathrm{cl}} = \frac{1}{6}\frac{g^2}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu} = G(x)\,. \tag{5.24}$$

This result immediately implies the breaking of gauge invariance. We consider the gauge variation of S_{cl}. We obtain:

$$\begin{aligned} \delta\,[S_{\mathrm{cl}}] &= -\int dx\,\alpha(x)\partial_\mu j^\mu_{\mathrm{cl}}\,, \\ &= -\int dx\,\alpha(x)G(x)\,, \\ &\neq 0\,, \end{aligned} \tag{5.25}$$

where we have denoted by $[S_{\mathrm{cl}}]$ the full one-loop effective action induced by S_{cl}.

There is not much freedom to repair this situation. The only possibility is that ΔS in Eq. (5.22), which is a genuine $O(\hbar)$ term, has a gauge variation which exactly compensates the one given in Eq. (5.25), namely:

$$\delta(\Delta S) = \int dx\,\alpha(x)G(x)\,. \tag{5.26}$$

Indeed, we may try to find the general — CP invariant — solution to the equation (5.26). To this end it is useful to split ΔS into a parity violating part $\Delta S_{\rm PV}$ and a parity conserving term $\Delta S_{\rm PC}$. Indeed it is not restrictive to require that $\Delta S_{\rm PC}$ is gauge invariant so that it does not contribute to the previous equation. The term $\Delta S_{\rm PC}$, analogous to the term $(\Delta\mathcal{L})_B$ of Eq. (4.4), can be determined via suitable matching conditions, as explained in Section 4, but is irrelevant to the present discussion. The general solution to Eq. (5.26) has now the form:

$$\Delta S_{\rm PV} = \Delta S^0_{\rm PV} + \Delta S^1_{\rm PV}\,, \qquad (5.27)$$

where $\Delta S^0_{\rm PV}$ is the general solution of the homogeneous equation $\delta(\Delta S^0_{\rm PV}) = 0$, *i.e.* the set of all possible gauge and CP invariant, P violating operators. Such operators do not exist.[7]. So we remain with $\Delta S^1_{\rm PV}$, a particular solution of Eq. (5.26). Long ago Wess and Zumino found the solution [41]:

$$\Delta S^1_{\rm PV} = \frac{1}{v}\int dx\ \xi(x)G(x)\,,$$

which indeed satisfies Eq. (5.26), as can be seen by using the transformation properties of ξ in Eq. (5.15) and the invariance of G.

The final gauge invariant effective action is given by:

$$S_{\rm eff} = S_{\rm cl} + \Delta S_{\rm PC} + \frac{1}{6v}\int dx\ \xi F_{\mu\nu}\tilde{F}^{\mu\nu}\,. \qquad (5.29)$$

Notice that $S_{\rm eff}$ is non-renormalizable, as in fact it should be, since otherwise, we would have "integrated away" the anomaly. The non-renormalizability is related to the restricted domain of applicability of the effective theory. Such domain is bounded in energy by some critical value E_c, beyond which the breakdown of perturbative unitarity signals the inadequacy of $S_{\rm eff}$ to approximate the full theory.

A similar mechanism of anomaly cancellation is active in the low energy effective Lagrangian $\mathcal{L}_{\rm eff}$ of the previous section, obtained from the SM in the heavy m_t limit [14]. In this case we have to deal with the additional constraint given by the lightness of the bottom quark, which belongs to the low-energy part of the spectrum. At first sight this new element seems to lead to a contradictory situation. On one hand the consistency conditions which the $SU(2)_L\otimes U(1)_Y$ anomalies must satisfy require for the bottom contribution to the anomaly to be fully included in the Wess-Zumino term ΔS. On the other hand, being the bottom a light field, nothing prevents the

[7] We are using the possible assignment: $P = +1$ and $C = -1$, for the field ξ.

separate evaluation of the bottom contribution to the modified Ward identities in $S_{\rm cl}$. It is a property of the nonlinear realization of the $SU(2)_L \otimes U(1)_Y$ symmetry that makes the latter vanish, solving the paradox.

Similar mechanisms take also place in supersymmetric extensions of the SM analyzed in the large m_t limit [42].

To conclude this section we illustrate the physical relevance of the Wess–Zumino term by discussing the cancellation of the gauge dependence in the scattering amplitude for $l\bar{l} \to AA$. This amplitude, beyond the tree-level contributions, receives from the lepton loop three independent one-loop corrections: the exchange A_V of the vector boson A in the s-channel, the exchange A_ξ of a would-be Goldstone boson ξ and finally the contribution A_{WZ} of the Wess–Zumino term through the exchange of ξ. In a generic R_λ gauge (λ denoting the gauge parameter) we have:

$$A_V = \bar{v}(ig\gamma^\mu a_-)u \cdot \frac{-i}{p^2 - M_A^2} \cdot \left[g_{\mu\nu} - \frac{1-\lambda^{-1}}{p^2 - \frac{M_A^2}{\lambda}} p_\mu p_\nu \right] \langle igj^\nu \rangle \, , \tag{5.30}$$

where $\langle igj_\mu \rangle$ denotes the insertion of the leptonic current between two vector bosons via the lepton loop. From the anomalous Ward identity one obtains:

$$ip^\mu \langle j_\mu \rangle = -im_l \left\langle \bar{l}\gamma_5 l \right\rangle + \langle G \rangle \, , \tag{5.31}$$

so that one finds:

$$A_V = A_V^1 + A_V^2 \, , \tag{5.32}$$

with

$$A_V^1 = \bar{v}(ig\gamma^\mu a_-)u \, \frac{-i}{p^2 - M_A^2} \, \langle igj_\mu \rangle \, , \tag{5.33}$$

and

$$A_V^2 = -g^2 m_l (\bar{v}\gamma_5 u) \, \frac{1}{p^2 - M_A^2} \, \frac{1-\lambda^{-1}}{p^2 - \frac{M_A^2}{\lambda}} \left[\langle G \rangle - im_l \left\langle \bar{l}\gamma_5 l \right\rangle \right] . \tag{5.34}$$

Notice that A_V^1 does not depend on λ. The other contributions are given by:

$$A_\xi = \frac{m_l}{v} (\bar{v}\gamma_5 u) \, \frac{i}{p^2 - \frac{M_A^2}{\lambda}} \, \frac{m_l}{v} \left\langle \bar{l}\gamma_5 l \right\rangle \, , \tag{5.35}$$

and

$$A_{WZ} = \frac{m_l}{v} (\bar{v}\gamma_5 u) \, \frac{i}{p^2 - \frac{M_A^2}{\lambda}} \, \frac{i}{v} \langle G \rangle$$

Finally the sum of the contributions A_V^2, A_ξ and A_{WZ} is given by:

$$A_V^2 + A_\xi + A_{WZ} = -\frac{1}{v^2} m_l (\bar{v}\gamma_5 u) \frac{1}{p^2 - M_A^2} \left[\langle G\rangle - i m_l \langle \bar{l}\gamma_5 l\rangle\right] \quad (5.37)$$

displaying the desired independence on λ. It is only the sum of the three contributions that does not depend on λ. In particular, neglecting the contribution from the Wess–Zumino term we would obtain an unacceptable gauge-dependent amplitude.

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