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POLYLOGARITHMS AND LOGARITHMIC DIVERSION IN STATISTICAL MECHANICS*

M. Howard Lee

Department of Physics and Astronomy University of Georgia
Athens, GA 30602 USA†
and
Korea Institute for Advanced Study, Seoul 130-012, Korea
`MHLee@uga.edu`

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Sommerfeld's work on the low temperature theory of an ideal Fermi gas cannot be easily extended nor generalized. By recognizing an underlying logarithmic structure in ideal quantum gases, a new approach has been developed and this work is reviewed with some new insights. A unified formulation of the grand partition function provides several special solutions such as thermodynamic equivalence in two dimensions and μ singularity in null dimension.

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1. Introduction

Since the early days it has been customary to study the statistical mechanics of ideal Fermi and Bose gases separately. This is entirely natural as these quantum particles obey different statistics. As we now know well, the different statistics manifest themselves in different thermodynamic behavior. Perhaps the most notable for these gases in $3d$ is: For a Fermi gas the existence of the Fermi energy and, as a result, the zero point pressure; for a Bose gas the occurrence of Bose–Einstein Condensation (BEC). It is thus not surprising that statistical mechanics has divided itself into two branches, one for Fermi and the other for Bose.

Given this history and the weight of evidence, probably no one seriously entertained any idea of "unifying" the two branches of statistical mechanics. But it could have happened much earlier, had the underlying mathematics

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[†] Permanent address.

was understood. The basic mathematical structure is logarithmic. When logarithms are encountered *e.g.* $\log x$ say, a common tendency is to "de-log" it by finding a way to replace it with dx/x. Such an elementary step will often enable one to solve a log integral. But if a problem comes naturally "enlogged", this practice may keep us from seeing an underlying structure. Knowing it might lead us to a big picture. One aim of this work is to illustrate this point through the statistical physics of ideal quantum gases.

2. Fermi gas model and Sommerfeld

As is well known, the electrons in normal metals are nearly free and ideal, which give rise to an ideal Fermi gas model. The quantum theory of normal metals is essentially the physics deduced from the ideal Fermi gas model. We do not intend to review this rich subject, but focus on some special aspects, especially on the calculation of the chemical potential μ as a function of temperature T. In quantum statistical mechanics μ is perhaps the most important thermodynamic variable. It contains information on statistics. As a result, it reveals its true identity at or near the ground state as we shall see below. Far from it, such as at high temperatures, the physical meaning becomes blurred because of its relationship to the entropy.

In 1928 Sommerfeld [1] obtained the chemical potential of an ideal Fermi gas as a low temperature expansion. Since that time, this work has been a standard in almost all textbooks on quantum statistical mechanics. We will briefly outline the basic idea since it will be our starting point.

It should be emphasized that Sommerfeld's work on the thermodynamics of a Fermi gas shows the importance of Fermi statistics. His results are in perfect accord with thermodynamic principles, which as we know were formulated before quantum mechanics.

The work itself does not explicitly contain log terms, but they are there hidden. Sommerfeld's idea applies to odd numbered d's but not to even numbered d's, already suggesting its limitation. At the time of Sommerfeld, it was probably deemed unimportant since only $3d$ would have been considered relevant.

3. Sommerfeld's method

There are N Fermi particles in a volume V in thermal equilibrium at temperature T. We shall have in mind that N and V are both large but the number density $\rho = N/V$ remains finite. If these Fermi particles are ideal, their energy (in units of $\hbar = 1$) is $\varepsilon_k = k^2/2m$, where m is the mass of a Fermi particle like an electron. If they are neutrinos, $\varepsilon_k = ck$, where c is the speed of light. We shall be concerned mostly with non-relativistic behavior, but there will be an occasion to consider the thermodynamics of a neutrino gas, which is also an ideal quantum gas.

3.1. Fermi gas in 3d

For an ideal spinless Fermi gas in $3d$

$$\rho = \frac{1}{V}\sum_{\varepsilon} n(\varepsilon)\,, \tag{3.1}$$

where $n(\varepsilon)$ is the Fermi function

$$n(\varepsilon) = \langle \hat{n}(\varepsilon)\rangle = \{1 + z^{-1}e^{\beta\varepsilon}\}^{-1}\,, \tag{3.2}$$

where $z = \exp\beta\mu$, the fugacity. Observe that since ρ, T and ε are all given, the only unknown is μ (or z). Hence one can extract μ from 3.1 as a function of these given quantities. It is a simple idea but the process is not simple because of the presence of the Fermi function. For a review, see Ref. [2].

Since the energy levels are continuous at this limit, we may convert the sum in (3.1) into an integral. After some re-arrangements, letting $\beta\varepsilon = x$, we obtain

$$\rho \;=\; \lambda^{-3}\int\limits_0^\infty dx\, x^{1/2}\, n(x)\,, \tag{3.3a}$$

where

$$n(x) \;=\; \{1 + e^{(-a+x)}\}^{-1}\,, \tag{3.3b}$$

where $a = \beta\mu$ and $\lambda = (2\pi\beta/m)^{1/2}$ the thermal wavelength. The right-hand side still cannot be integrated.

To approximately evaluate it, let us proceed as follows: with $x^{1/2} = 2/3\, d/dx\, x^{3/2}$, we can put d/dx on the Fermi function by partial integration and obtain

$$\rho = -\frac{2}{3}\,\lambda^{-3}\int\limits_0^\infty dx\, x^{3/2}\frac{d}{dx}\,n(x)\,. \tag{3.4}$$

In obtaining (3.4), we have used the fact that $n(\infty) = 0$ for a positive energy spectrum. Thus far everything is exact. But 3.4 is turned into a form more amenable for approximation.

At the ground state (*i.e.* $T = 0$), the Fermi function is a step function: $n(x) = 1$ if $x \leq x_0$ and $n(x) = 0$ if $x > x_0$, where $x_0 = \beta\varepsilon_{\rm F}$, $\mu(T = 0) = \varepsilon_{\rm F}$, where $\varepsilon_{\rm F}$ is the Fermi energy. Thus $dn(x)/dx = -\delta(x - x_0)$. If the gas were at near the ground state, $n(x)$ would be very sharply peaked at about $x \approx x_0$, still nearly a delta function. Hence for a product function like $f(x)dn/dx$, $f(x)$ gives weight only in the neighborhood of where dn/dx is peaked and gives hardly any weight elsewhere.

One may thus express $f(x)$ by a Taylor expansion in the neighborhood of $x = a = \beta\mu(T)$, which should generate a low T expansion, provided that $f(x)$ is not a rational or meromorphic function. Also since dn/dx is symmetric with respect to $x = a$, the resulting series should be even in T.

Near the ground state the convergence of the series would be fast. Far away from the ground state the expansion would converge too slowly to be useful. If one is interested in the low T behavior of μ, this is a great idea due to Sommerfeld.

The left-hand side of (3.4) is basically $x_0 = \beta\varepsilon_{\mathrm{F}}$. For the right-hand side, by expanding $f = x^{3/2}$ about $x = a$, one can now solve it term by term and obtain a series in powers of kT/μ. This series may be reverted by a standard procedure to obtain the μ series in powers of $kT/\varepsilon_{\mathrm{F}}$ given below:

$$\frac{\mu(T)}{\varepsilon_{\mathrm{F}}} = 1 - \frac{1}{2}\zeta_2\left(\frac{kT}{\varepsilon_{\mathrm{F}}}\right)^2 - \frac{9}{8}\zeta_4\left(\frac{kT}{\varepsilon_{\mathrm{F}}}\right)^4 - \dots\,, \tag{3.5}$$

where $\zeta_2 = \pi^2/6,\ \zeta_4 = \pi^4/90$, which are the Riemann zeta functions.

It is a remarkable result, full of significance. Most obvious is that even at room temperature the expansion should converge very fast. The ε_{F}'s for metals are in the range of $T = 10^5 k$. As T increases the expansion goes in the direction of the classical result $\mu/kT = \log(\rho\lambda^3) < 0$.

More subtle is that μ decreases with T. By the second law, $\mu = (\partial U/\partial N)_{V,S}$ where U and S are the internal energy and entropy, respectively. It is the energy required to add one more particle to a gas of N particles in a fixed volume but without affecting the entropy S. If the gas is very near the ground state, the entropy is nearly zero. For a gas near the ground state and in a rigid box, μ is simply the energy for adding a particle. At the ground state it is exactly the Fermi energy ε_{F}. At slightly above the ground state, this energy is less. Why?

When a gas is at somewhat above its ground state, the Fermi function no longer has a sharp edge. Near the ε_{F} it is rounded off such that now $n(\varepsilon_{\mathrm{F}}) < 1$. Thus a particle could be added below ε_{F}.

Losing the sharp edge at ε_{F} means that the gas will acquire entropy, which can also be calculated with help from (3.5)

$$\frac{S}{Nk} = 3\,\zeta_2\left(\frac{kT}{\varepsilon_{\mathrm{F}}}\right) - \frac{9}{2}\zeta_4\left(\frac{kT}{\varepsilon_{\mathrm{F}}}\right)^3 - \dots\,. \tag{3.6}$$

It is hardly necessary to point out that (3.6) fully satisfies the third law. Nernst proposed the third law in 1906. The work of Sommerfeld, coming more than 20 years later, has given a firm verification.

3.2. Fermi gas in 1*d*

Sommerfeld's method can easily be extended to the same spinless Fermi gas in 1d. Recall that the basic idea was to expand $f(\varepsilon) = \varepsilon^{3/2}$ in the neighborhood of $\varepsilon_{\rm F}$. If $f = \varepsilon^{d/2}$ and d is an odd positive integer, one can always expand it in the neighborhood of μ for the corresponding d. It is no more difficult to obtain $\mu(T)$ in 1d given below:

$$\frac{\mu(T)}{\varepsilon_{\rm F}} = 1 + \frac{1}{2}\zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right)^2 + \frac{5}{2}\zeta_4\left(\frac{kT}{\varepsilon_{\rm F}}\right)^4 + \dots , \tag{3.7}$$

where by $\varepsilon_{\rm F}$ we mean the Fermi energy in 1d. The expansion looks different from the expansion for 3d (3.5). Now μ increases with T, at least initially. Ultimately it must decrease as it becomes classical. The significance of the rise is not difficult to discern. At a small T the sharpness of the edge of the Fermi function is lost. In 3d it meant that a particle could be added at below $\varepsilon_{\rm F}$. But in 1d it is prevented from doing so because its paths are blocked by other excited particles. One would expect, therefore, that a gas in 1d is less well ordered than in 3d. The entropy per particle per degree of freedom must be greater in 1d than in 3d.

The entropy in 1d is

$$\begin{aligned}\frac{S}{Nk} &= \zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right) + 2\,\zeta_4\left(\frac{kT}{\varepsilon_{\rm F}}\right)^3 + \dots ,\\ &= \zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right)\left\{1 + \frac{4}{15}\zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right)^2 + \dots\right\} .\end{aligned} \tag{3.8}$$

Let us compare the above with (3.6) but divided by $d = 3$:

$$\frac{S}{3Nk} = \zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right)\left\{1 - \frac{3}{5}\zeta_2\left(\frac{kT}{\varepsilon_{\rm F}}\right)^2 + \dots\right\} . \tag{3.9}$$

The comparison suggests that the particles which are taken out of the Fermi sea are more dispersed in 1d resulting in greater entropy than in 3d. This observation seems supported by a recent numerical study of Grether *et al.* [3]. Note that it is not necessary to divide μ by d since it is expressed in terms of $\varepsilon_{\rm F}$, which depends on d itself.

3.3. Limitations of Sommerfeld's method

Sommerfel's work has dominated this subject for over a half century. To our knowledge, the first attempts to improve the calculations were provided by Barker [4] and Lee [5]. Theirs are more of a formal treatment still limited to low temperatures.

In spite of its great success, Sommerfeld's method has limitations. We have already noted that it cannot be applied to even numbered d's. Perhaps more fundamental is the apparent absence of logarithmic character in the thermodynamic functions. They should have acquired it from the Fermi function, an inverse function of z. In fact, it is present through the Riemann zeta functions, the coefficients in the expansions for μ and S, which trace to certain logarithmic integrals [6].

This insight suggests that Sommerfeld's method is not a natural one. It comes at a price of missing out on something perhaps larger or deeper. We shall show that a natural approach leads to an unsuspected symmetry in the statistical mechanics of ideal quantum gases.

4. Grand partition function

The grand partition function Q should be a most natural starting point for obtaining the thermodynamics of a Fermi gas as of any other similar ones. Q or $\log Q$ may be termed a thermodynamic source function, for if it is known, one can obtain from it most of the thermodynamic functions by simple operations.

4.1. Fermi gas

Let us start from this source function for a spinless Fermi gas:

$$Q = \mathrm{Tr}\, z^{\hat{N}}\, e^{-\beta H}\,, \tag{4.1}$$

where trace Tr is to be taken over Fermi states, $\hat{N}$ is the number operator, and $z \geq 0$. Since the thermodynamic potential is given by $\log Q$,

$$\log Q = \sum_{\varepsilon} \log(1 + z e^{-\beta\varepsilon})\,. \tag{4.2}$$

By converting the sum into an integral, if $d = 3$ (for explicitness)

$$\frac{1}{V}\log Q = \lambda^{-3}\,\frac{1}{\Gamma(3/2)}\int_0^\infty x^{1/2}\log(1 + z e^{-x})\,dx\,. \tag{4.3}$$

The integrand is logarithmic, so it is naturally enlogged. We do not try to remove it. In fact, we make it "worse" by enlogging it more through the substitution:

$$y \;=\; z e^{-x} \tag{4.4a}$$

or

$$x \;=\; \log z - \log y\,. \tag{4.4b}$$

We obtain

$$\frac{1}{V}\log Q = \lambda^{-3}\,\frac{1}{\Gamma(3/2)}\int\limits_0^z (\log z - \log y)^{1/2}\log(1+y)\,\frac{dy}{y}\,. \tag{4.5}$$

We can "unworsen" it a little by partial integration once to obtain the final form

$$\frac{1}{V}\log Q = \lambda^{-3}\,\frac{1}{\Gamma(5/2)}\int\limits_0^z (\log z - \log y)^{3/2}\,\frac{dy}{(1+y)}\,. \tag{4.6}$$

The log integral does not appear doable. Thus, we shall simply define it a log integral $-L(5/2;-z)$.

Although the above is for $d = 3$, the same procedure will work for any integer d. We can replace 3 by d to arrive at a general form:

$$\frac{1}{V}\log Q = -\lambda^{-d}\,L(d/2+1;\,-z)\,, \tag{4.7}$$

where L is as yet an unsolved log integral, defined as:

$$L(d/2+1;\,-z) = -\frac{1}{\Gamma(d/2+1)}\int\limits_0^z \frac{(\log z - \log y)^{d/2}\,dy}{(1+y)}\,. \tag{4.8}$$

There is one property we can obtain at once:

$$z\,\frac{z}{dz}L(d/2+1;\,-z) = L(d/2;-z)\,, \tag{4.9}$$

which is a recurrence relation. Thus one can obtain a lower-order one by differentiation once and a higher-order one by integration once.

4.2. Sommerfeld's result for the density

Given (4.7) for $\log Q$, we can recover the density obtained by Sommerfeld as follows: Since $Q = Q(z,\beta)$,

$$\langle\hat{N}\rangle = z\,\frac{\partial}{\partial z}\log Q\bigg|_{\beta}\,. \tag{4.10}$$

By (4.7), holding β fixed,

$$\rho = \frac{\langle\hat{N}\rangle}{V} = z\,\frac{\partial}{\partial z}\frac{1}{V}\log Q = -\lambda^{-d}\,L(d/2;-z)\,. \tag{4.11}$$

If the right-hand side is expanded for $d = 3$ and $z \to \infty$, Sommerfeld result (3.5) is readily recovered.

4.3. Bose gas

The grand partition function for an ideal Bose gas is

$$Q = \mathrm{Tr}\, z^{\hat{N}} e^{-\beta H}\,, \tag{4.12}$$

where for a Bose gas, $0 \leq z \leq 1$ since $\mu \leq 0$. Taking the trace for the Bose states, we obtain

$$\log Q = -\sum_{\varepsilon} \log(1 - z e^{-\beta\varepsilon})\,. \tag{4.13}$$

If the above is compared with the corresponding one for a Fermi gas (4.2), we see that

$$\log Q(z_{\mathrm{B}}) \to -\log Q(z_{\mathrm{F}}) \qquad \text{if} \qquad z_{\mathrm{B}} \to -z_{\mathrm{F}}\,.$$

Immediately we sense that there must be some sort of symmetry connecting the two fundamentally different quantum gases. Let us defer this possibility to a later time. Taking advantage of this transformation, we can at once write down $\log Q$ for a Bose gas in d dimensions as:

$$\frac{1}{V} \log Q = \lambda^{-d} L(d/2 + 1;\, z)\,. \tag{4.14}$$

4.4. Unification

We now introduce a new variable ζ defined on the interval $(-\infty, 1)$. Let

$$\zeta = \begin{cases} z_{\mathrm{B}} & \text{if } \zeta > 0\,, \\ -z_{\mathrm{F}} & \text{if } \zeta < 0\,. \end{cases} \tag{4.15}$$

In terms of a new variable ζ, a composite fugacity, $\log Q$ for both the Fermi and Bose gases may be stated as:

$$\frac{1}{V} \log Q = \lambda^{-d} \mathrm{sgn}\,(\zeta)\, L(d/2 + 1;\, \zeta)\,. \tag{4.16}$$

The implications of the unified formulation is clear: Fermi and Bose gases are in the same family! The two are not unrelated, not as we have been accustomed to think.

In 1964 May [7] proved that in $2d$ the specific heats of the two gases are the same. At that time this work seemed to have been viewed an accident or an anomaly. Nothing further followed and soon it was largely forgotten. To this date almost everyone has continued to think that there could not be any relationship between the two gases in any d.

It is our conjecture that had Sommerfeld not "delogged" the problem, he might have seen this big picture. Had he discovered it, given his great stature it might very well have altered the course of the statistical physics of quantum gases. It was a missed opportunity.

5. Log integrals $L(s;t)$ and polylogs $\mathrm{Li}_s(t)$

It was realized that the log integrals defined by $L(s;t)$ are not new functions but are identical to the polylogs $\mathrm{Li}_s(t)$, the simplest one being the dilog due to Euler. There is already considerable knowledge on the analytic properties of these functions. See Lewin [8].

Here we will give a very brief summary of these important but little known functions. As with many good things, Euler was the inventor of the dilogs. Several years later the English mathematician Landen expanded the subject by inventing the trilogs. Despite the auspicious beginning the polylog functions never seemed to have gained a wide interest. In physics or at least in statistical physics the application was limited to expressing the Bose function by a series expansion of the polylogs. What have prevented a wider application? Is it possibly due to an aversion to logarithmic integrals?

5.1. Euler's dilogs

The simplest of the polylogs are defined as:

$$\mathrm{Li}_2(t) = -\int_0^t \log(1-x)\,\frac{dx}{x}\,. \tag{5.1}$$

Euler was able to show that the dilogs have a number of transformation properties: 1. $t \to -t$ (duplication); 2. $t \to 1/t$ (inversion); 3. $t \to 1-t$ (reflection). He also obtained special values for $t = 1, -1, 1/2, (\sqrt{5}-1)/2$ (golden mean).

Suppose we define $\mathrm{Li}_1(t)$ as:

$$\mathrm{Li}_1(t) = -\log(1-t)\,, \qquad t \neq 1\,, \tag{5.2}$$

then, (5.1) may be expressed as:

$$\mathrm{Li}_2(t) = \int_0^t \mathrm{Li}_1(x)\,\frac{dx}{x}\,. \tag{5.3}$$

It implies that

$$t\,\frac{d}{dt}\,\mathrm{Li}_2(t) = \mathrm{Li}_1(t)\,, \tag{5.4}$$

a recurrence relation that takes Li_2 down to Li_1. The same relation will take Li_3 to Li_2 *etc.* Thus a general version of (5.4) is

$$t\,\frac{d}{dt}\,\mathrm{Li}_{s+1}(t) = \mathrm{Li}_s(t)\,. \tag{5.5}$$

In generating the recurrence relation, Li_1 plays a pivotal role. To recognize its importance and to bring it into the family of polylogs, we have dubbed it the mono-log!

We see that (5.5) is exactly the same as (4.9). Are the L functions the same as the polylog functions Li?

5.2. New integral representation

In an attempt to evaluate the $\log Q$ for a Fermi gas we arrived at an integral. Unable to obtain a closed form, we simply defined it as a log-integral of the order of $s+1$ in t, or $L(s+1;\,t)$:

$$L(s+1;\,t) = \frac{1}{\Gamma(s+1)} \int_0^t (\log t - \log x)^s \, \frac{dx}{(1-x)}\,, \qquad s > -1\,. \tag{5.6}$$

If $s=0$ in (5.6), we see at once that $L(1,t) = -\log(1-t)$, which is $\mathrm{Li}_1(t)$ the monolog defined by (5.2). If $s=1$ in (5.6), we see that, after one partial integration, $L(2,t)$ reduces exactly to the integral form for $\mathrm{Li}_2(t)$ the dialog of (5.1). In addition, $L(1,t)$ and $L(2,t)$ are related by exactly the same form of a recurrence relation which relates $\mathrm{Li}_1(t)$ and $\mathrm{Li}_2(t)$. Since the higher polylogs are all defined by the recurrence relation (5.5), see Ref. [8], Eq. (6.2), it is sufficient to conclude that $L(s,t) = \mathrm{Li}_s(t)$ for all applicable s and t. For a more general and extended proof, see Ref. [9], Secs. II and III. Thus, (5.6) may be said to be another integral representation of the polylog function $\mathrm{Li}_s(t)$.

5.3. Analytic properties

Since the two functions are the same, we can at once avail ourselves of the known analytic properties of the polylogs. The most useful for the quantum gases are as follows:

- *(i)* For $\mathrm{Li}_s(t)$, s and t are complex numbers although for our purposes it is sufficient to take s a real number greater than or equal to zero.
- *(ii)* $\mathrm{Li}_s(t)$ is analytic everywhere save $1 \le \mathrm{Re}\, t \le \infty$. This strip is also a branch cut with an end point singularity at $t=1$.
- *(iii)* $\mathrm{Li}_s(t)$ is real on the real axis of t for the interval $(-\infty, 1)$. It is complex on the branch cut, for the interval $(1, \infty)$.
- *(iv)* $\mathrm{Li}_s(t=1) = \zeta_s$ where if $s \ne 1$, ζ_s is the Riemann ζ function of order s.
- *(v)* Since $\mathrm{Li}_s(t)$ is analytic everywhere save on the branch cut, it is possible to continue it to wherever designed.

5.4. Thermodynamics by polylogs

We now express the unification in terms of the polylogs: For $d > 0$,

$$\frac{1}{V}\log Q(\zeta; d) = \lambda^{-d}\,\mathrm{sgn}(\zeta)\mathrm{Li}_{d/2+1}(\zeta)\,, \tag{5.7}$$

where $\zeta = z_{\mathrm{B}}$ if $0 < \zeta < 1$ and $\zeta = z_{\mathrm{F}}$ if $-\infty < \zeta < 0$. The unification was originally given for the density [9]. But this version is more general as the grand partition function is the source function, not the density.

The grand partition function is real as it must be for the domain of the composite fugacity. The domain is divided into the Fermi branch $(-\infty, 0)$ and the Bose branch $(0, 1)$. The two branches meet at the classical limit $\zeta \to \pm 0$. It is a fixed point of the polylogs, therewith shedding log character. See Ref. [9].

The end point $\zeta \to -\infty$ corresponds to $T = 0$ for the Fermi gas. The low-T expansions obtained by Sommerfeld is a consequence of the analyticity of the polylogs in the neighborhood of $\zeta = -\infty$. The end point of the branch cut $\zeta = 1$ is a singularity of the polylogs. It is the underpinning of BEC.

5.5. Dimensionality d

The grand partition function Q depends on two independent variables the composite fugacity ζ and dimensionality d. They are, respectively, the argument and order of the polylogs. As shown above, the argument ζ determines the denizen of a particular domain whether fermions or bosons.

For a fixed d, let ζ go from say ζ_1 to ζ_2 in the same domain. If $\Delta\zeta = \zeta_2 - \zeta_1$ is small, the change in $\log Q$ should be slight. It is all about the analyticity property.

The order d denotes the disposition of its denizen. If d is limited to integers, the change in Q brought about by a change in d must be in accord with a fixed density due to the thermodynamic limit (TL) having been taken initially for a given d.

For the sake of explicitness, let us assume that TL were taken at $3d$. Let us fix ξ and suppress it since it is not at issue. $Q(3d)$ is the thermodynamic source function for a gas of N particles in volume $V = L^3$. By an analytic continuation of the polylogs in Q, we obtain $Q(2d)$, now the source function of a gas in area L^2.

The process of analytic continuation does not "flatten" the original N particles in a box of volume L^3 into a sheet of area L^2. Instead it is as if it decomposes a box of volume L^3 into L identical sheets of area L^2. The particles are confined to these sheets and are not free to move in or out of their own sheet. For them these sheets are disconnected as if they belong to different Hilbert spaces. Each sheet is a replication, represented by the same source function $Q(2d)$.

A composite of L $Q(2d)$'s does not make one $Q(3d)$. But a composite of the areal density of L sheets is equal to the initial fixed value of the volume density. This process may be compared with how one could calculate $Q(d)$ for each d (*e.g.* $d = 2$ or 3) separately. Each calculation would entail taking TL also separately.

In the polylog formulation, d can be changed from one value to another by analytic continuation. But since TL is initially set at a given d, the density is fixed. It does not change when d changes. The process of analytic continuation is accomplished by a decomposition of space as described above.

If d goes from 2 to 1, the picture is now obvious. The area L^2 is decomposed into L disconnected lines. If d goes from 1 to 0, there are L disconnect dots.

Given the analytic continuation property, it is sufficient to obtain $Q(d)$ once for one convenient value of d. One can then access $Q(d)$ for any other values of d including some for which a direct calculation for Q might pose a challenge.

5.6. Polylogs and related functions

Since our work on unification by means of the polylogs, there have appeared several papers studying the polylog function and connecting it to other functions and models.

A partial list of more mathematically related papers are: A connection is made to a blending function by Boyd [10], to polynomials by Schmidt and Schnack [11]. Maximon [12] studies the polylog function of a complex argument. Ciccariello [13,14] relates it Lerch's function and Hallerberg *et al.*, [15] to Ruelle function. Jodra [16] relates it to probability distributions in the Bass model.

A partial list of more physically related and applied papers are: Use is made of in relativistic thermodynamics by Blas and Pimentel [17], in entropic currents by Blencowe and Koschnick [18], in solid state devices by Ulrich and Barnes [19] and Ulrich *et al.* [20], in hot plasmas by Ayala *et al.* [21] in interacting Bose gas by Li *et al.* [22], in Casimir effect by Tonchev [23], in phase transitions in generalized statistics by Anghel [24], and in a Bose gas in a harmonic potential by Kirsten and Toms [49].

6. Application of polylog formulation

The unified form of the grand partition function via the polylogs (5.7) shows an appealing simplicity. But is it useful? Can it predict something new as a result of this formulation? We shall first illustrate its usefulness and then provide some unexpected consequences.

6.1. Thermodynamics by polylogs

We re-state (5.7) below:

$$\frac{1}{V}\log Q(\xi,d) = \lambda^{-d}\,\mathrm{sgn}(\xi)\,\mathrm{Li}_{d/2+1}(\xi)\,, \tag{6.1}$$

where ξ is the composite fugacity. We shall now generate standard thermodynamic functions from the source function the grand partition function by elementary operations.

(a) Number density ρ

$$N = \langle\hat{N}\rangle = z\,\frac{\partial}{\partial z}\,\log Q|_{\beta}\,, = \lambda^{-d}V\mathrm{sgn}(\xi)\mathrm{Li}_{d/2}(\xi)\,, \tag{6.2}$$

which is obtained by the recurrence relation (5.5). One may also write the above result as

$$\rho\,\lambda^{d} = \mathrm{sgn}(\xi)\mathrm{Li}_{d/2}(\xi)\,. \tag{6.3}$$

(b) Internal energy U

$$U = \frac{-\partial}{\partial\beta}\log Q\bigg|_{\xi} \tag{6.4a}$$

$$= \frac{d}{2}\,\beta^{-1}\,\lambda^{-d}V\mathrm{sgn}(\xi)\mathrm{Li}_{d/2+1}(\xi)\,. \tag{6.4b}$$

Hence,

$$\beta U = \frac{d}{2}\log Q\,. \tag{6.4c}$$

By (6.2),

$$\frac{\beta U}{N} = \frac{d}{2}\,\frac{\mathrm{Li}_{d/2+1}(\xi)}{\mathrm{Li}_{d/2}(\xi)}\,. \tag{6.5}$$

(c) Pressure P

$$PV = \beta^{-1}\,\log Q\,. \tag{6.6}$$

Hence,

$$\beta P\rho^{-1} = \frac{\mathrm{Li}_{d/2+1}(\xi)}{\mathrm{Li}_{d/2}(\xi)}\,. \tag{6.7}$$

By comparing (6.6) with (6.4)

$$U = \frac{d}{2}\,PV\,. \tag{6.8}$$

(d) Entropy S
It is simplest to use the definition

$$\frac{S}{Nk}=\beta\left(\frac{U}{N}+\frac{PV}{N}-\mu\right)=\beta\left(\frac{d}{2}+1\right)\frac{PV}{N}-\mu\,. \tag{6.9}$$

Hence,

$$\frac{S}{Nk}=\left(\frac{d}{2}+1\right)\frac{1}{N}\log Q-\log|\xi|\,. \tag{6.10}$$

We shall use these results to see how useful is the unified formulation.

6.2. Classical ideal gas

If $|t|<1$, the polylogs (5.6) may be given an expansion in t as

$$\mathrm{Li}_s(t)=\sum_{k=1}\frac{t^k}{k^s}\,. \tag{6.11}$$

For proof see [8,9]. Suppose $t\to 0$, so that only the first term on the right-hand side of (6.11) needs be kept:

$$\mathrm{Li}_s(t)=t\,. \tag{6.12}$$

Now the right-hand side of (6.12) is independent of $s=d/2+1$, which thus corresponds to the classical limit. But also observe that $t=\zeta$ is the fixed point of the polylogs. The fixed point has no logarithmic character. If the polylogs are represented by the fixed point, $\log Q$ sheds spin statistic and turns itself over into one for Boltzmann statistics as shown below: Under the classical limit, (6.1) becomes simply

$$\frac{\lambda^d}{V}\log Q=z \tag{6.13}$$

for both the Bose and Fermi gasses. Thus, under the classical limit, $\mu\to-\infty$ since $z\to 0$. Eq. (6.13) directly yields $\rho\lambda^d=z$ and by (6.6), for example, $PV=NkT$ and all other familiar results of an idea classic gas.

6.3. Cosmic thermodynamics

In the early universe, the neutrinos are thought to have played an important role in its evolution. As the universe cooled and freely expanded, the neutrinos also became decoupled from matter and thermalized in much the same way as the photons. To study the neutrino contributions to the cosmological evolution, the thermodynamic functions of a neutrino gas have been calculated. We can show by use of the unified formulation that most of it could have been avoided.

The unified formulation for the relativistic particles like the neutrino requires a few adjustments: The order of the polylogs $d/2$ is replaced by d and the thermal wavelength λ replaced by η defined by [25]

$$\eta = \pi^{1/2}\,\hbar\beta c \left\{ \frac{\Gamma(1/2)}{\Gamma(d/2+1/2)} \right\}^{1/d} , \tag{6.14}$$

where c is the speed of light. If $d = 3$, $\eta = \pi^{2/3}\hbar\beta c$. The same form would apply for particles with a linear dispersion relation (*e.g.* phonons, graphenes) with c replaced by ν.

Thus (6.1) now reads:

$$\frac{1}{V}\,\log Q(\xi, d) = \eta^{-d}\,\mathrm{sgn}(\xi)\,\mathrm{Li}_{d+1}(\xi)\,, \tag{6.15}$$

where ξ is the same composite fugacity.

The neutrinos are massless Fermi particles. Lately experimental evidence seems to suggest that the neutrinos are not massless [26]. Should a neutrino have a mass, it could not but be very small. If very small, it should have little practical consequences for thermodynamics. Thus we shall continue to assume that the neutrinos are massless.

Subject to this assumption, for the neutrinos we take $\xi = -1$. For photons, which are massless Bose particles, $\xi = 1$.

By the unified grand partition function (6.15), the thermodynamics of a neutrino gas is simply related to the thermodynamics of a photon gas by the structural relationship of $\mathrm{Li}_4(-1)/\mathrm{Li}_4(1) = 7/8$. If V is the same,

$$\log Q(\text{neutrino}) = k\,\log Q(\text{photon})\,, \tag{6.16}$$

where $k = 1/2(7/8)$, where $1/2$ comes from a polarization ratio between two massless particles [25]. Note that the unified grand partition function is without degeneracy factors due to *e.g.* spin for the sake of structural simplicity.

Hence, for the energy density $u = U/N$,

$$u(\text{neutrino}) = k\,u(\text{photon})\,. \tag{6.17}$$

The specific heat and the entropy are all simply related by the factor k. The neutrino thermodynamics is the same as what one would see in a black body scaled by k. The unified formulation obviates the need for a separate study for a neutrino gas given the knowledge of the black body radiation.

6.4. Thermodynamic equivalence in 2d

Returning to the non-relativistic version, let $d = 2$ in (6.1). For $\xi = -z_{\rm F}$ and $\xi = z_{\rm B}$, where $z_{\rm F}$ and $z_{\rm B}$ are the fugacities, respectively, for Fermi and Bose gases, we obtain:

$$\frac{1}{V}\log Q(z_{\rm F}) \;=\; -\lambda^{-2}{\rm Li}_2(-z_{\rm F})\,, \tag{6.18}$$

$$\frac{1}{V}\log Q(z_{\rm B}) \;=\; \lambda^{-2}{\rm Li}_2(z_{\rm B})\,, \tag{6.19}$$

where $V = L^2$ the area.

The grand partition functions are in dilogs. Could they be related through some properties of the dilogs? If so, the two ideal gases would be thermodynamically related. One could hardly expect such a relationship to exist between the gases of different statistics.

For the dilogs, there is a special relation due to Landen: If x and y are real numbers and are related by

$$y \;=\; \frac{-x}{(1-x)}\,, \tag{6.20}$$

$${\rm Li}_2(x) \;=\; -{\rm Li}_2(y) - \tfrac{1}{2}\{{\rm Li}_1(y)\}^2\,. \tag{6.21}$$

An elementary proof of Landen's relation is given by us [27].

Let $x = z_{\rm B}$ and $y = -z_{\rm F}$. Then,

$$\frac{1}{V}\log Q(z_{\rm B}) = \frac{1}{V}\log Q(z_{\rm F}) - \tfrac{1}{2}\lambda^{-2}\{{\rm Li}_1(-z_{\rm F})\}^2\,. \tag{6.22}$$

By (6.3), setting $d = 2$ therein,

$$\rho\lambda^2 = -{\rm Li}_1(-z_{\rm F})\,. \tag{6.23}$$

The left-hand side of (6.23) is $\beta\varepsilon_{\rm F}$, $\varepsilon_{\rm F}$ the Fermi energy in $2d$, which is twice the ground state energy per particle $(U_0^{\rm F}/N)$ of a spinless Fermi gas in $2d$: $U_0^{\rm F}/N = 1/2\varepsilon_{\rm F}$. This gives $\beta\, U_0^{\rm F}/V$ for the second term on the right-hand side of (6.22). Thus,

$$\log Q(z_{\rm B}) = \log Q(z_{\rm F}) - \beta\, U_0^{\rm F}\,. \tag{6.24}$$

By (6.4a) or (6.4c), we obtain

$$U(z_{\rm B}) = U(z_{\rm F}) - U_0^{\rm F}\,. \tag{6.25}$$

The above means that the constant-volume specific heats of the two gases are identically the same at all T, a remarkable result first obtained by May [7] in 1964 but without any physical insight. Also observe that $U(z_{\rm B}) \geq 0$.

Our result (6.25) was obtained by use of Landen's relation of x and y. See (6.20). It is premised on letting $x = z_{\rm B}$ and $y = -x_{\rm F}$, seemingly an arbitrary choice, which would connect the two unrelated variables $z_{\rm B}$ and $z_{\rm F}$ by (6.20). What physical conditions could yield such a special relationship?

To obtain possible physical conditions, let

$$\rho_{\rm B}\, \lambda_{\rm B}^2 = \rho_{\rm F}\, \lambda_{\rm F}^2 \,. \tag{6.26}$$

Then, by (6.3) setting $d = 2$,

$$\mathrm{Li}_1(z_{\rm B}) = -\mathrm{Li}_1(-z_{\rm B})\,. \tag{6.27}$$

Since $\mathrm{Li}_1(t) = -\log(1-t)$, we obtain at once,

$$z_{\rm F} \;=\; \frac{z_{\rm B}}{(1-z_{\rm B})}\,, \tag{6.28a}$$

or

$$z_{\rm B} \;=\; \frac{z_{\rm F}}{(1+z_{\rm F})}\,. \tag{6.28b}$$

Eq. (6.28a) corresponds exactly to (6.20) with $x = z_{\rm B}$ and $y = -z_{\rm F}$. Thus the equivalence (6.25) can exist by the physical condition of (6.26). The simplest way to satisfy it is to take $\rho_{\rm B} = \rho_{\rm F}$ and $\lambda_{\rm B} = \lambda_{\rm F}$. If in addition, $m_{\rm B} = m_{\rm F}$, it would mean that $T_{\rm B} = T_{\rm F}$. It would thus mean that the equivalence (6.25) would hold at any temperature.

The resulting relation (6.28a) or (6.28b) implies a special relationship for the chemical potentials. From (6.28b),

$$\mu_{\rm B} = \mu_{\rm F} - \beta^{-1}\log(1+z_{\rm F})\,. \tag{6.29}$$

As to the second term on the right-hand side, $\log(1+z_{\rm F}) = -\mathrm{Li}_1(-z_{\rm F}) = \rho\lambda^2 = \beta\varepsilon_{\rm F}$. Hence,

$$\mu_{\rm B} = \mu_{\rm F} - \varepsilon_{\rm F}\,. \tag{6.30}$$

This is a unique relationship that underlies the thermodynamic equivalence in 2d. Observe that $\mu_{\rm F} \leq \varepsilon_{\rm F}$ in 2d. Hence, $\mu_{\rm B} \leq 0$. But $U(z_{\rm B}) \geq 0$.

Since $\mu = \mu(T, P)$, it is like a differential surface, projected from the plane of T and P. The slope of the curvature of this surface along a given direction, T-axis or P-axis, has a meaning according to the Gibbs–Duhem equation [28]:

$$\frac{S}{N} = -\frac{\partial\mu}{\partial T}\Big|_{\rm P}\,. \tag{6.31}$$

and

$$\rho^{-1} = \frac{\partial \mu}{\partial P}\Big|_{\mathrm{T}} . \tag{6.32}$$

In this geometrical picture, (6.30) says that the two surfaces, one for Bose and the other for Fermi, are everywhere separated by a fixed distance of ε_{F}.

If (6.30) is applied to (6.31), we immediately obtain that the two entropies per particle are the same. If (6.32) is applied, we recover the condition that the two densities are the same. One can also obtain the equality of the two entropies per particle by (6.10) for $d = 2$ by using (6.24) and (6.30).

In this way we have established a thermodynamic equivalence between the two gases of different statistics [27]. As far as we know, no one has ever suggested such a possibility since the dawn of the statistical mechanics of quantum gases. Our work has generated considerable attention [29–45], [50–52].

7. Physics in null dimension

As stated earlier at some lengths (see Sec. 5.5), the grand partition function $Q(\xi; d)$ by polylogs has meant that d may take on any values. When $d = 2$ (see Sec. 6.4), it resulted in something new, a thermodynamic equivalence existing only in $2d$. When $d = 1$, there emerge some other features in the thermodynamic behavior. But they are more or less not out of the ordinary.

What if we let $d = 0$? To do statistical mechanics in null dimension by conventional methods would not be easy. It can be done easily by the polylog formalism since, by analytic continuation, we can reach $d = 0$.

In null dimension there is an important physical principle to consider: the uncertainty principle (UP). While somewhat quiescent in finite dimensions, it acts a dominant factor in null dimension.

7.1. Uncertainty principle and μ-singularity

If $d \to 0$, $V = L^d = 1$. There is no length scale. The volume of null dimension is unity, which refers to a point or a dot. For a particle placed in it, there can be no uncertainty in position: As $d \to 0$, $\Delta x \to \pm 0$. By UP, $\Delta p \to \pm\infty$. In addition, $\Delta t(\text{time}) \to \pm 0$ and $\Delta\varepsilon(\text{energy}) \to \pm\infty$.

If $\Delta\varepsilon \to \pm\infty$, $\Delta U \to \pm\infty$ since $\Delta U \to \sum \Delta\varepsilon\, n(\varepsilon)$. By the second law, $\mu = \Delta U/\Delta N|_{V,S}$. If S can be held fixed such as near the ground state where it is nearly zero, we can conclude that becomes $\mu \to \pm\infty$ as $d \to 0$. For a Fermi gas near the ground state, $\mu > 0$. For a Bose gas, $\mu \leq 0$. Hence, as $d \to 0$, $\mu(F) \to +\infty$ and $\mu(B) \to -\infty$. They become singular in null dimension. When μ becomes singular, the temperature T becomes irrelevant. It is a quantum phenomenon with no classical analog [5].

If $\mu(F) = \infty$, $n(F) = 1$, where $n(F)$ means the Fermi function. If $\mu(B) = -\infty$, $n(B) = 0$ where $n(B)$ is the Bose function. That $n(F) = 1$ at any T is merely a re-statement of the Pauli principle (PP) in the coordinate space: there may be one spinless Fermi particle in a dot space and no more. It is comparable to the action of PP in the momentum space in *e.g.* the 1s state of the H atom.

Why Bose particles cannot be confined to a dot, *i.e.* $n(B) = 0$, is harder to understand. An analogy might be of some help here. For a classical gas, $\mu \to -\infty$ if $T \to \infty$. When T is very large, a particle becomes too energetic to be confined in a finite volume.

7.2. Grand partition function for Fermi gas

Let us now see what the grand partition formalism will give us. Let in (6.1) $d = 0$ and $\xi = -z_{\mathrm{F}}$ in (6.1). Since $V = 1$,

$$\log Q = -\mathrm{Li}_1(-z_{\mathrm{F}}) = \log(1 + z_{\mathrm{F}})\,. \tag{7.1}$$

Hence, $N/V = N \equiv n$ is obtained from (7.1) as:

$$n = z\frac{\partial}{\partial z}\log Q = \frac{z_{\mathrm{F}}}{1 + z_{\mathrm{F}}}\,. \tag{7.2}$$

The above is also obtained from (6.2) or (6.3) by $\mathrm{Li}_0(-x) = -x/(1 + x)$. The left-hand side is a pure number, which cannot depend on T. The only way this requirement can be satisfied is if $z_{\mathrm{F}} \to \infty$, or $\mu_{\mathrm{F}} \to \infty$, precisely the same result obtained by UP. See Sec.7.1. If $z_{\mathrm{F}} = \infty$, $n = 1$, also given by the Fermi function when $\mu_{\mathrm{F}} = \infty$. See Ref. [46].

If, in the thermodynamic functions given in Sec. 6.1, we set $d = 0$, $z_{\mathrm{F}} = \infty$, $N = 1$ and $V = 1$, Fermi "thermodynamics" in null dimension follows:

By (6.4c),

$$\beta U = \lim_{z_{\mathrm{F}}\to\infty,\, d\to\infty} \frac{d}{2}\log z_{\mathrm{F}} = \infty\,. \tag{7.3}$$

For a proof, see Ref [46]. This seems to be the limit of blue-shift of the energy in size reduction [46].

By (6.6),

$$\beta P = \log z_{\mathrm{F}} = \infty \tag{7.4}$$

evidently related to the energy behavior.

By (6.10),

$$\frac{S}{k} = \log(1 + z_{\mathrm{F}}) - \log z_{\mathrm{F}} = \frac{1}{z_{\mathrm{F}}} - \dots\,. \tag{7.5}$$

The entropy vanishes since one Fermi particle in null dimension gives a perfectly ordered configurational state.

One might wonder what has happened to the thermodynamic limit when $0d$ is reached by analytic continuation. To answer this question, let us begin with $1d$: There are N particles (N large number) in the volume of L (L large length) such as to allow the thermodynamic limit. When d goes down to 0, the overall system is still $1d$-like but it now consists of N disconnected dots, each of which contains just 1 Fermi particle. Thus the total number of particles is still preserved. As noted earlier in Sec. 5.5, it is analogous to *e.g.* $2d \rightarrow 1d$, when the area becomes decomposed into L disconnected lines.

7.3. Meaning of μ singularity

In grand ensemble theory the chemical potential acts a control on the flow of particles into or out of a system. It is measured from the zero of energy. It is like the zero of energy for the electron in the H atom, where the zero is the ionization level if seen from one of its bound states.

A Fermi particle cannot enter into a Fermi gas at the zero of energy owing to the Pauli principle (PP). As a result, μ has to be positive until it goes classical. A particle may enter or leave only at the energy value matching $\mu(\mu = \varepsilon_{\mathrm{F}}$ if $T = 0)$. It is like a ceiling through which particles may enter or leave, not through the floor, which is at the zero of energy.

A Fermi particle may not leave through the floor because of the stability of a Fermi gas. Suppose one were to remove the particle at the floor having energy $\varepsilon = 0$. As a result, the one immediately above it with $\varepsilon = \Delta\varepsilon$ would move down. The same process would bring the next one down and it would continue until the highest one falling to $\varepsilon = \varepsilon_{\mathrm{F}} - \Delta\varepsilon$. The total energy required to achieve it is exactly the ground state energy $U_0 = d/(d+2)N\varepsilon_{\mathrm{F}}$. The energy would be just ε_{F} if simply the one at $\varepsilon = \varepsilon_{\mathrm{F}}$ were removed, yielding an identical configuration. Removing a particle at the floor is much like trying to remove an object at the base weighted down.

As there is no PP to contend with for a Bose gas, the bosonic μ is different. It is non-positive, measured from the afore mentioned zero of energy. When $\mu = 0$, where BEC can occur, Bose particles may enter or leave at no cost of energy (in contrast to the price of ε_{F} to be paid by Fermi particles if at $T = 0$). For Bose particles, their ceiling is at the zero of energy.

When μ has a negative value (say $\mu = -\varepsilon_0$, $\varepsilon_0 > 0$), it is as if this value ($-\varepsilon_0$) sets the floor for Bose particles. To remove a Bose particle, it has to be raised from this floor to its ceiling, the zero of energy. (It is much like ionizing the H atom in a bound state at $-\varepsilon_0$.)

It is the same if a Bose particle is to enter at the zero of energy. It has to give up the energy $-\varepsilon_0$ (as measured from the zero of the energy) to reach the floor from the ceiling. It is much like a free electron being captured by a hydrogen nucleus.

Let us now see how this picture explains the behavior of μ when $d \to 0$. For fermions, a dot contains but 1 particle. Thus, assuming that the entropy remains zero,

$$\mu_{\mathrm{F}} = U(2) - U(1) = \infty\,. \tag{7.6}$$

The ceiling is infinitely high. Another Fermi particle may not enter a dot already occupied. As we said, it is just PP at work. The same high ceiling also acts to confine a particle already within, preventing its escape.

For bosons, a dot cannot contain any particles. Hence there is no meaningful thermodynamics *e.g.* $U(0) = 0$. But one can still define μ as:

$$\mu_{\mathrm{B}} = U(1) - U(0) = -\infty\,. \tag{7.7}$$

The bosonic floor is infinitely deep. This is why a dot cannot contain any bosonic particles. As $d \to 0$, the bosonic floor seemingly drops out of sight into a bottomless pit. It is like the ground state of the H atom going from $-13.6\,\mathrm{eV}$ to $-\infty\,\mathrm{eV}$. There would not be enough energy to ionize such an H atom. When $d \to 0$, the Bose particles go into their own black hole, which gives the appearance of having escaped and unconfinable.

8. Concluding remarks

When the pioneering work of Sommerfeld is viewed from today's perspectives, certain limitations are apparent. They prevent from being generalizable. Hence it is difficult to gain a deeper insight into the statistical thermodynamics of quantum gases.

The problem is inherently logarithmic. Apart from the fact that, it is $\log Q$, not Q itself, which ties into thermodynamics. More deeply, both the Fermi and Bose functions are inverse functions of the fugacity z, implying logarithmic in character. The polylog formulation of $\log Q$ is a reflection of this character. Indeed at the classical limit, the polylogs are replaced by their fix point, losing the log character entirely. At the classical limit, both the Fermi and Bose functions are no longer an inverse function of the fugacity. The power of the polylogs seems almost limitless even in prosaic analysis [47, 48].

The polylog formulation of the grand partition function is sufficiently general to yield several special solutions. Of particular interest is the existence of thermodynamic equivalence in $2d$. It has already provided considerable stimulation and perhaps there will be more to come yet. The physics of null dimension presents some unusual features, still to be experimentally verified. In all the following lines may capture the spirit of this work:

When something goes inversely,
Logs cannot be far away.

Logs are strange math,
They have their own rules.

When they are embraced,
Logs can brighten an unseen side.

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