SOLUTION OF THE SPECIFIC MODEL OF FIVE-BODY PROBLEM TO INVESTIGATE THE EFFECTIVE ALPHA–NUCLEON INTERACTION IN A PARTIAL-WAVE ANALYSIS

E. Ahmadi Pouya[†], A.A. Rajabi[‡]

Physics Department, Shahrood University of Technology
P.O. Box 3619995161-316, Shahrood, Iran

(Received February 6, 2017; accepted May 18, 2017)

In this paper, we have solved a simple specific model of the five-body problem in the framework of the Yakubovsky equations, restricted to the configurations of the alpha–nucleon types only, to investigate the effective interaction between an inert alpha-particle and a neutron. In the general case, the Yakubovsky scheme for the solution of the five-body system leads to a set of four coupled equations related to four independent configurations, which can be restricted to two coupled ones, to describe the effective alpha–nucleon structure model, namely an inert four-body alpha–core and a nucleon. Hence, in such a model, the other configurations will not be taken into account. To calculate the binding energies of the five-body system in the model of alpha–nucleon structure, the two coupled equations are represented in the momentum space on the basis of the Jacobi momenta. After an explicit evaluation of the two coupled integral equations in a partial-wave analysis, the obtained equations are the starting point for a numerical calculation as an eigenvalue equation form, using typical iteration method. In the first step to the calculations, *i.e.* applying some spin-independent potential models, some obtained binding energy differences between the four-body as an alpha-particle and the five-body as an alpha–nucleon systems suggest that a simple effective interaction between an inert alpha-particle and a nucleon is attractive and of about 13 MeV. In addition, the represented binding energy results with respect to the regarded spin-independent potentials are in a fair agreement with the results obtained from other methods.

DOI:10.5506/APhysPolB.48.1279

[†] E.Ahmady.ph@ut.ac.ir
[‡] A.A.Rajabi@shahroodut.ac.ir

1. Introduction

The subject of effective alpha–nucleon (αN) interaction plays an important role in nuclear structure of few-body problems, and an entire understanding of this interaction is interesting and necessary. Moreover, the investigation of light nuclei and the study of the identity of the governing effective interactions, in addition to the specific properties of the bound and scattering states, are very interesting and relevant topics in nuclear few-body systems, as well as the atomic community. The main interest in the few-body problems is finding an accurate solution for the systems, as well as looking for unknown interactions governing these systems. To this regard, the investigation of few-nucleon bound systems interacting via simple and realistic interactions has been always in the center of interest. The description of light nuclei, and the effective αN interactions especially require well-established methods to solve the non-relativistic Schrödinger equation, in addition to the description of the relevant models of such interactions.

In the past few decades, considerable efforts have been made to investigate the effective αN interaction and applied to the exploration of the structure of light and heavy nuclei with this interaction, such as multichannel αN and $\alpha\alpha$ interactions [1], bound-state properties of the ^{6}He and ^{6}Li in a 3-body model, with an investigation of the αN interactions [2], interactions of αN in an elastic scattering [3], a survey of the αN interaction [4], peripheral αN scattering with NN potential [5]. Significant attempts have been made to obtain accurate ground-state properties of the few-nucleon systems even for $A > 4$ with simple and realistic potentials, namely the Stochastic Variational Monte Carlo (SVM) method [6] which still used simplified forces without realistic nuclear interactions, and the Nonsymmetrized Hyperspherical Harmonics (HH) approach [7] appears to be quite promising to deal with permutational-symmetry breaking terms in the Hamiltonian. The HH calculational scheme is usually based on the partial-wave (PW) representation. The SVM, however, is performed directly using position vectors in the configuration space. All these methods have proved to be of great accuracy and they have been tested using different benchmarks. The successful outcomes in [6, 7] suggest that a direct study of the five-body systems and beyond is now accessible on today's computers with high computational speed. Now, a direct treatment of the five-body problem for the model of effective alpha–nucleon (α–N) structure is required to investigate the effective interaction between an inert alpha-particle and an attractive nucleon (αN). Therefore, we feel that in order to solve the five-body systems, an accurate and old reliable method, the solution within the Yakubovsky scheme, would be interesting.

Now, after the experiences with 4- and 6-body bound systems within the Yakubovsky scheme in a typical PW analysis [8], and a three-dimensional formalism [9] whose technical expertise has been developed and the very strong increase of computational power recently achieved allow to study the five-body problems in the framework of the Yakubovsky equations for the model of effective α–N structure to investigate the effective αN interaction. It is worthwhile to mention that a realistic five-nucleon problem is not allowed for a bound state. However, in order to investigate the effective interactions between the two particles, namely alpha-particle and an attractive nucleon, we consider the five-body problem for the model of effective α–N structure as a bound system. Also, an objection to the use of simple phenomenological potentials for αN scattering arises from the fact that these potentials allow a bound state for the 5-body system which is forbidden by the exclusion principle. Therefore, in order to calculate the effective αN interaction in the special specific α–N structure of the five-body model system, we study the Yakubovsky scheme, extending the applications to systems with $A = 5$ and in order to calculate the binding energy results, we evaluate the coupled equations in momentum space based on a PW representation. Next, we have developed a particular representation of the high-dimension eigenvalue matrix, which is systematic with respect to the number of components and well-suited for a numerical implementation. In pursuit of this goal, we investigate the convergence of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points and calculate the expectation value of the Hamiltonian operator, which is systematic with respect to the number of components and well-suited for a numerical implementation.

This paper is organized as follows. In Sect. 2, the Yakubovsky formalism to the five-body problem using the standard notation [10] is explicitly derived. In addition, the identity of the particles is added which leads to a set of four coupled equations related to 4 different sequential subclusters of 5 particles. In Sect. 3, corresponding Jacobi coordinates of each Yakubovsky component is defined and the relevant configurations are selected to approximate the effective α–N structure. By these selections, a set of four coupled equations leads to two coupled ones and the irrelevant components will not be taken into account. In Sect. 4, the integral representation of each wavefunction (WF) component is represented by introducing the PW basis states based on Jacobi momenta. We describe also details for numerical techniques which are considered useful for a numerical performance. In Sect. 5, in order to compare and discuss our obtained results for binding energies of the four-body in the model of alpha-particle and the five-body system in the model of effective α–N structure, and to describe the effective αN interaction, the binding energy results are presented in tables where they are listed together with those obtained from other methods. In addition, in order to

test our calculations, we investigate the convergence of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points and calculate the expectation value of the Hamiltonian operator. Finally, the conclusions are provided in Sect. 6.

2. The five-body Yakubovsky formalism

In the five-body system, there are ten different two-body forces, or ten different cluster decompositions having 4-body fragments. They are labeled with (a_4), *e.g.* $a_4 = 12 \equiv 12+3+4+5$. To solve a typical five-body bound system in the Yakubovsky scheme using the subcluster notation [10], the idea is to first sum up the pair forces in each 4-body fragment (a_4), in a second step, among all 3-body fragments (a_3), and then in a third step, among all 2-body fragments (a_2). We work out this formalism ending with two-body subclusters in the spirit of the usually used approximate effective α–N structure model. To this end, we start with the non-relativistic Schrödinger equation for the five-body system, as follows:

$$\left(H_0 + \sum_{a_4} V_{a_4}\right) \Phi = E\,\Phi\,, \tag{2.1}$$

where H_0 stands for the free Hamiltonian operator of the five-body system, and $\sum_{a_4} V_{a_4} \equiv V_{12}+\cdots+V_{45}$ is the summation of the all 2-body interactions having ten terms. According to the Faddeev scheme, Eq. (2.1) is rewritten as an integral equation

$$\Phi = G_0 \sum_{a_4} V_{a_4}\,\Phi\,, \tag{2.2}$$

where G_0 is the five-body free Green's function operator and in the case of scattering states, we have $G_0 = [E - H_0 \pm i\varepsilon]^{-1}$. In investigating the bound states, there is no $i\varepsilon$ needed since $E < 0$. The first step is the summation of each pair force to infinite order, so we can define $\Phi = \sum_{a_4} \varphi_{a_4} \equiv \varphi_{12} + \cdots + \varphi_{45}$ for total WF, where we have $\varphi_{a_4} \equiv G_0 V_{a_4} \Phi$. By inserting that decomposition for Φ into the right-hand side, we have

$$\varphi_{a_4} \equiv G_0 V_{a_4}\,\varphi_{a_4} + G_0 V_{a_4} \sum_{b_4} \bar{\delta}_{a_4 b_4}\,\varphi_{b_4}\,. \tag{2.3}$$

The first term is related to a renewed interaction V_{a_4}, whereas in the second term, the next interaction $V_{b_4} \neq V_{a_4}$ and we use the anti-delta function as $\bar{\delta}_{a_4 b_4} = 1 - \delta_{a_4 b_4}$. Equation (2.3) can be packed by using the Faddeev-like equation

$$\varphi_{a_4} \equiv\; G_0 t_{a_4} \sum_{b_4} \bar{\delta}_{a_4 b_4}\;\varphi_{b_4}\,, \tag{2.4}$$

where t_{a_4} is a two-body t-matrix operator that obeys the Lippmann–Schwinger equation as $t_{a_4} = V_{a_4} + V_{a_4} G_0 t_{a_4}$. Next, we can describe main subclusters with definition of new components, as follows:

$$\varphi_{a_4 a_3} = G_0 t_{a_4} \sum_{b_4 \subset a_3} \bar{\delta}_{a_4 b_4} \varphi_{b_4} \,, \tag{2.5}$$

where (a_3) refers to any 3-body fragment containing the pair (a_4) and the sum runs over pairs $b_4 \subset a_3$. Next, we have $\varphi_{a_4} = \sum_{a_4 \subset a_3} \varphi_{a_4 a_3}$ and this relation is used to obtain a closed set of equations for $\varphi_{a_4 a_3}$

$$\varphi_{a_4 a_3} = G_0 t_{a_4} \sum_{b_4 \subset a_3} \bar{\delta}_{a_4 b_4} \sum_{b_4 \subset b_3} \varphi_{b_4 b_3} \,. \tag{2.6}$$

Now, we separate the components $\varphi_{a_4 a_3}$ for a given (a_3) from the rest

$$\varphi_{a_4 a_3} - G_0 t_{a_4} \sum_{b_4 \subset a_3} \bar{\delta}_{a_4 b_4} \varphi_{b_4 a_3} = G_0 t_{a_4} \sum_{b_4 \subset a_3} \bar{\delta}_{a_4 b_4} \sum_{b_4 \subset b_3} \bar{\delta}_{a_3 b_3} \varphi_{b_4 b_3} \,, \tag{2.7}$$

defining for a fixed (a_3) the column vectors φ^{a_3} and $\varphi^{(a_3)}$ with the components $(\varphi^{a_3})_{a_4} = \varphi_{a_4, a_3}$ and $(\varphi^{(a_3)})_{a_4} = \sum_{b_4 \subset a_3} \bar{\delta}_{a_3 b_4} \varphi_{b_4 b_3}$, respectively. Introducing the matrix M^{a_3} with the elements $M^{a_3} \equiv t_{a_4} \bar{\delta}_{a_4 b_4}$, Eq. (2.7) leads to

$$\varphi^{a_3} = (1 - G_0 M^{a_3})^{-1} G_0 M^{a_3} \varphi^{(a_3)} \equiv G_0 \mathcal{T}^{a_3} \varphi^{(a_3)} \,. \tag{2.8}$$

It is well-known that we achieve a Lippmann–Schwinger-like equation in the above Faddeev-like equation as $\mathcal{T}^{a_3} = M^{a_3} + M^{a_3} G_0 \mathcal{T}^{a_3}$. In the primal explicit notation, Eq. (2.8) leads to

$$\varphi_{a_4 a_3} = G_0 \sum_{b_4 \subset a_3} \mathcal{T}^{a_3}_{a_4 b_4} \left(\varphi^{a_3}\right)_{a_4} = G_0 \sum_{b_4 \subset a_3} \mathcal{T}^{a_3}_{a_4 b_4} \sum_{b_4 \subset b_3} \bar{\delta}_{a_3 b_3} \varphi_{b_4 b_3} \,, \tag{2.9}$$

where

$$\mathcal{T}^{a_3}_{a_4 b_4} = t_{a_4} \bar{\delta}_{a_4 b_4} + G_0 \sum_{c_4 \subset a_3} t_{a_4} \bar{\delta}_{a_4 c_4} \mathcal{T}^{a_3}_{c_4 b_4} \,. \tag{2.10}$$

We note that there are two types of $\mathcal{T}$-matrix. For (a_3) of the type 123+4+5, $\mathcal{T}^{a_3}_{a_4 b_4}$ is a 3×3 matrix and for (a_3) of the type $12 + 34 + 5$, $\mathcal{T}^{a_3}_{a_4 b_4}$ is a 2×2 matrix. Next, subsequent decompositions of the right-hand side of Eq. (2.9) according to 2-body fragments are given as

$$\varphi^{a_2}_{a_4, a_3} = G_0 \sum_{b_4 \subset a_3} \mathcal{T}^{a_3}_{a_4 b_4} \sum_{\substack{b_4 \subset b_3 \\ b_3 \subset a_2}} \bar{\delta}_{a_3 b_3} \varphi_{b_4 b_3} \,. \tag{2.11}$$

We remind that for 2-body fragments of $a_2 = 1234{+}5$ type, there are 18 pairs of (a_4), (a_3) and for 2-body fragments of $a_2 = 123{+}45$ type, there are 6 pairs of (a_4), (a_3). This defines the dimensions of the different $\mathcal{T}$-matrix. In the following, the single particles in subclusters 4-, 3- and 2-body fragments, *i.e.* (a_4), (a_3) and (a_2), respectively, will no longer be displayed.

Next, we implement the identity of the particles that leads to a set of four coupled equations related to four different structures of 5 particles. After implementing the identity of the particles, in Appendix A, we end up with four independent components: $\varphi^{1234}_{12;123}$, $\varphi^{1234}_{12;12+34}$, $\varphi^{123+45}_{12,123}$ and $(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34})$ coupled in equations, (A.35), (A.37) and (A.39). The linear and final form of the Yakubovsky coupled equations for a general model of the five-body system yields

$$\varphi^{1234}_{12;123} = G_0\mathcal{T}^{123}\Big((P_{34}P_{45} - P_{34})\,\varphi^{1234}_{12;123} - P_{34}\varphi^{123+45}_{12,123} + \Big(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\Big) + \varphi^{1234}_{12;12+34}\Big), \tag{2.12}$$

$$\varphi^{1234}_{12;12+34} = G_0\mathcal{T}^{12+34}\Big((1-P_{34})\,\big((1-P_{45})\varphi^{1234}_{12;123}\big) + (1-P_{34})\,\varphi^{123+45}_{12,123}\Big), \tag{2.13}$$

$$\varphi^{123+45}_{12,123} = G_0\mathcal{T}^{123}\,(-P_{35})\,\Big(\Big(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\Big) + \varphi^{1234}_{12;12+34}\Big), \tag{2.14}$$

$$\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34} = G_0\mathcal{T}^{12+34}\Big((-P_{35}-P_{45})\,\Big(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\Big) - P_{45}\,(1-P_{34})\;\varphi^{1234}_{12;12+34} - P_{35}\,\Big((1-P_{34})\,\varphi^{1234}_{12;123} + \varphi^{123+45}_{12,123}\Big)\Big). \tag{2.15}$$

In the next step, we describe the configuration of each independent component and select the specific configurations that describe the five-body system in the model of effective α–N structure.

3. Coupled equations of the effective alpha–nucleon structure

Regarding the subcluster underlying the four components, only two of them, that is $\varphi^{1234}_{12;123}$ and $\varphi^{1234}_{12;12+34}$, are related to the very approximate effective two-body configuration of α–N model, where the alpha-particle and attractive nucleon approximation is valid, according to the first two configurations in Fig. 1. The component $\varphi^{123+45}_{12,123}$ refers to an inert 3-body together with a 2-body subclustering. The linear combinations $(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34})$ refer again to an inert 3-body together with a 2-body subclustering, where the underlying fragmentation related to 2-body fragments differs from $\varphi^{123+45}_{12,123}$ (see Fig. 1).

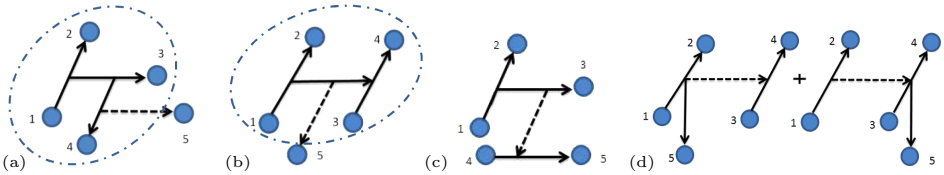

Fig. 1. Diagramatic configuration of the four independent components of 5-body system in Jacobi coordinates. (a), (b), (c) and (d) are respectively the configurations of the components $\varphi^{1234}_{12;123}$, $\varphi^{1234}_{12;12+34}$, $\varphi^{123+45}_{12,123}$ and $(\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34})$. In the bag, the alpha–core acts as a 4-body subsystem.

Now, we explain why we chose some specific components, and which components are related to the effective α–N structure. It is worthwhile to mention that for full solution of the 5-body system, in a general model, we need modern super-computers organized with grid parallel and we must consider all the configurations. However, in this project, we are interested to study the 5-body system for the specific model of effective α–N structure. Therefore, according to the above discussions, we choose the first two relevant configurations, and further, the other configurations will not be taken into account in the specific α–N structure. Moreover, according to Fig. 1, the effective interaction of alpha-particle is governor and concealed in the remained components (see first two configurations in Fig. 1 and compare them with Figs. 1 and 2 in Ref. [11]). Therefore, for approximating the effective α–N structure, we selected $\varphi^{1234}_{12;123} \equiv K$ and $\varphi^{1234}_{12;12+34} \equiv H$. As a result, the corresponding first two coupled equations, namely Eq. (2.12) and Eq. (2.13) lead to

$$K = G_0\, \mathcal{T}^{123} \left((P_{34}\, P_{45} - P_{34})\, K + H \right) , \tag{3.1}$$

$$H = G_0\, \mathcal{T}^{12+34} \left(1 - P_{45} - P_{34} + P_{34}\, P_{45} \right) K . \tag{3.2}$$

It is well-known that such a nuclear system should be treated in the fermionic approaches, *i.e.* the five-body total WF follows $\Phi = -P_{ij}\Phi$, and the Pauli principle is taken into account, even for spinless particles. Here however, as a simplification, we switch off spin and isospin degrees of freedom and study the effective five-body system in $L = 1$ states as spinless particles, (see Appendix B). According to the above-mentioned first two specific configurations, Fig. 1, after removing the interaction of the fifth nucleon in the above-mentioned coupled equations, namely $P_{45} \equiv 0$, the five-body system leads to a typical four-body problem [11] as follows:

$$K = -G_0\, \mathcal{T}^{123}\, P_{34}\, K + G_0\, \mathcal{T}^{123}\, H , \tag{3.3}$$

$$H = G_0\, \mathcal{T}^{12+34} \left(1 - P_{34} \right) K . \tag{3.4}$$

Such a reduction confirms that extending the Yakubovsky formulations for the specific model of the five-body system is a reasonable approximation to describe the effective α–N structure as a five-body system. Therefore, in the calculation step, for binding energies of specific five-body system, we can typically calculate the four-body binding energies for a comparison.

4. Numerical implementation

In this step, in order to implement the numerical techniques, the two coupled equations, Eq. (3.1) and Eq. (3.2), are represented in momentum space. We also introduce the standard Jacobi momentum vectors on the basis of distinct configuration according to Fig. 1, represented in Appendix B. Let us now represent the coupled equations, Eq. (3.1) and Eq. (3.2), to the basis states introduced in Appendix B. By inserting the completeness relations, Eq. (B.9), between the permutation operators, we receive

$$\begin{aligned}\langle a|K\rangle &= \int a'^2 \mathrm{d}a' \int a''^2 \mathrm{d}a'' \langle a|G_0\mathcal{T}^{123}|a'\rangle\langle a'|\left(P_{34}P_{45}-P_{34}\right)|a''\rangle\langle a''|K\rangle \\ &\quad + \int a'^2 \mathrm{d}a' \int b'^2 \mathrm{d}b' \langle a|G_0\mathcal{T}^{123}|a'\rangle\langle a'|b'\rangle\langle b'|H\rangle\,, \qquad (4.1)\\ \langle b|H\rangle &= \int b'^2 \mathrm{d}b' \int a'^2 \mathrm{d}a' \langle b|G_0\mathcal{T}^{12+34}|b'\rangle\langle b'|\left(1-P_{45}-P_{34}+P_{34}P_{45}\right)|a'\rangle \\ &\quad \times\langle a'|K\rangle\,. \qquad (4.2)\end{aligned}$$

The various terms appearing in the right hand-side of Eqs. (4.1) and (4.2) are explicitly evaluated in Appendix C. After evaluation of each term in the above-mentioned coupled integral equations in the standard PW analysis, the obtained equations are the starting point for numerical calculations as an eigenvalue equation form. In order to reduce the high dimension of the problem, we first choose an appropriate coordinate system. In this selection, the third vector A_3 has been chosen parallel to z-axis, the second vector A_2 in the x–z plane, and the first vector A_1 and fourth vector A_4 are arbitrary in the space. Therefore, we need nine variables to uniquely specify the geometry of the four vectors $A_i\,(i=1,\ldots,4)$ with three spherical angles and two azimuthal angles variables between them. By these considerations, the dimension of the eigenvalue problem is

$$N = N_{\mathrm{Jac}}^4 \times N_{\mathrm{sph}}^3 \times N_{\mathrm{azi}}^2 \times 2\,. \qquad (4.3)$$

The dependence on the continuous momentum and angle variables should be replaced in the numerical treatment by a dependence on certain discrete values. The large matrix eigenvalue equation requires an iterative solution method. We use a Lanczos-like scheme that is proved to be very efficient for

nuclear few-body problems [12, 13]. This technique reduces the dimension of the eigenvalue problem to the number of iteration minus one. The evaluated coupled set of Eqs. (4.1) and (4.2) in a matrix notation has the following schematic structure as an eigenvalue equation:

$$\eta(E)\ \varphi(K,H) = k(E)\ \varphi(K,H)\ , \tag{4.4}$$

where E is the energy eigenvalue at which the auxiliary Yakubovsky kernel eigenvalue $\eta(E)$ is equal to one. The Yakubovsky kernel of the linear equations $k(E)$ is energy-dependent, and $\eta(E)$ is its eigenvalue with φ as the corresponding eigenvector. In order to solve the eigenvalue equation, Eq. (3.4), we use the Gaussian quadrature grid points. The coupled equations represent a set of homogeneous integral equations, which, after discretization, turn into a large matrix eigenvalue equation. Starting from an arbitrary initial $\varphi \equiv \varphi_0$, by consecutive applications of $k(E)$, one generates a sequence of amplitudes φ_n, which, after orthogonalization, form a basis into which φ is expanded. It turns out that a reasonably small number of k-applications (of the order of 10–20) is sufficient, which leads to an algebraic eigenvalue problem of rather low dimension. Then the energy is varied such that one reaches $\eta(E) = 1$. More similar discussions can be found in Refs. [8, 14].

5. Results

5.1. Binding energy

In this section, in order to investigate the effective αN interaction, we present numerical results for binding energies of the five-body system in the model of effective α–N structure, and compare them with the four-body binding energies in the alpha-particle model, because the binding energy differences between four- and five-body systems, in such a model, reflects the value of effective αN interaction. Bound-state results of the four- and five-body systems are shown in Table II and III, respectively. We also draw comparisons with results obtained by results of other methods. In order to compare our calculations with results obtained by other techniques, we use the spin-independent simple potential models, as follows:

I. Gauss-type Volkov potential [15]

$$V(r) = V_{\mathrm{R}} \exp\left[-\mu_{\mathrm{R}} r^2\right] - V_{\mathrm{A}} \exp\left[-\mu_{\mathrm{A}}\ r^2\right]\ [\mathrm{MeV}]\,. \tag{5.1}$$

II. Yukawa-type Malfliet–Tjon V potential [16]

$$V(r) = V_{\mathrm{R}} \frac{\exp\left[-\mu_{\mathrm{R}}\ r\right]}{r} - V_{\mathrm{A}}\ \frac{\exp\left[-\mu_{\mathrm{A}}\ r\right]}{r}\ [\mathrm{MeV}]\,. \tag{5.2}$$

In the above-mentioned potentials, label V_{R} and V_{A} stand for repulsive- and attractive-part coefficients, respectively, and μ is the exchanged pion mass. The parameters of each potential are given in Table I. It is well-known that such simple potentials mentioned above applied in the calculations allow a bound state for the five-body system and they are naturally expected. In the calculations, we have used the operator form of the above potentials.

TABLE I

List of the parameters of the simple potential models applied in the calculations. The potential strengths $(V_{\mathrm{R}}, V_{\mathrm{A}})$ are in MeV for Volkov and MeV×fm for Malfliet–Tjon V, and the range parameters, exchanged pion masses $(\mu_{\mathrm{R}}, \mu_{\mathrm{A}})$, are in fm^{-2} for Volkov and fm^{-1} for Malfliet–Tjon V.

Potential	Type	V_{R}	μ_{R}	V_{A}	μ_{A}
Volkov [15]	Gauss	144.86	1.487	83.34	0.3906
Malfliet–Tjon V [16]	Yukawa	1458.05	3.11	578.09	1.55

For the Volkov potential, our calculations for four- and five-body binding energies yield the values −30.39 and −44.02 MeV, respectively, which as shown in Table II, are also in a good compatibility with those obtained from other calculations.

TABLE II

Four- and five-body binding energies for the Volkov potential in MeV.

Method	E_4	E_5
HH [17, 18]	−30.420	−43.032
HH [19]	−30.406	−42.383
SVM [16]	−30.424	−43.00
Present work	−30.39	−44.02

Our calculations for Malfliet–Tjon V yield the value −31.36 MeV for four-body binding energy, which is in a good agreement with HH [17], SVM [6] and VMC [20] results. Our results for five-body binding energy with value −44.30 MeV are also in a fair compatibility with those obtained from other methods.

Comparison of our numerical results for binding energies with respect to the regarded spin-independent potentials are in a good agreement with results of other methods in the first step calculations. Also some obtained binding energy differences between the four-body as an alpha-particle and five-body as an effective alpha–nucleon model systems suggest that an effective αN interaction in such a model is attractive and its value is about 13 MeV.

5.2. Expectation value energy

In this section, we have implemented the numerical stability of our algorithm and our representation of the five-body Yakubovsky components in PW analysis. We have specially investigated the convergence of the eigenvalue of the Yakubovsky kernel with respect to the number of grid points for Jacobi momenta, azimuthal and spherical angle variables. We have also investigated the quality of our representation of the Yakubovsky components and, consequently, WF by calculation of the expectation value of the five-body Hamiltonian operator. We have applied the Malfliet–Tjon V potential in our investigations. In Table IV, we present the obtained eigenvalue results for the five-body binding energy given in Table III for suitable different grids. We label the number of grid points for K and H WFs Jacobi momenta respectively as $N^a_{\rm Jac}$ and $N^b_{\rm Jac}$, for spherical angles as $N_{\rm sph}$ and for azimuthal angles as $N_{\rm azi}$. As demonstrated in Table IV, the calculations of the eigenvalue η converge to the value one for $N^a_{\rm Jac} = N^b_{\rm Jac} = 20$ and $N_{\rm sph} = N_{\rm azi} = 14$.

TABLE III

Four- and five-body binding energies for the Malfliet–Tjon V potential in MeV.

Method	E_4	E_5
SVM [6]	−31.360	−43.48
VMC [20]	−31.3	−42.98
HH [17]	−31.347	
Present work	−31.36	−44.30

TABLE IV

Convergence of the eigenvalue η of the Yakubovsky kernel with respect to the number of grid points in Jacobi momenta $N^a_{\rm Jac}$ and $N^b_{\rm Jac}$, spherical angles $N_{\rm sph}$ and azimuthal angles $N_{\rm azi}$, where $E_5 = -44.30$ MeV.

$N^a_{\rm Jac}$	$N^b_{\rm Jac}$	$N_{\rm sph} = N_{\rm azi}$	η
10	10	14	0.926
14	10	14	0.963
16	14	14	0.987
20	16	14	0.998
20	20	14	1.000

The solution of the coupled integral equations in momentum space allows estimating numerical errors reliably. In order to demonstrate reliability of our calculations, we evaluated the expectation value of the five-body Hamiltonian operator and compared this value to the calculated binding energy of the eigenvalue equation, Eq. (4.4) — results are given in Table III. The expectation values of the five-body kinetic energy $\langle H_0 \rangle$, the all 2-body interaction $\langle V \rangle$ and the five-body Hamiltonian operator $\langle H \rangle$ for the five-body system are given in Table V for Malfliet–Tjon V interactions calculated in PW analysis. The little differences between the expectation value of the five-body Hamiltonian $\langle H \rangle$ and the eigenvalue energy E_5 show that the results are in a fair agreement. However, a better agreement could be reached if we considered a larger number of grid points in our calculations.

TABLE V

Expectation values of the five-body kinetic energy $\langle H_0 \rangle$ and all 2-body interactions $\langle V \rangle$. Additionally, the expectation values of the five-body Hamiltonian operator $\langle H \rangle$ are compared to the binding energy results from the eigenvalue equation. All values are given in MeV.

Method	$\langle H_0 \rangle$	$\langle V \rangle$	$\langle H \rangle$	E
Malfliet–Tjon V	72.43	−118.71	−46.28	−44.30

6. Conclusions

The subject of alpha–nucleon interaction plays such an important role in nuclear structure problems that an entire understanding of this interaction is necessary. Therefore, in order to investigate the effective αN interaction, we have solved the coupled Yakubovsky equations for the five-body system in the model of approximating effective alpha–nucleon structure in a PW analysis that is implemented in the basis of momentum variables. To this end, we formulated the coupled equations for the spinless particles as the function of Jacobi momenta, namely the magnitudes of the momenta and angles between them. The coupled integral equations for a bound-state calculation can be handled in PW representation and solved by a numerically reliable standard method. Our numerical results of binding energies with respect to the regarded spin-independent simple potentials are in a fair agreement with the results of other methods in the first step calculations, and also some obtained binding energy differences between the four-body as an alpha-particle and the five-body as an alpha–nucleon model systems suggest that an effective interaction of αN is attractive and occurs at about 13 MeV. In addition, the stability of our algorithm has been achieved with the calculation of the eigenvalue of Yakubovsky kernel, where a different number of grid points for

Jacobi momenta and angle variables have been used. We have also calculated the expectation value of the five-body Hamiltonian operator. This test of calculation has been done with Malfliet–Tjon V potential and we have achieved a good compatibility between the obtained eigenvalue energy and the expectation value of the Hamiltonian operator.

It is worthwhile to mention that by including the spin effects in the implementation of the four-body system in the model of alpha-particle and five-body system in the specific model of effective α–N structure, both binding energy results will be almost equally improved, so correspondingly, the results of the effective αN interaction will remain almost unchanged when the spin-dependent interactions are used. In addition to the solution of the five-body system in the model of effective α–N structure, according to Fig. 1, only the first two relevant configurations, in terms of two first Yakubovsky components, are considered. Obviously, the irrelevant configurations/components will not be taken into account, even for spin-dependent potentials though for a full solution of the general model of five-body bound systems, such as constituent quark models (pentaquark) or atomic five-boson bound systems (pentamer), the incorporation of the all components is required. This is very promising and nourishes our hope for performing calculations with spin-dependent nucleon–nucleon potential models in a PW analysis and also 3-dimentional formalism based on the Yakubovsky method.

Appendix A

Implementation of the identity of the particles

We start from Eq. (2.9) and choosing the case of 4-body fragments $a_4 = 12$ with 3-body fragments $a_3 = 123$, one obtains

$$\varphi_{12,123} = G_0\mathcal{T}^{123}_{12,12}\left(\varphi^{(123)}\right)_{12} + G_0\mathcal{T}^{123}_{12,23}\left(\varphi^{(123)}\right)_{23} + G_0\mathcal{T}^{123}_{12,31}\left(\varphi^{(123)}\right)_{31} \tag{A.1}$$

according to the second term of Eq. (2.9)

$$\begin{aligned}\varphi_{12,123} \;=\; & G_0\mathcal{T}^{123}_{12,12}\left(\varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+34} + \varphi_{12,12+35} + \varphi_{12,12+45}\right) \\ & + G_0\mathcal{T}^{123}_{12,23}\left(\varphi_{23,234} + \varphi_{23,235} + \varphi_{23,23+14} + \varphi_{23,23+15} + \varphi_{23,23+45}\right) \\ & + G_0\mathcal{T}^{123}_{12,31}\left(\varphi_{31,314} + \varphi_{31,315} + \varphi_{31,31+24} + \varphi_{31,31+25} + \varphi_{31,31+45}\right) .\end{aligned} \tag{A.2}$$

It is easily seen, going back to the definitions in Eq. (2.3) and Eq. (2.6) together with the anti-symmetry requirement for the total state wave function,

that

$$\varphi_{23,234} + \varphi_{23,235} + \varphi_{23,23+14} + \varphi_{23,23+15} + \varphi_{23,23+45}$$
$$= P_{12}P_{23}\left(\varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+34} + \varphi_{12,12+35} + \varphi_{12,12+45}\right), \quad \text{(A.3)}$$
$$\varphi_{31,314} + \varphi_{31,315} + \varphi_{31,31+24} + \varphi_{31,31+25} + \varphi_{31,31+45}$$
$$= P_{13}P_{23}\left(\varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+34} + \varphi_{12,12+35} + \varphi_{12,12+45}\right). \quad \text{(A.4)}$$

Therefore, Eq. (A.2) turns into

$$\varphi_{12,123} = G_0\left(\mathcal{T}^{123}_{12,12} + \mathcal{T}^{123}_{12,23}P_{12}P_{23} + \mathcal{T}^{123}_{12,31}P_{13}P_{23}\right) \times \left(\varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+34} + \varphi_{12,12+35} + \varphi_{12,12+45}\right). \quad \text{(A.5)}$$

The coupled sets of Eq. (2.10) for using $a_3 = 123$ with relations like $P_{12}P_{23}t_{13}P_{23}P_{12} = P_{13}P_{23}t_{23}P_{23}P_{13} = t_{12}$, reveals that

$$\mathcal{T}^{123} = \mathcal{T}^{123}_{12,12} + \mathcal{T}^{123}_{12,23}P_{12}P_{23} + \mathcal{T}^{123}_{12,31}P_{13}P_{23}, \quad \text{(A.6)}$$

where $\mathcal{T}^{123}$ obeys $\mathcal{T}^{123} = t_{12}P + t_{12}PG_0\mathcal{T}^{123}$ and where $P = P_{12}P_{23} + P_{13}P_{23}$. Then, Eq. (A.5) simplifies to

$$\varphi_{12,123} = G_0\mathcal{T}^{123}\left(\varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+34} + \varphi_{12,12+35} + \varphi_{12,12+45}\right). \quad \text{(A.7)}$$

Starting again from Eq. (2.9) but now for the case of 4-body fragments $a_4 = 12$ with 3-body fragments $a_3 = 12 + 34$, one obtains

$$\varphi_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\left(\varphi^{(12+34t)}\right)_{12} + G_0\mathcal{T}^{12+34}_{12,34}\left(\varphi^{(12+34)}\right)_{34} \quad \text{(A.8)}$$

according to the second term of Eq. (2.9)

$$\varphi_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\left(\varphi_{12,123} + \varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right)$$
$$+G_0\mathcal{T}^{12+34}_{12,34}\left(\varphi_{34,134} + \varphi_{34,234} + \varphi_{34,345} + \varphi_{34,34+15} + \varphi_{34,34+25}\right). \quad \text{(A.9)}$$

Then, we use permutation operator properties as follows:

$$\varphi_{12,12+34} = 7G_0\left(\mathcal{T}^{12+34}_{12,12} + \mathcal{T}^{12+34}_{12,34}P_{13}P_{24}\right) \times \left(\varphi_{12,123} + \varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right) \quad \text{(A.10)}$$

defining

$$\mathcal{T}^{12+34} = \mathcal{T}^{12+34}_{12,12} + \mathcal{T}^{12+34}_{12,34}P_{13}P_{24}, \quad \text{(A.11)}$$

where $\mathcal{T}^{12+34}$ obeys the equation $\mathcal{T}^{12+34} = t_{12}P + t_{12}PG_0\mathcal{T}^{12+34}$ and where $P = P_{13}P_{24}$. Therefore, Eq. (A.9) simplifies to

$$\varphi_{12,12+34} = G_0\mathcal{T}^{12+34}\left(\varphi_{12,123} + \varphi_{12,124} + \varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right) . \tag{A.12}$$

The next step is to decompose $\varphi_{12,123}$ according to Eq. (2.11). For 4-body fragments $a_4 = 12$, with 3-body fragments $a_3 = 123$, the possible 2-body fragments (a_2) are $1234,\ 1235,\ 123 + 45$. Let us begin with

$$\begin{aligned}\varphi^{1234}_{12,123} \ &= \ G_0\mathcal{T}^{123}_{12,12}\left(\varphi_{12,124} + \varphi_{12,12+34}\right) + G_0\mathcal{T}^{123}_{12,23}\left(\varphi_{23,234} + \varphi_{23,23+14}\right) \\ &\quad + G_0\mathcal{T}^{123}_{12,31}\left(\varphi_{31,134} + \varphi_{31,31+24}\right) .\end{aligned} \tag{A.13}$$

Since

$$\varphi_{23,234} + \varphi_{23,23+14} \ = \ P_{12}P_{23}\left(\varphi_{12,124} + \varphi_{12,12+34}\right) , \tag{A.14}$$

$$\varphi_{31,134} + \varphi_{31,31+24} \ = \ P_{13}P_{23}\left(\varphi_{12,124} + \varphi_{12,12+34}\right) , \tag{A.15}$$

Eq. (A.13) simplifies according to Eq. (A.6) what leads to

$$\varphi^{1234}_{12,123} = G_0\mathcal{T}^{123}\left(\varphi_{12,124} + \varphi_{12,12+34}\right) . \tag{A.16}$$

Similarly,

$$\varphi^{1235}_{12,123} = G_0\mathcal{T}^{123}\left(\varphi_{12,125} + \varphi_{12,12+35}\right) . \tag{A.17}$$

Again, using symmetry properties, one gets

$$\varphi^{123+45}_{12,123} = G_0\mathcal{T}^{123}\varphi_{12,12+45} , \tag{A.18}$$

all summed up to

$$\varphi_{12,123} = \varphi^{1234}_{12,123} + \varphi^{1235}_{12,123} + \varphi^{123+45}_{12,123} , \tag{A.19}$$

which when written out agrees with Eq. (A.7). Similarly, next, we decompose $\varphi_{12,12+34}$ according to Eq. (2.11). For the 4-body fragments $a_4 = 12$, with 3-body fragments $a_3 = 12 + 34$, the possible 2-body fragments (a_2) are $1234,\ 125 + 34,\ 12 + 345$, which are now, in turn, regarded as

$$\varphi^{1234}_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\left(\varphi_{12,123} + \varphi_{12,124}\right) + G_0\mathcal{T}^{12+34}_{12,34}\left(\varphi_{34,234} + \varphi_{34,134}\right) \tag{A.20}$$

since we use $\varphi_{34,234} + \varphi_{34,134} = P_{13}P_{24}(\varphi_{12,123} + \varphi_{12,124})$. One can use Eq. (A.11) and gets

$$\varphi^{1234}_{12,12+34} = G_0\mathcal{T}^{12+34}\left(\varphi_{12,123} + \varphi_{12,124}\right) . \tag{A.21}$$

Next,

$$\varphi^{125+34}_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\varphi_{12,125} + G_0\mathcal{T}^{12+34}_{12,34}\left(\varphi_{34,15+34} + \varphi_{34,25+34}\right), \quad \text{(A.22)}$$

$$\varphi^{12+345}_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\left(\varphi_{12,12+35} + \varphi_{12,12+45}\right) + G_0\mathcal{T}^{12+34}_{12,34}\varphi_{34,345}. \quad \text{(A.23)}$$

The two above amplitudes cannot be related by permutations, but their sum can be used

$$\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34} = G_0\mathcal{T}^{12+34}_{12,12}\left(\varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right) + G_0\mathcal{T}^{12+34}_{12,34}\left(\varphi_{34,15+34} + \varphi_{34,25+34} + \varphi_{34,345}\right), \quad \text{(A.24)}$$

in the case of

$$\varphi_{34,15+34} + \varphi_{34,25+34} + \varphi_{34,345} = P_{13}P_{24}\left(\varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right), \quad \text{(A.25)}$$

this leads to

$$\varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34} = G_0\mathcal{T}^{12+34}\left(\varphi_{12,125} + \varphi_{12,12+35} + \varphi_{12,12+45}\right). \quad \text{(A.26)}$$

Thus, Eq. (A.20) and Eq. (A.26) summarize to

$$\varphi_{12,12+34} = \varphi^{1234}_{12,12+34} + \varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}, \quad \text{(A.27)}$$

which when written out agrees with Eq. (A.12). The two amplitudes $\varphi^{1234}_{12,123}$ and $\varphi^{1234}_{12,12+34}$ expressed in Eq. (A.16) and Eq. (A.21) are connected to each other as has been shown. Equation (A.19) can easily be converted to $\varphi_{12,124}$ and using in addition Eq. (A.27), with Eq. (A.16), one finds

$$\varphi^{1234}_{12,123} = G_0\mathcal{T}^{123}\left(\left(\varphi^{1234}_{12,124} + \varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124}\right) + \left(\varphi^{1234}_{12,12+34} + \varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\right)\right). \quad \text{(A.28)}$$

One separates the components $\varphi^{1234}_{12,124}$ and $\varphi^{1234}_{12,12+34}$ from the rest

$$\varphi^{1234}_{12,123} - G_0\mathcal{T}^{123}\left(\varphi^{1234}_{12,124} + \varphi^{1234}_{12,12+34}\right) = G_0\mathcal{T}^{123}\left(\varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124} + \varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\right), \quad \text{(A.29)}$$

correspondingly, Eq. (A.21) yields

$$\varphi^{1234}_{12,12+34} = G_0\mathcal{T}^{12+34}\left(\left(\varphi^{1234}_{12,123} + \varphi^{1235}_{12,123} + \varphi^{123+45}_{12,123}\right) + \left(\varphi^{1234}_{12,124} + \varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124}\right)\right). \quad \text{(A.30)}$$

Again, one separates the components $\varphi^{1234}_{12,123}$ and $\varphi^{1234}_{12,124}$ from the rest

$$\begin{aligned}&\varphi^{1234}_{12,12+34} - G_0\mathcal{T}^{12+34}\left(\varphi^{1234}_{12,123} + \varphi^{1234}_{12,124}\right)\\ &= G_0\mathcal{T}^{12+34}\left(\varphi^{1235}_{12,123} + \varphi^{123+45}_{12,123} + \varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124}\right).\end{aligned} \tag{A.31}$$

With $\varphi^{1234}_{12,124} = -P_{34}\varphi^{1234}_{12,123}$, we can put Eq. (A.29) and Eq. (A.31) into a matrix form

$$\begin{aligned}&\begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix} - G_0\begin{pmatrix}\mathcal{T}^{123}(-P_{34}) & \mathcal{T}^{123}\\ \mathcal{T}^{12+34}(1-P_{34}) & 0\end{pmatrix}\begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix}\\ &= G_0\begin{pmatrix}\mathcal{T}^{123}\left(\varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124} + \varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\right)\\ \mathcal{T}^{12+34}\left(\varphi^{1235}_{12,123} + \varphi^{123+45}_{12,123} + \varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124}\right)\end{pmatrix}.\end{aligned} \tag{A.32}$$

Since

$$\varphi^{1245}_{12,124} = -P_{34}\varphi^{1235}_{12,123},\ \varphi^{124+35}_{12,124} = -P_{34}\varphi^{123+45}_{12,123}, \tag{A.33}$$

the right-hand side of Eq. (A.32) can be factored and achieves the form of

$$\begin{aligned}&\begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix} - G_0\begin{pmatrix}\mathcal{T}^{123}(-P_{34}) & \mathcal{T}^{123}\\ \mathcal{T}^{12+34}(1-P_{34}) & 0\end{pmatrix}\begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix}\\ &= G_0\begin{pmatrix}\mathcal{T}^{123}(-P_{34}) & \mathcal{T}^{123}\\ \mathcal{T}^{12+34}(1-P_{34}) & 0\end{pmatrix}\\ &\times\begin{pmatrix}\varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124} + \varphi^{125+34}_{12,12+34} + \varphi^{12+345}_{12,12+34}\\ \varphi^{1235}_{12,123} + \varphi^{123+45}_{12,123} + \varphi^{1245}_{12,124} + \varphi^{124+35}_{12,124}\end{pmatrix},\end{aligned} \tag{A.34}$$

the right-hand side can be reduced applying permutations and obtains the final form of

$$\begin{aligned}\begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix} &= G_0\begin{pmatrix}\mathcal{T}^{123}(-P_{34}) & \mathcal{T}^{123}\\ \mathcal{T}^{12+34}(1-P_{34}) & 0\end{pmatrix}\\ &\times\left[\begin{pmatrix}-P_{45}\varphi^{1234}_{12;123} + \varphi^{123+45}_{12,123}\\ \varphi^{125+34}_{12;12+34} + \varphi^{12+345}_{12;12+34}\end{pmatrix} + \begin{pmatrix}\varphi^{1234}_{12;123}\\ \varphi^{1234}_{12;12+34}\end{pmatrix}\right].\end{aligned} \tag{A.35}$$

After an adequate permutation of Eq. (A.27), one obtains a form of Eq. (A.18) as

$$\varphi^{123+45}_{12,123} = G_0\mathcal{T}^{123}\left(\varphi^{1245}_{12,12+45} + \varphi^{123+45}_{12,12+45} + \varphi^{12+345}_{12,12+45}\right), \tag{A.36}$$

or

$$\varphi_{12,123}^{123+45} = G_0 \mathcal{T}^{123} \left(-P_{35}\right) \left(\left(\varphi_{12;12+34}^{125+34} + \varphi_{12;12+34}^{12+345} \right) + \left(\varphi_{12,12+34}^{1234} \right) \right) . \tag{A.37}$$

Further, Eq. (A.26) yields inserting the decomposition of the right-hand side related to

$$\varphi_{12,12+34}^{125+34} + \varphi_{12,12+34}^{12+345} = G_0 \mathcal{T}^{12+34} \left(\varphi_{12,125}^{1235} + \varphi_{12,125}^{1245} + \varphi_{12,125}^{125+34} + \varphi_{12,12+35}^{1235} \right.$$
$$\left. + \varphi_{12,12+35}^{124+35} + \varphi_{12,12+35}^{12+345} + \varphi_{12,12+45}^{1245} + \varphi_{12,12+45}^{123+45} + \varphi_{12,12+45}^{12+345} \right) . \tag{A.38}$$

Here, quite a few amplitudes can be related to previous ones by permutations leading to

$$\begin{aligned} \left(\varphi_{12,12+34}^{125+34} + \varphi_{12,12+34}^{12+345} \right) &= G_0 \mathcal{T}^{12+34} \left(-P_{35} - P_{45}\right) \\ &\times \left(\left(\varphi_{12,12+34}^{125+34} + \varphi_{12,12+34}^{12+345} \right) - P_{45} (1 - P_{34}) \, \varphi_{12,12+34}^{1234} \right. \\ &\left. - P_{35} \left((1 - P_{34}) \, \varphi_{12;123}^{1234} + \varphi_{12,123}^{123+45} \right) \right) . \end{aligned} \tag{A.39}$$

Appendix B

Definition of the Jacobi momenta and partial-wave basis states

Here, we display some Jacobi momenta related to the 5-body system in the case of two specific components. For $\varphi_{12;123}^{1234}$ in terms of the first configuration in Fig. 1, we choose

$$\begin{aligned} \boldsymbol{a}_1 &= 1/2 \left(\boldsymbol{p}_1 - \boldsymbol{p}_2 \right) , \\ \boldsymbol{a}_2 &= 1/3 \left(2\boldsymbol{p}_3 - (\boldsymbol{p}_1 + \boldsymbol{p}_2) \right) , \\ \boldsymbol{a}_3 &= 1/4 \left(3\boldsymbol{p}_4 - (\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3) \right) , \\ \boldsymbol{a}_4 &= 1/5 \left(4\boldsymbol{p}_5 - (\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4) \right) . \end{aligned} \tag{B.1}$$

In the non-relativistic case, we may express the kinetic energy operator by two equivalent forms. So, the kinetic energy in terms of $\boldsymbol{a}$-set Jacobi momenta, is given as

$$H_0^a = \sum_{i=1}^{5} \frac{p_i^2}{2m} \equiv \frac{a_1^2}{m} + \frac{3}{4}\frac{a_2^2}{m} + \frac{2}{3}\frac{a_3^2}{m} + \frac{5}{8}\frac{a_4^2}{m} , \tag{B.2}$$

where p_i is an individual particle momentum in the center-of-mass form (under the condition $\sum_i \boldsymbol{p}_i = 0$) that is described by relative Jacobi momenta $a_i; (i = 1,\ 2,\ 3,\ 4)$. In the conventional Yakubovsky treatment, the

total Hamiltonian, according to Eq. (2.1), is first split into the free Hamiltonian H_0 and the interaction Hamiltonian (summation of all pair interactions).

Similarly, to $\varphi^{1234}_{12;12+34}$ in terms of the second configuration in Fig. 1 belongs

$$\begin{aligned} \boldsymbol{b}_1 &= 1/2\,(\boldsymbol{p}_1 - \boldsymbol{p}_2)\,, \\ \boldsymbol{b}_2 &= 1/2\,(\boldsymbol{p}_3 - \boldsymbol{p}_4)\,, \\ \boldsymbol{b}_3 &= 1/2\,((\boldsymbol{p}_1 + \boldsymbol{p}_2) - (\boldsymbol{p}_3 + \boldsymbol{p}_4))\,, \\ \boldsymbol{b}_4 &= 1/5\,(4\boldsymbol{p}_5 - (\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3 + \boldsymbol{p}_4))\,. \end{aligned} \tag{B.3}$$

Correspondingly, the kinetic energy in terms of $\boldsymbol{b}$-set Jacobi momenta, is given as

$$H_0^b = \sum_{i=1}^{5} \frac{p_i^2}{2m} \equiv \frac{b_1^2}{m} + \frac{b_2^2}{m} + \frac{1}{2}\frac{b_3^2}{m} + \frac{5}{8}\frac{b_4^2}{m}\,. \tag{B.4}$$

Now, we introduce the basis states corresponding to the two specific independent components. The partial-wave basis states suitable for $\varphi^{1234}_{12;123}$ are given as

$$|a\rangle \equiv |a_1 a_2 a_3 a_4;\ \gamma_a\rangle\,, \tag{B.5}$$

and we represent the basis states for $\varphi^{1234}_{12;12+34}$ Jacobi momenta as

$$|b\rangle \equiv |b_1 b_2 b_3 b_4;\ \gamma_b\rangle\,. \tag{B.6}$$

We apply the usage of these basis states without angular momentum, spin and isospin effects, *i.e.* $\gamma_a = \gamma_b = 0$, and here, we study the spinless particles. Though, in the numerical techniques, we describe dependent on angular grid points by choosing the relevant coordinate systems, because total angular momentums of the 5-body system are restricted in $L = 1$ state, and the Pauli principle will be taken into account (see Sect. 4).

Clearly, all basis states are complete in the five-body Hilbert space

$$\int A^2 DA\ |A_1 A_2 A_3 A_4\rangle\langle A_1 A_2 A_3 A_4| \equiv 1\,, \tag{B.7}$$

where A_i indicates each one of a_i and b_i magnitude of vectors, and

$$A^2 DA \equiv A_1^2 dA_1\ A_2^2 dA_2\ A_3^2 dA_3\ A_4^2 dA_4\,. \tag{B.8}$$

They are normalized according to

$$\left\langle A_1 A_2 A_3 A_4 | A_1' A_2' A_3' A_4' \right\rangle = \frac{\delta\,(A_1 - A_1')}{A_1^2}\frac{\delta\,(A_2 - A_2')}{A_2^2}\frac{\delta\,(A_3 - A_3')}{A_3^2}\frac{\delta\,(A_4 - A_4')}{A_4^2}\,. \tag{B.9}$$

Appendix C

Explicit partial-wave evaluation of the coupled equations

To evaluate the coupled equations, Eq. (4.1) and (4.2), the following matrix elements need to be evaluated in a PW analysis

$$\langle a|\, G_0\mathcal{T}^{123}\,|a'\rangle\ , \tag{C.1}$$
$$\langle a'|\,(P_{34}P_{45}-P_{34})\,|a''\rangle\ , \tag{C.2}$$
$$\langle a'|b'\rangle\ , \tag{C.3}$$
$$\langle b|\, G_0\mathcal{T}^{12+34}\,|b'\rangle\ , \tag{C.4}$$
$$\langle b'|\,(1-P_{45}-P_{34}+P_{34}P_{45})\,|a'\rangle\ . \tag{C.5}$$

To evaluate the first term, Eq. (C.1), we need to solve the first subcluster Faddeev-like equation to obtain $\mathcal{T}^{123}$ by using the Padé approximation [11] as follows:

$$G_0\mathcal{T}^{123}=G_0t_{12}P+G_0t_{12}PG_0t_{12}P+G_0t_{12}P\ G_0t_{12}PG_0t_{12}P+\ldots \tag{C.6}$$

To evaluate the first term of Eq. (C.6), once more a completeness relation has to be inserted between the two-body t-matrix and the permutation operators

$$\langle a|G_0t_{12}P|a'\rangle=G_0\int a''^2Da''\,\langle a\,|t_{12}|\,a''\rangle\,\langle a''\,|P|\,a'\rangle\ , \tag{C.7}$$

where

$$\langle a\,|t_{12}|\,a''\rangle=\langle a_1\,|t_{12}|\,a_1''\rangle\,\langle a_2|a_2''\rangle\,\langle a_3|a_3''\rangle\,\langle a_4|a_4''\rangle\ , \tag{C.8}$$

and

$$\langle a'\,|P|\,a''\rangle=\langle a'\,|P_{12}P_{23}|\,a''\rangle+\langle a'\,|P_{13}P_{23}|\,a''\rangle\ . \tag{C.9}$$

Furthermore,

$$\begin{aligned}\langle a\,|t_{12}|\,a''\rangle\ &=\ \langle a_1\,|t_{(\epsilon)}|\,a_1''\rangle\,\frac{\delta\,(a_2-a_2'')}{(a_2'')^2}\,\frac{\delta\,(a_3-a_3'')}{(a_3'')^2}\,\frac{\delta\,(a_4-a_4'')}{(a_4'')^2}\,;\\ \epsilon\ &=\ E-\frac{3}{4}\frac{a_2^2}{m}-\frac{2}{3}\frac{a_3^2}{m}-\frac{5}{8}\frac{a_4^2}{m}\,,\end{aligned} \tag{C.10}$$

where ϵ is the energy of two-body subsystem in a-set configuration, and

$$\begin{aligned}\langle a''|\,P\,|a'\rangle\ &=\ \frac{\delta\,(a_3''-a_3')}{(a_3')^2}\,\frac{\delta\,(a_4''-a_4')}{(a_4')^2}\\ &\quad\times\int_{-1}^{1}\mathrm{d}x_{2''2'}\,\frac{\delta\left[a_1'-\left|-\frac{1}{2}\boldsymbol{a}_2''-\boldsymbol{a}_2'\right|\right]}{[a_1']^2}\,\frac{\delta\left[a_1''-\left|\frac{1}{2}\boldsymbol{a}_2'+\boldsymbol{a}_2''\right|\right]}{[a_1'']^2}\,.\end{aligned} \tag{C.11}$$

To evaluate the term of Eq. (C.2), there is a relation between Jacobi momenta in different chains, $(123+4+5;12)$ and $(124+5+3;12)$, which leads to

$$\begin{aligned}\left\langle a'\left|P_{34}P_{45}\right|a''\right\rangle &= \frac{1}{2^3}\frac{\delta\left(a_1'-a_1''\right)}{\left(a_1''\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}a_{23}\int_{-1}^{1}\mathrm{d}a_{24}\int_{-1}^{1}\mathrm{d}a_{34}\frac{\delta\left[a_2'-\left|\frac{1}{3}\boldsymbol{a}_2''+\frac{2}{9}\boldsymbol{a}_3''+\frac{5}{12}\boldsymbol{a}_4''\right|\right]}{\left(a_2''\right)^2}\\ &\times\frac{\delta\left[a_3''-\left|\boldsymbol{a}_2''+\frac{1}{12}\boldsymbol{a}_3''-\frac{5}{16}\boldsymbol{a}_4''\right|\right]}{\left(a_3''\right)^2}\frac{\delta\left[a_4''-\left|\boldsymbol{a}_3''-\frac{1}{5}\boldsymbol{a}_4''\right|\right]}{\left(a_4''\right)^2},\end{aligned} \tag{C.12}$$

and $(123+4+5;12)$ and $(124+3+5;12)$,

$$\begin{aligned}\left\langle a'\left|P_{34}\right|a''\right\rangle &= \frac{1}{2}\frac{\delta\left(a_1'-a_1''\right)}{\left(a_1''\right)^2}\frac{\delta\left(a_4'-a_4''\right)}{\left(a_4''\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}a_{23}\frac{\delta\left[a_2''-\left|\frac{1}{3}\boldsymbol{a}_2''+\frac{8}{9}\boldsymbol{a}_3''\right|\right]}{\left(a_2''\right)^2}\frac{\delta\left[a_3''-\left|\boldsymbol{a}_2''-\frac{1}{3}\boldsymbol{a}_3''\right|\right]}{\left(a_3''\right)^2}.\end{aligned} \tag{C.13}$$

To evaluate the term of Eq. (C.3), there is a relation between Jacobi momenta in different chains, $(123+4+5;12)$ and $(12+34+5;12)$, which leads to

$$\begin{aligned}\left\langle a'|b'\right\rangle &= \frac{1}{2}\frac{\delta\left(a_1'-b_1'\right)}{\left(b_1'\right)^2}\frac{\delta\left(a_4'-b_4'\right)}{\left(b_4'\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}a_{23}\frac{\delta\left[b_2'-\left|\frac{1}{2}\boldsymbol{a}_2''-\frac{2}{3}\boldsymbol{a}_3''\right|\right]}{\left(b_2'\right)^2}\frac{\delta\left[b_3'-\left|\boldsymbol{a}_2''-\frac{2}{3}\boldsymbol{a}_3''\right|\right]}{\left(b_3'\right)^2}.\end{aligned} \tag{C.14}$$

Correspondingly, to evaluate Eq. (C.4), we need to solve the subcluster Faddeev-like equation to obtain T^{12+34} by using the Padé approximation [11] as follows:

$$G_0\mathcal{T}^{12+34}=G_0t_{12}\tilde{P}+G_0t_{12}\tilde{P}\,G_0t_{12}\tilde{P}+G_0t_{12}\tilde{P}\,G_0t_{12}P\,G_0t_{12}\tilde{P}+\ldots \tag{C.15}$$

To evaluate the first term of Eq. (C.15), once more a completeness relation has to be inserted between the two-body t-matrix and the permutation operators

$$\left\langle b\left|G_0t_{12}\tilde{P}\right|b'\right\rangle=G_0\int b''^2Db''\left\langle b\left|t_{12}\right|b''\right\rangle\left\langle b''\left|\tilde{P}\right|b'\right\rangle, \tag{C.16}$$

where

$$\left\langle b\left|t_{12}\right|b''\right\rangle = \left\langle b_1\left|t_{12}\right|b_1''\right\rangle \left\langle b_2|b_2''\right\rangle \left\langle b_3|b_3''\right\rangle \left\langle b_4|b_4''\right\rangle \tag{C.17}$$

and

$$\left\langle b''\left|\tilde{P}\right|b'\right\rangle = \left\langle b''\left|P_{13}P_{24}\right|b'\right\rangle . \tag{C.18}$$

The matrix elements of two-body t-matrix and the permutation operator $\tilde{P}$ are evaluated as

$$\begin{aligned}\left\langle b\left|t_{12}\right|b''\right\rangle &= \left\langle b_1\left|t_{(\epsilon^*)}\right|b_1''\right\rangle \frac{\delta\left(b_2''-b_2\right)}{\left(b_2''\right)^2}\frac{\delta\left(b_3''-b_3\right)}{\left(b_3''\right)^2}\frac{\delta\left(b_4''-b_4\right)}{\left(b_4''\right)^2};\\ \epsilon^* &= E-\frac{b_2^2}{m}-\frac{1}{2}\frac{b_3^2}{m}-\frac{5}{8}\frac{b_4^2}{m},\end{aligned} \tag{C.19}$$

where ϵ^* is the energy of two-body subsystem in b-set configuration. To evaluate the matrix elements of the permutation operator $\tilde{P}$, there is a relation between Jacobi momenta in different chains, $(12+34+5;12)$ and $(34+12+5;12)$,

$$\left\langle b''\left|\tilde{P}\right|b'\right\rangle = \frac{\delta\left(b_1''-b_2'\right)}{\left(b_2'\right)^2}\frac{\delta\left(b_2''-b_1'\right)}{\left(b_1'\right)^2}\frac{\delta\left(b_3''-b_3'\right)}{\left(b_3'\right)^2}\frac{\delta\left(b_4''-b_4'\right)}{\left(b_4'\right)^2}. \tag{C.20}$$

To evaluate the first term of Eq. (C.5), there is a relation between Jacobi momenta in different chains, $(12+34+5;12)$ and $(123+4+5;12)$,

$$\begin{aligned}\left\langle b'|a'\right\rangle &= \frac{1}{2}\frac{\delta\left(b_1'-a_1'\right)}{\left(a_1'\right)^2}\frac{\delta\left(b_4'-a_4'\right)}{\left(a_4'\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}b_{32'}\frac{\delta\left[b_2'-\left|\frac{2}{3}\boldsymbol{b}_2'-\frac{2}{3}\boldsymbol{b}_3'\right|\right]}{\left(b_2'\right)^2}\frac{\delta\left[b_3'-\left|-\boldsymbol{b}_2'-\frac{1}{2}\boldsymbol{b}_3'\right|\right]}{\left(b_3'\right)^2}.\end{aligned} \tag{C.21}$$

To evaluate the second term of Eq. (C.5), there is a relation between Jacobi momenta in different chains, $(12+34+5;12)$ and $(123+5+4;12)$,

$$\begin{aligned}\left\langle b'\left|P_{45}\right|a'\right\rangle &= \frac{1}{2^3}\frac{\delta\left(a_1'-b_1'\right)}{\left(a_1'\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}b_{32'}\int_{-1}^{1}\mathrm{d}b_{42'}\int_{-1}^{1}\mathrm{d}b_{34}\frac{\delta\left[a_2'-\left|\frac{2}{3}\boldsymbol{b}_2'-\frac{2}{3}\boldsymbol{b}_3'\right|\right]}{\left(a_2'\right)^2}\\ &\times\frac{\delta\left[a_3'-\left|-\frac{1}{4}\boldsymbol{b}_2'-\frac{1}{8}\boldsymbol{b}_3'+\frac{3}{4}\boldsymbol{b}_4'\right|\right]}{\left(a_3'\right)^2}\frac{\delta\left[a_4'-\left|-\frac{1}{2}\boldsymbol{b}_2'-\frac{1}{4}\boldsymbol{b}_3'+\frac{1}{2}\boldsymbol{b}_4'\right|\right]}{\left(a_4'\right)^2}.\end{aligned} \tag{C.22}$$

To evaluate the third term of Eq. (C.5), there is a relation between Jacobi momenta in different chains, $(12+34+5;12)$ and $(124+3+5;12)$,

$$\begin{aligned}\left\langle b'\left|P_{34}\right|a'\right\rangle &= \frac{1}{2}\frac{\delta\left(a'_1-b'_1\right)}{\left(a'_1\right)^2}\frac{\delta\left(a'_4-b'_4\right)}{\left(a'_4\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}b_{32'}\frac{\delta\left[a'_2-\left|-\frac{2}{3}\boldsymbol{b}'_2-\frac{2}{3}\boldsymbol{b}'_3\right|\right]}{\left(a'_2\right)^2}\frac{\delta\left[a'_3-\left|\boldsymbol{b}'_2-\frac{1}{2}\boldsymbol{b}'_3\right|\right]}{\left(a'_3\right)^2}\,. \quad \text{(C.23)}\end{aligned}$$

To evaluate the fourth term of Eq. (C.5), there is a relation between Jacobi momenta in different chains, $(12+34+5;12)$ and $(124+5+3;12)$,

$$\begin{aligned}\left\langle b'\left|P_{34}P_{45}\right|a'\right\rangle &= \frac{1}{2^3}\frac{\delta\left(a'_1-b'_1\right)}{\left(a'_1\right)^2}\\ &\times\int_{-1}^{1}\mathrm{d}b_{32'}\int_{-1}^{1}\mathrm{d}b_{42'}\int_{-1}^{1}\mathrm{d}b_{34}\frac{\delta\left[a'_2-\left|-\frac{2}{3}\boldsymbol{b}'_2-\frac{2}{3}\boldsymbol{b}'_3\right|\right]}{\left(a'_2\right)^2}\\ &\times\frac{\delta\left[a'_3-\left|\frac{1}{4}\boldsymbol{b}'_2-\frac{1}{8}\boldsymbol{b}'_3+\frac{3}{4}\boldsymbol{b}'_4\right|\right]}{\left(a'_3\right)^2}\frac{\delta\left[a'_4-\left|\frac{1}{2}\boldsymbol{b}'_2-\frac{1}{4}\boldsymbol{b}'_3+\frac{1}{2}\boldsymbol{b}'_4\right|\right]}{\left(a'_4\right)^2}\,. \quad \text{(C.24)}\end{aligned}$$

In Appendix C, the quantities a_{ij} (b_{ij}) indicate the angle variable between $\boldsymbol{a}_i$ and $\boldsymbol{a}_j$ ($\boldsymbol{b}_i$ and $\boldsymbol{b}_j$), namely $a_{ij}\equiv\cos(a_i,\ a_j)$ and $b_{ij}\equiv\cos(b_i,\ b_j)$, respectively.

REFERENCES

[1] D.S. Chuu, C.S. Han, *J. Phys. G: Nucl. Phys.* **3**, 555 (1977).

[2] D.R. Lehman, *Phys. Rev. C* **25**, 3146 (1982).

[3] N. Alexander, K. Amos, B. Apagyi, D.R. Lun, *Phys. Rev. C* **53**, 88 (1996).

[4] S. Ali, A.A.Z. Ahmad, N. Ferdous, *Rev. Mod. Phys.* **57**, 923 (1985).

[5] L.A. Barreiro, R. Higa, C.L. Lima, M.R. Robilotta, *Phys. Rev. C* **57**, 2142 (1998).

[6] K. Varga, Y. Suzuki, *Phys. Rev. C* **52**, 2885 (1995).

[7] M. Gattobigio, A. Kievsky, M. Viviani, *Few-Body Syst.* **50**, 463 (2011).

[8] E. Ahmadi Pouya, A.A. Rajabi, *Eur. Phys. J. Plus* **131**, 240 (2016).

[9] E. Ahmadi Pouya, A.A. Rajabi, *Int. J. Mod. Phys. E* **25**, 1650072 (2016).

[10] W. Glöckle, *The Quantum Mechanical Few-Body Problem*, Springer-Verlag, New York 1983.

[11] H. Kamada, W. Glöckle, *Nucl. Phys. A* **548**, 205 (1992).

[12] A. Stadler, W. Glöckle, P.U. Saur, *Phys. Rev. C* **44**, 2319 (1991).
[13] A. Nogga, H. Kamada, W. Glöckle, B.R. Barrett, *Phys. Rev. C* **65**, 054003 (2002).
[14] D. Huber *et al.*, *Few-Body Syst.* **22**, 107 (1997).
[15] A.B. Volkov, *Nucl. Phys.* **74**, 33 (1965).
[16] R.A. Malfliet, J.A. Tjon, *Nucl. Phys. A* **127**, 161 (1969).
[17] M. Viviani, A. Kievsky, S. Rosati, *Phys. Rev. C* **71**, 024006 (2005).
[18] M. Gattohigio, A. Kievsky, M. Viviani, *Phys. Rev. C* **83**, 024001 (2011).
[19] N. Barnea, *Phys. Rev. A* **59**, 1135 (1999).
[20] R.B. Wiringa, private communication.